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Heuristic Solving Linear Programming Problems

Heurystyczne rozwiązywanie problemów programowania liniowego

Эвристическое решение проблем линейного программирования

The exact solutions of linear programming problems are obtained by means of simplex method and the electronic calculating machines. The paper is aimed at showing that finding the approximate solutions of certain important practically linear programming problems by means of heuristic approach requires using neither the sophisticated algorithm nor the electronic computer. Let us notice that the advisability of such an approach to solving linear programming problems as well as the practical usefulness of the approximate solutions had been first emphasized by L. V. Kantorowicz (1).

The paper is divided into two parts. The general exposition of the heuristic approach is followed by several examples of its application.

1. Let us consider the following linear programming problem:

maximize the objective function

$$S = X^T R \tag{1}$$

subject to

$$AX + Y = P_0 \tag{2}$$

$$X \geqslant 0 \tag{3}$$

$$Y \geqslant 0 \tag{4}$$

where

X — the column vector $(n \times 1)$ of the decision variables,

X^T — the transposition of the vector X,

Y — the column vector $(m \times 1)$ of the slack variables,
R — the column vector $(n \times 1)$ of the objective function parameters,
A — the matrix $(m \times n)$ of the coefficients,
P_o — the column vector $(m \times 1)$ of the constants.

Assuming that some coefficients in the matrix A are equal to zero [1] the approximate solution of the problem (1)—(4) can be found by means of the following heuristic procedure:

a) **Finding the first solution:**

— choosing the first sequence of the decision variables:

$$x_j^1,\ x_i^2\ \ldots\ldots\ x_i^n$$

on the basis of

$$r_j^1 > r_j^2 > \ldots\ldots > r_j^n$$

where

$$j \in N$$

N — the set of the decision variables subscripts,
r_j — the j-th element of the vector R.

— finding the maximum value, of the variable x_j^1:

$$\max,\ x_j^1 = \min.\left(\frac{b_1}{a_{1j}},\ \frac{b_2}{a_{2j}}\ \ldots\ldots\ \frac{b_m}{a_{mj}}\right)$$

where

a_{ij} — the element of the matrix A standing in the i-th row and the j-th column,
b_i — the i-th element of the vector P_o.

— finding the vector P'_{o1} of the constants:

$$P'_{o1} = P_o - x_j^1 P_j^1$$

where

P_j — the j-th column vector of the matrix A corresponding with the variable x_j^1

[1] Let us notice that it das not pay to use the heuristic approach when there are all coefficients bigger than zero in the matrix A.

— finding the maximum value of the variable x_j^2:

$$\max.\ x_j^2 = \left(\frac{b_1'}{a_{1j}},\quad \frac{b_2'}{a_{2j}} \ \cdots\cdots\ \frac{b_m'}{b_{mj}}\right)$$

where

b_i' — the i-th element of the vector P_{o1}'

— finding the vector P_{o1}'' of the constants:

$$P_{o1}'' = P_{o1}' - x_j^2 P_j^2$$

where

P_j^2 — the j-th column vector of the matrix A corresponding to the variable x_j^2

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— finding the maximum value of the variable x_j^n

$$\max.\ x_j^n = \text{mtn.}\left(\frac{b_1''}{a_{1j}},\quad \frac{b_2''}{a_{2j}},\ \cdots\cdots\ \frac{b_m''\cdots}{a_{mj}}\right)$$

where

$b_i''\cdots$ — the i-th element of the vector $P_{o1}''\cdots$.

— finding the vector $P_{o1}''\cdots'$ of the constants:

$$P_{o1}''\cdots' = P_{o1}''\cdots - x_j^n P_j^n$$

where

P_j^n — the j-th column vector of the matrix A corresponding to the variable x_j^n

— finding the values of the slack variables:

$$Y = P_{o1}''\cdots'$$

b) Finding the second solution:

— choosing the second sequence of the decision variables on the basis of the first sequence and the results of the first solution [2]

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[2] The way of performing that operation will be further explained when solving an example below.

— performance of the operations shown above when looking for the first solution.

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(s) **Finding the s-th solution** [3] in the way shown above.

(s+1) Choosing the approximate solution of the problem (1)—(4):

— finding the approximate maximum value of the objective function on the basis of

$$\text{appr. max. } S = \text{max. } ({}^1X^TR, {}^2X^TR, \ldots, {}^sX^TR)$$

where

${}^pX^T$ $(p = 1, 2, \ldots, s)$ — the vector (1xn) of the values assigned to the decision variables in the p-th solution

— defining the approximate solution:

assuming that the appr. max. S has been achieved in the p-th solution then the approximate solution is given by the vectors: ${}^pX^T$ and $Y = P'_{Op}{}^{\cdots}$ where $P'_{Op}{}^{\cdots}$ is the vector of the constants obtained in the p-th solution [3].

An example.

Maximize the objective function

$$S = \sum_{j=1}^{4} x_j z_j$$

subject to

$$\begin{array}{llllllll}
a_{11}x_1 & & +a_{13}x_3+a_{14}x_4+y_1 & & & & & = b_1 \\
 & a_{22}x_2 & & +y_2 & & & & = b_2 \\
 & a_{32}x_2 & +a_{34}x_4 & & +y_3 & & & = b_3 \\
a_{41}x_1 & & & & & +y_4 & & = b_4 \\
 & & +a_{53}x_3+a_{54}x_4 & & & & +y_5 & = b_5
\end{array}$$

$$x_j \geqslant 0 \quad (j = 1 \ldots 4)$$

$$y_{j+k} \geqslant 0 \quad (k = 1 \ldots 5)$$

[3] As our experience shows, at least two solutions are necessary for choosing an approximate solution satisfactory practically.

(a) **Finding the first solution:**

— assuming that $z_4 > z_3 > z_1 > z_2$, then the first sequence of the decision variables is as follows:

$$x_4 \; x_3 \; x_1 \; x_2$$

— assuming also that

$$\max. \; x_4 = \min. \left(\frac{b_1}{a_{14}}, \quad \frac{b_3}{a_{34}}, \quad \frac{b_5}{a_{54}}\right) = \frac{b_1}{a_{14}}$$

$$\text{— then } P'_{01} = P_0 - x_4 P_4 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} - \frac{b_1}{a_{14}} \begin{bmatrix} a_{14} \\ 0 \\ a_{34} \\ 0 \\ a_{54} \end{bmatrix} = \begin{bmatrix} 0 \\ b_2 \\ b_3 - a_{34}\dfrac{b_1}{a_{14}} \\ b_4 \\ b_5 - a_{54}\dfrac{b_1}{a_{14}} \end{bmatrix}$$

$$\text{— } \max. \; x_3 = \min. \left(\frac{0}{a_{13}}, \quad \frac{b_5 - a_{54}\dfrac{b_1}{a_{14}}}{a_{53}}\right) = 0$$

$$\text{— } \max. \; x_1 = \min. \left(\frac{0}{a_{11}}, \quad \frac{b_4}{a_{41}}\right) = 0$$

$$\text{— assuming that } \max. \; x_2 = \min. \left(\frac{b_2}{a_{22}}, \quad \frac{b_3 - a_{34}\dfrac{b_1}{a_{14}}}{a_{32}}\right) = \frac{b_3 - a_{34}\dfrac{b_1}{a_{14}}}{a_{32}}$$

$$\text{— then } P''_{01} = P'_{01} - x_2 P_2 = \begin{bmatrix} 0 \\ b_2 \\ b_3 - a_{34}\dfrac{b_1}{a_{14}} \\ b_4 \\ b_5 - a_{54}\dfrac{b_1}{a_{14}} \end{bmatrix} - \frac{b_3 - a_{34}\dfrac{b_1}{a_{14}}}{a_{32}} \begin{bmatrix} 0 \\ a_{22} \\ a_{32} \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ b_2 - a_{22}x_2 \\ 0 \\ b_4 \\ b_5 - a_{54}x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

(b) **Finding the second solution:**

— as the second sequence of the decision variables we assume:

$$x_3 \quad x_4 \quad x_1 \quad x_2$$

because in the first solution $x_2 > 0$ and $x_3 = x_1 = 0$ in spite of that $z_3 > z_1 > z_2$.

— assuming that

$$\max. x_3 = \min. \left(\frac{b_1}{a_{13}}, \quad \frac{b_5}{a_{53}}\right) = \frac{b_5}{a_{53}}$$

$$\text{— then } P'_{02} = P_0 - x_3 P_3 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} - \frac{b_5}{a_{53}} \begin{bmatrix} a_{13} \\ 0 \\ 0 \\ 0 \\ a_{53} \end{bmatrix} = \begin{bmatrix} b_1 - a_{13}x_3 \\ b_2 \\ b_3 \\ b_4 \\ 0 \end{bmatrix}$$

$$\text{— } \max. x_4 = \min. \left(\frac{b_1 - a_{13}x_3}{a_{14}}, \quad \frac{b_3}{a_{34}}, \quad \frac{0}{a_{54}}\right) = 0$$

— assuming next that

$$\max. x_1 = \min. \left(\frac{b_1 - a_{13}x_3}{a_{14}}, \quad \frac{b_4}{a_{41}}\right) = \frac{b_4}{a_{41}}$$

$$\text{— then } P''_{02} = P'_{02} - x_1 P_1 \begin{bmatrix} b_1 - a_{13}x_3 \\ b_2 \\ b_3 \\ b_4 \\ 0 \end{bmatrix} - \frac{b_4}{a_{41}} \begin{bmatrix} a_{11} \\ 0 \\ 0 \\ a_{41} \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 - (a_{13}x_3 + a_{11}x_1) \\ b_2 \\ b_3 \\ 0 \\ 0 \end{bmatrix}$$

— and assuming lastly that

$$\max. x_2 = \min. \left(\frac{b_2}{a_{22}}, \quad \frac{b_3}{a_{32}}\right) = \frac{b_2}{a_{22}}$$

$$\text{— then } P'''_{02} = P''_{02} - x_2 P_2 = \begin{bmatrix} b_1 - (a_{13}x_3 + a_{11}x_1) \\ b_2 \\ b_3 \\ 0 \\ 0 \end{bmatrix} - \frac{b_2}{a_{22}} \begin{bmatrix} 0 \\ a_{22} \\ a_{32} \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} b_1-(a_{13}x_3+a_{11}x_1) \\ 0 \\ b_3-a_{32}x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

(c) **Choosing the approximate solution:**

— assuming that

$$\text{appr. max. } S = \text{max. } ({}^1X^TZ, {}^2X^TZ) = {}^2X^TZ$$

— then the approximate solution is as follows:

$$x_1 = \frac{b_4}{a_{41}} \quad x_2 = \frac{b_2}{a_{22}} \quad x_3 = \frac{b_5}{a_{53}} \quad y_1 = b_1-(a_{13}x_3+a_{11}x_1) \quad y_3 = b_3-a_{32}x_2$$

Let us notice that there are exactly five values larger than zero in the approximate solution which signify that the solution is basic.

2. We shall now turn to the applications of the presented heuristic approach. The nature of the first problem we are going to deal with is as follows:

to find such production program $X^T = [x_1, x_2, ..., x_n]$ which maximizes approximately the objective function

$$E = \frac{\sum_{j=1}^{n} w_j x_j}{\sum_{j=1}^{n} n_j x_j}$$

and fulfills the conditions

$$AX \leqslant P_o$$
$$X \geqslant 0$$

where

w_j, n_j — the unit output, input, respectively

$\frac{w_j}{n_j}$ — the unit effectiveness

$$E = \frac{\dfrac{\sum_{j=1}^{n} w_j x_j}{\sum_{j=1}^{n} x_j}}{\dfrac{\sum_{j=1}^{n} n_j x_j}{\sum_{j=1}^{n} x_j}} = \frac{\bar{w}}{\bar{n}} \text{ — the average effectiveness}$$

where

$\bar{w}$, $\bar{n}$ — the average output, input, respectively

Thus

$$\left(\frac{w_j}{n_j}\right)' \leqslant E^a \leqslant \left(\frac{w_j}{n_j}\right)''$$

where

E^a — the maximum value of the objective function,

$\left(\frac{w_j}{n_j}\right)'$ $\left(\frac{w_j}{n_j}\right)''$ — the smallest, largest unit effectiveness, respectively

Let us now consider the following numerical example:
maximize the objective function

$$E = \frac{37x_1+28x_2+30x_3+25x_4+17x_5+18x_6}{148x_1+122x_2+140x_3+136x_4+100x_5+120x_6}$$

subject to

$$\begin{array}{llr} x_1 \qquad\qquad + y_1 & = & 8\,000 \\ x_2 \qquad\qquad + y_2 & = & 25\,000 \\ 90x_1+80x_2+75x_3+50x_4+52x_5+60x_6 \quad + y_3 & = & 2\,870\,000 \\ 30x_1+20x_2+20x_3+18x_4+46x_5+34x_6 \quad + y_4 & = & 775\,000 \end{array}$$

$$x_j \ (j = 1, 2 \ldots 6) \geqslant 0$$

$$y_k \ (k = 1, 2 \ldots 4) \geqslant 0$$

(a) **Finding the first solution:**

— the first sequence of the decision variables is

$$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6$$

because the unit effectiveness indicators r_j are

j	1	2	3	4	5	6
r_j	25	23	21	18	17	15

where

$$r_j = [w_j/n_j]\,100$$

— then $\max.\ x_1 = \min.\left(\frac{8\,000}{1},\ \frac{2\,870\,000}{90} = 31\,889,\ \frac{775\,000}{30} = \right.$

$$\left. = 25\,833\right) = 8000$$

— then $P_{01}' = P_o - x_1P_1 = \begin{bmatrix} 8\,000 \\ 25\,000 \\ 2\,870\,000 \\ 775\,000 \end{bmatrix} - 8\,000 \begin{bmatrix} 1 \\ 0 \\ 90 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 25\,000 \\ 2\,150\,000 \\ 535\,000 \end{bmatrix}$

— then $\max.\ x_2 = \min.\left(\frac{25\,000}{1},\ \frac{2\,150\,000}{80} = 26\,875,\ \frac{535\,000}{20} = \right.$

$$\left. = 26\,750\right) = 25\,000$$

— then $P_{01}'' = P_{01}' - x_2P_2 = \begin{bmatrix} 0 \\ 250\ \ 00 \\ 2\,150\,000 \\ 535\,000 \end{bmatrix} - 25\,000 \begin{bmatrix} 0 \\ 1 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 150\,000 \\ 35\,000 \end{bmatrix}$

— then $\max.\ x_3 = \min.\left(\frac{150\,000}{75} = 20000,\ \frac{35\,000}{20} = 1\,750\right) = 1\,750$

— then $P_{01}''' = P_{01}'' - x_3P_3 = \begin{bmatrix} 0 \\ 0 \\ 150\,000 \\ 35\,000 \end{bmatrix} - 1\,750 \begin{bmatrix} 0 \\ 0 \\ 75 \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 18\,750 \\ 0 \end{bmatrix}$

— then $x_4 = x_5 = x_6 = 0$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 18\,750 \\ 0 \end{bmatrix}$

As can be inferred from the foregoing discussion the solution just obtained can be already considered as the one looked for. The corresponding value of the objective function is the maximum value E^a, assuming that the values of the slack variables are minimized.

The next problem to be considered here is given in Table 1. When solving it, let us first notice and take advantage of its specific structure. Namely, the set of 16 decision variables can be divided into two parts. To the first part belong those variables values of which may be determined independently on the basis of only one appropriate constraint:

x_5 (2)

$x_6, x_7, x_8, x_{10}, x_{11}, x_{16}$ (3) [4]

x_{12} (5)

and to the second part those remaining, values of which cannot be found in such a way:

(a) **Finding the values of the decision variables belonging to the first category:**

— $\max. x_5 = \frac{120}{60} = 2$ and $y_3 = 0$

— $\max. x_{12} = \frac{322}{32{,}2} = 10$ and $y_5 = 0$

— the sequence of the variables $x_6, x_7, x_8, x_{10}, x_{11}, x_{16}$ is as follows:

j 16 6 7 8 11 10

because the standardized parameters of the objective function are

j	6	7	8	10	11	16
e_j	0,028	0,025	0,0224	0,016	0,0215	0,085

where $e_j = r_j/a_{2j}$

where r_j — the j-th parameter in the objective function,
a_{2j} — the j-th coefficient in the second constraint.

— hence $\max. x_{16} = \frac{2430}{81} = 30$

and $x_6 = x_7 = x_8 = x_{10} = x_{11} = 0$

and $y_2 = 0$

[4] The reference number of the constraint.

(b) **Determining the values of the variables belonging to the second category:**

— to maximize

$$F' = 0{,}88x_1 + 1{,}94x_2 + 3{,}8x_3 + 5{,}44x_4 + 6{,}86x_9 + 1{,}94x_{13} + 10{,}5x_{14} + 10{,}93x_{15}$$

subject to

(1) $89{,}3x_1 + 164x_2 + 132{,}2x_3 + 199{,}7x_4 + 501x_9 + 67{,}6x_{13} + 603{,}6x_{14} + 469{,}3x_{15} + y_1 = 3424$

(4) $49{,}93x_9 + 26{,}04x_{14} + 26{,}04x_{15} + y_4 = 46$

(6) $16{,}11x_3 + 33{,}83x_4 + y_6 = 35$

(7) $16x_1 + y_7 = 20$

— the sequence of the decision variables is

$$x_3 \; x_{13} \; x_4 \; x_{15} \; x_{14} \; x_9 \; x_2 \; x_1$$

because the standardized parameters of the objective function are

j	1	2	3	4	9	13	14	15
f_j	0,010	0,012	0,03	0,027	0,014	0,028	0,017	0,023

where $f_j = r_j/a_{1j}$

where a_{1j} — the j-th coefficient in the first constraint

— $$\max. x_3 = \min. \left(\frac{3\,424}{132{,}2} = 25{,}9, \quad \frac{35}{16{,}11} = 2{,}17\right) = 2{,}17$$

— $$P_0' = P_0 - x_3P_3 = \begin{bmatrix} 3\,424 \\ 46 \\ 35 \\ 20 \end{bmatrix} - 2{,}17 \begin{bmatrix} 132{,}2 \\ 0 \\ 16{,}11 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\,137{,}13 \\ 46 \\ 0 \\ 20 \end{bmatrix}$$

Tab. 1.

To maximize the objective function

$$F = 0{,}88x_1+1{,}94x_2+3{,}8x_3+5{,}44x_4+3{,}52x_5+0{,}48x_6+0{,}58x_7+0{,}72x_8+6{,}86x_9+3{,}5x_{10}+2{,}28x_{11}+28x_{12}+1{,}94x_{13}+10{,}5x_{14}+10{,}93x_{15}+6{,}91x_{16}$$

subject to

(1) $89{,}3x_1+164x_2+132{,}2x_3+199{,}7x_4+501x_9+67{,}6x_{13}+603{,}6x_{14}+469{,}3x_{15}+y_1 = 3424$

(2) $17{,}2x_6+22{,}9x_7+32{,}1x_8+213x_{10}+106x_{11}+81x_{16}+y_2 = 2430$

(3) $60x_5+y_3 = 120$

(4) $49{,}93x_9+26{,}04x_{14}+26{,}04x_{15}+y_4 = 46$

(5) $32{,}2x_{12}+y_5 = 322$

(6) $16{,}11x_3+33{,}83x_4+y_6 = 35$

(7) $16x_1+y_7 = 20$

(8) $x_j \geqslant 0 \quad (j = 1, 2 \ldots 16)$

(9) $y_k \geqslant 0 \quad (k = 1, 2 \ldots 7)$

$$-\ \text{max. } x_{13} = \frac{3\,137,13}{67,6} = 46,41$$

$$-\ P_0'' = P_0' - x_{13}P_{13} = \begin{bmatrix} 3\,137,13 \\ 46 \\ 0 \\ 20 \end{bmatrix} - 46,41 \begin{bmatrix} 67,5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 46 \\ 0 \\ 20 \end{bmatrix}$$

$$-\ \text{hence } x_1 = x_2 = x_4 = x_9 = x_{14} = x_{15} = 0$$

$$\text{and} \begin{bmatrix} y_1 \\ y_4 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 46 \\ 0 \\ 20 \end{bmatrix}$$

Taking into account the shown specific structure of the problem and the performed standardization of the objective function parameters we shall assume as the approximate solution the one just obtained:

$$x_3 = 2,17 \qquad x_5 = 2 \qquad x_{12} = 10 \qquad x_{13} = 46,41 \qquad x_{16} = 30$$

$$y_4 = 46 \qquad y_7 = 20$$

Let us notice moreover that the outlined here general idea of the heuristic solving linear programming problems has — as our experience proves — many applications of practical importance, e. g. in finding the production assortment at the factory level.

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STRESZCZENIE

Celem tego artykułu jest ukazanie, że aby znaleźć przybliżone rozwiązania pewnych praktycznie istotnych problemów programowania liniowego za pomocą podejścia heurystycznego, nie jest konieczne stosowanie ani skomplikowanego algorytmu, ani komputera elektronicznego.

Artykuł dzieli się na dwie części. Po ogólnym przedstawieniu proponowanej procedury heurystycznej następuje kilka przykładów jej zastosowania.

РЕЗЮМЕ

Цель статьи — показать, что для того, чтобы найти приближенное решение некоторых существенных с практической точки зрения преблем минейного программирования с помощью эвристического подхода, необязательно применять ни сложный логарифм, ни электронный компьютер.

Статья состоит из двух частей. После общего представления предлагаемой эвристической процедуры приводятся примеры ее применения.