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**Pairing Approximation for  $p_{3/2}^+$  and  $f_{7/2}^-$  Nuclear Shells**

**Przybliżenie sił pairing dla powłok jądrowych  $p_{3/2}^+$  i  $f_{7/2}^-$ .**

**Приближение парных сил для  $p_{3/2}^+$  и  $f_{7/2}^-$  оболочек**

1. INTRODUCTION

The pairing forces treated as a residual interaction between nucleons moving in the common potential are known as having the properties of a short-range interaction. The idealisation of the interaction is rewarded by the possibility of exact calculation of their matrix elements between any of the states in mixing configurations of identical nucleons  $j_{11}^{n_1} j_{22}^{n_2} \dots j_k^{n_k}$ . The structure of the ground and excited states depends on the allowed transition of nucleon pairs coupling to  $J = 0$ . After having calculated all matrix elements one can diagonalise the pairing Hamiltonian.

In this paper we restricted the problem to the consideration of two shells only. The main properties of the pairing interaction can be seen even under this restriction, and, on the other hand, uncertainty introduced by the assumption of pairing interaction seems to be much greater than the uncertainty made by the assumption of two shells.

To diagonalise the pairing Hamiltonian we exploited the quasi-spin formalism first used for the same purpose by Kerman, Lawson, and Macfarlane [1]. Pairing Hamiltonian allows for introducing of the Racah [2] and Flowers [3] seniority scheme. The low energy levels are those with seniority equal to zero (even nuclei) and one (odd nuclei). Both cases were considered but the attempt of comparison with few experimental data was carried out for odd nuclei with seniority equal to one and with identical nucleons on  $p_{3/2}^+$  and  $f_{7/2}^-$  shells.

## 2. THE QUASI-SPIN FORMALISM

The quasi-spin formalism introduced by Anderson [4] and Wada, Takano, Fokuda [5] in the theory of superconductivity was applied in the nuclear problem by Kerman, Lawson, Macfarlane [1] and Helmers [6]. We will briefly repeat the method in the form appropriate to our case.

The pairing Hamiltonian for the  $j^n$  configuration of equivalent nucleons can be written as

$$H = -GQ_+ Q_- \quad (1)$$

where

$$Q_+ = \sum_{m>0} (-1)^{j-m} a_{jm}^+ a_{j-m}^+ \quad Q_- = (Q_+)^+, \quad (2)$$

$G$  is the strength of the pairing forces, and  $a_{jm}^+$ ,  $a_{jm}$  are the creation and annihilation operators of the particle in the state  $j$ ,  $m$ .

If we add to the operators (2) all those of the type  $a + a$  i. e.,

$$\begin{aligned} \sum_{m>0} a_{jm}^+ a_{jm} & \quad \sum_{m>0} a_{jm}^+ a_{j-m} \equiv q_+ \\ \sum_{m>j} a_{j-m}^+ a_{j-m} & \quad \sum_{m>0} a_{j-m}^+ a_{jm} \equiv q_- \end{aligned} \quad (3)$$

we can easily check that they form a complete set of infinitesimal operators whose algebra is the Lie algebra associated with the orthogonal group  $R(4)$ . The  $R(4)$  group is, however, isomorphic to the direct product of  $R(3) \times R(3)$ . This can be seen also in the frame of the Lie algebra after taking the linear combinations

$$\begin{aligned} Q_0 &= \frac{1}{2} \left( \sum_{m>0} a_m^+ a_m + \sum_{m>0} a_{-m}^+ a_{-m} - j - \frac{1}{2} \right) \\ q_0 &= \frac{1}{2} \left( \sum_{m>0} a_m^+ a_m - \sum_{m>0} a_{-m}^+ a_{-m} \right) \end{aligned} \quad (4)$$

where we omitted the repeating index  $j$ . The operators  $Q_+$ ,  $Q_-$ ,  $Q_0$  as well as  $q_+$ ,  $q_-$ ,  $q_0$  form the complete sets of infinitesimal operators of the  $R(3)$  group, and thus they have property of angular momentum operators. Moreover, the Hamiltonian given by (1) does not depend on the vector  $q$  and commutes with it. Therefore, eigenstates of the  $H$  with different eigenvalues  $q$  are degenerated.

The eigenvalue of the  $H$  in the state labelled by quantum numbers  $Q$  and  $Q_0$  is

$$E = -G \left\{ Q(Q+1) - Q_0^2 + Q_0 \right\} \quad (5)$$

where 
$$Q_0 = \frac{1}{2}(n - \Omega) \quad Q = \frac{1}{2}(\Omega - s), \quad (6)$$

$n$  is the number of particles of the system,

$s$  is the seniority number,

$$\Omega = j + 1/2.$$

For the configuration of  $n$  particles within two shell  $j_1, j_2$  ( $\Omega_1, \Omega_2$ ) the pairing Hamiltonian is

$$H = 2\varepsilon \left( P_0 + \frac{1}{2} \Omega_2 \right) - G(Q_+ + P_+)(Q_- + P_-) \quad (7)$$

where  $\varepsilon$  is the single particle difference of the energy of the two levels  $j_1, j_2$  i. e.  $\varepsilon = E(j_2) - E(j_1)$ . The operators  $P_+, P_-, P_0$  are the quasi-spin operators for the  $j_2$  ( $\Omega_2$ ) shell. The single particle energy in (7) is taken with respect to the  $j_1$  level.

If we introduce the vector

$$J = Q + P \quad 8.$$

we get 
$$H = 2\varepsilon \left( P_0 + \frac{\Omega_2}{2} \right) - GJ_+J_- \quad (9)$$

and the problem becomes similar to the problem of adding two quasi-spin vectors. But if we try to construct the base diagonal in  $Q^2, P^2, J^2, J_0$  instead of that diagonal in  $Q^2, Q_0, P^2, P_0$  by means of the Clebsch-Gordan coefficients

$$|QP, JJ_0\rangle = \sum_{Q_0 P_0} C_{Q_0 P_0}^{JJ_0} |QQ_0, PP_0\rangle, \quad (10)$$

we gain nothing as the  $H$  is diagonal neither in the  $|Q Q_0, P P_0\rangle$  nor in the  $|QP, JJ_0\rangle$  base. Thus we start with the  $|Q Q_0, P P_0\rangle$  base and calculate all the non-vanishing matrix elements of the  $H$ . The states  $|Q Q_0, P P_0\rangle$  differ from one another by the excitation of one pair, two pairs, etc. to a higher level. The excitation of one pair is described by the change of  $Q_0$  by  $-1$  and  $P_0$  by  $+1$ . The Hamiltonian (9) has non-zero matrix elements between those states which either are the same or differ from one another by the excitation of one pair, because the operator  $J_+J_- \equiv J^2 - J_0^2 + J_0$  produces the changes in  $Q_0$  and  $P_0$  just of  $0$  or  $\pm 1$ .

Writing explicitly we get

$$\begin{aligned} J_+J_- |QQ_0, PP_0\rangle &= \left\{ (Q+Q_0)(Q-Q_0+1) + (P+P_0)(P-P_0+1) \right\} |QQ_0, PP_0\rangle \\ &+ \left\{ (Q+Q_0+1)(Q-Q_0)(P+P_0)(P-P_0+1) \right\}^{1/2} |QQ_0+1, PP_0-1\rangle \\ &+ \left\{ (Q+Q_0)(Q-Q_0+1)(P+P_0+1)(P-P_0) \right\}^{1/2} |QQ_0-1, PP_0+1\rangle. \quad (11) \end{aligned}$$



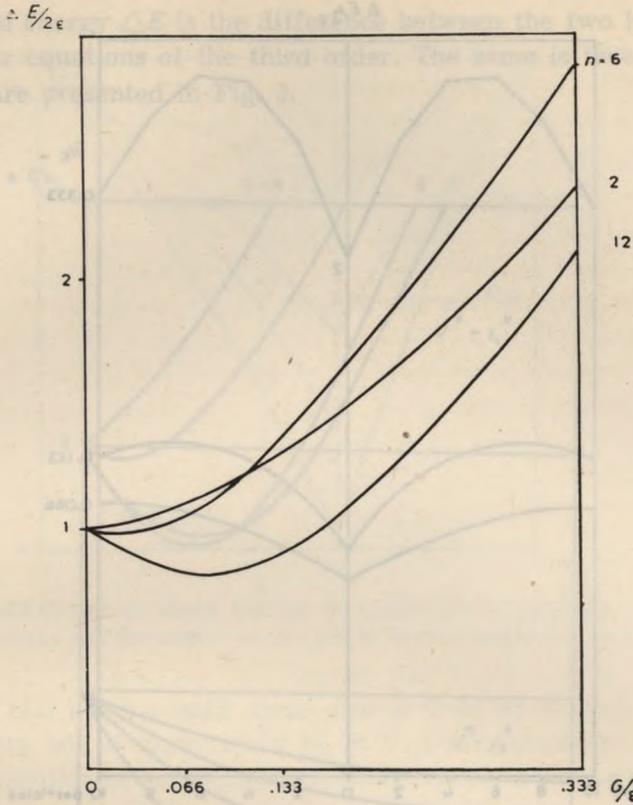


Fig. 1. The energy of the first excited  $0+$  state with seniority equal to zero, and  $\Omega_1 = \Omega_2 = 6$ , as a function of the strength  $G$  of pairing forces;  $n$  is the number of particles,  $\epsilon = E(\Omega_2) - E(\Omega_1)$

We have to consider separately the case with the unpaired particle on the lower  $j_1$  level, and on the higher,  $j_2$  level. These two groups of states are not connected by the Hamiltonian. We can set two secular problems (12) and obtain two groups of energy levels, say  $E_1, E_2, \dots$ ;  $E'_1, E'_2, \dots$ . The energy of the lowest excited state with  $J = j_1$  (or  $j_2$ ) will be given by  $E'_1 - E_1$ . We still can express that difference as a function of the number of particles. Let us notice that the difference  $E_2 - E_1$  gives the energy of the excited level with the same spin as that of the ground state. In the unperturbed approximation the energy  $E_2 - E_1$  is exactly twice as large as  $E'_1 - E_1$ .

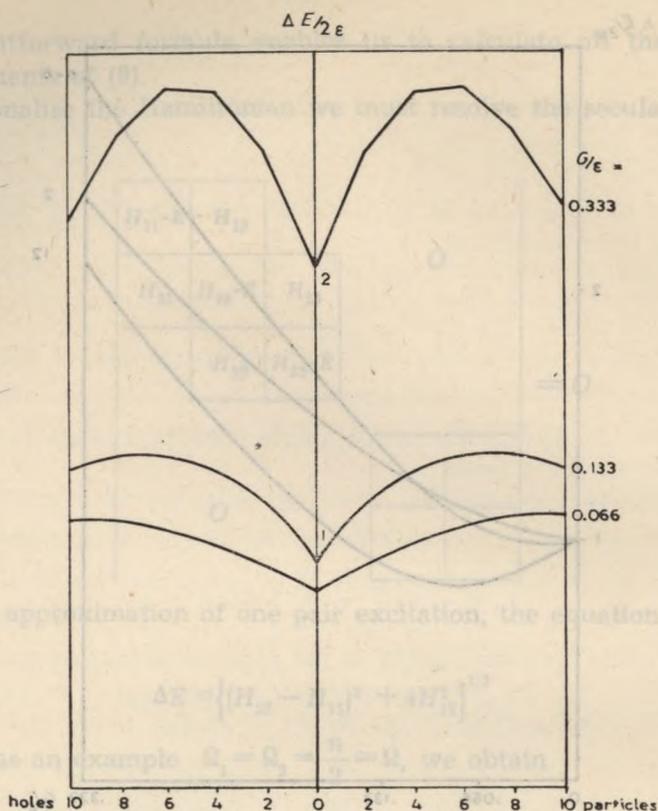


Fig. 2. The energy of the first excited  $0^+$  state with seniority equal to zero and  $\Omega_1 = \Omega_2 = 6$  as a function of the number of particles (holes). The zero particles means that the  $\Omega_1$  level is fully occupied and the  $\Omega_2$  level is completely empty.  $G$  is the strength of pairing forces,  $\epsilon = E(\Omega_2) - E(\Omega_1)$

### 3. THE RESULTS

If we now restrict the problem to the case of  $\Omega_1 = 2$ ,  $\Omega_2 = 4$  ( $j_1 = 3/2$ ,  $j_2 = 7/2$ ), the excited level with the energy  $E' - E_1$  will have the spin  $J = j_1 = 7/2$  for  $n = 1, 3$  and  $J = j_2 = 3/2$  for  $n = 5, 7, 9, 11$ . For  $n = 1, 11$  the energy is just the difference between the energies of the two single particle shells  $j_1$  and  $j_2$ . Then, as the secular equation (12) is of the second order in the unknown  $E$  (or  $E'$ ) for  $n = 3, 9$ , one can obtain the exact algebraic formulas for the energy of the excited level:

$$y = 1 + (1 - 3x + 6.25x^2)^{1/2} - (1 - x + 6.25x^2)^{1/2} \quad \text{for } n = 3,$$

$$y = 2 + (1 + x + 6.25x^2)^{1/2} - (1 + 3x + 6.25x^2)^{1/2} \quad \text{for } n = 9,$$

where  $y = \Delta E / \epsilon$   $x = G / \epsilon$

For  $n = 5$  the energy  $\Delta E$  is the difference between the two lowest roots of the secular equations of the third order. The same is true for  $n = 7$ . The curves are presented in Fig. 3.

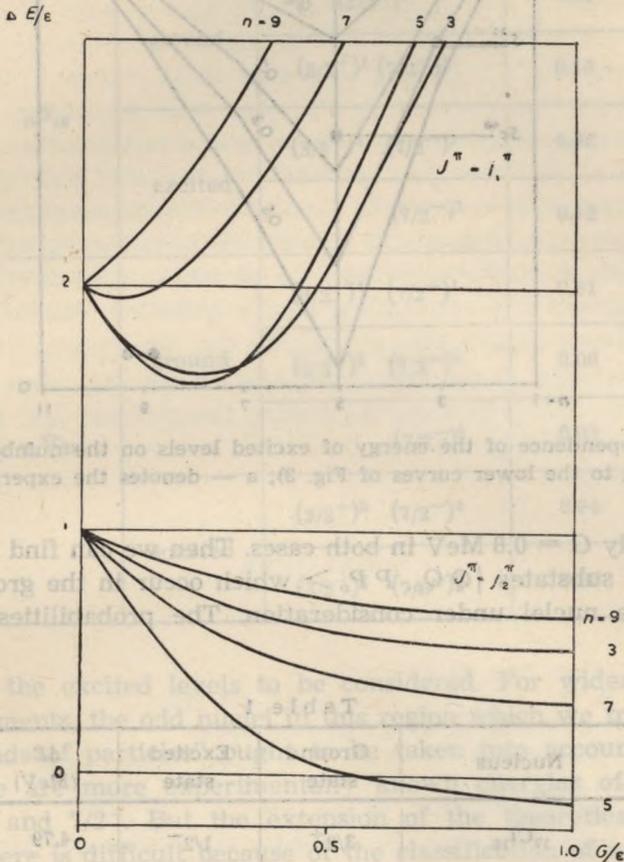


Fig. 3. The energy of the excited levels of odd nuclei in the mixed configuration  $(3/2^+)^{n_1} (7/2^-)^{n_2}$  with the pairing approximation (seniority = 1). The spin and the parity of the ground state are taken to be  $j_1^\pi(3/2^+ \text{ or } 7/2^-)$

The experimental energies of the excited levels under consideration, i. e., the levels  $3/2^+$ ,  $7/2^-$  of the nuclei with 16 or 28 protons or neutrons are presented in Table 1 [9]. The energies 4.79 MeV and 2.937 MeV were regarded as the energy differences of the single particle levels  $3/2^+$  and  $7/2^-$  for the proton and neutron, respectively. The remaining energies of  $S^{35}$  and  $Sc^{49}$  gave (Fig. 4)  $G/\varepsilon_n = 0.3$  and  $G/\varepsilon_p = 0.17$ . Thus we obtain

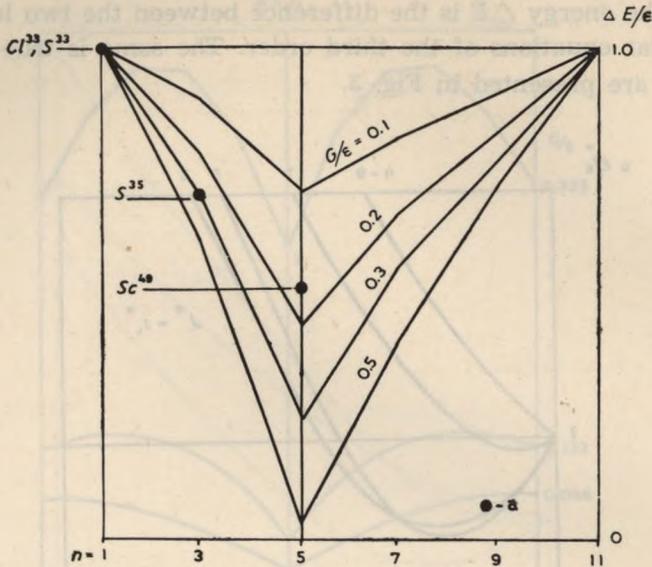


Fig. 4. The dependence of the energy of excited levels on the number of particles (corresponding to the lower curves of Fig. 3); a — denotes the experimental points

approximately  $G = 0.8$  MeV in both cases. Then we can find probabilities of particular substates  $|Q Q_0, P P_0\rangle$  which occur in the ground excited states of the nuclei under consideration. The probabilities are shown in Table 2.

Table 1

Nucleus	Ground state	Excited state	$\Delta E$ (MeV)
$^{17}\text{Cl}_{16}$	$3/2^+$	$1/2^-$	4.79
$^{21}\text{Sc}_{28}$	$7/2^-$	$3/2^+$	2.5
$^{15}\text{S}_{17}$	$3/2^+$	$7/2^-$	2.937
$^{16}\text{S}_{19}$	$3/2^+$	$7/2^-$	1.99

The value  $G = 0.8$  MeV, the same for both protons and neutrons, seems to be too high. It can be due to a) the interpretation of the experimentally known energies as differences of two levels  $3/2^+$ ,  $7/2^-$  for protons and neutrons, and b) rather few experimentally known

Table 2

Nucleus	State	Configuration	Probability
$^{16}\text{S}_{19}$	ground	$(3/2^+)^3$	0.84
		$(3/2^+)^1 (7/2^-)^2$	0.16
	excited	$(3/2^+)^2 (7/2^-)^1$	0.88
		$(7/2^-)^3$	0.12
$^{21}\text{Sc}_{28}$	ground	$(3/2^+)^4 (7/2^-)^1$	0.91
		$(3/2^+)^2 (7/2^-)^3$	0.06
		$(7/2^-)^5$	0.03
	excited	$(3/2^+)^3 (7/2^-)^2$	0.94
		$(3/2^+)^1 (7/2^-)^4$	0.06

energies of the excited levels to be considered. For wider comparison with experiments, the odd nuclei in this region which we treat as nuclei "of two kinds of particles" ought to be taken into account. For those nuclei there are more experimentally known energies of the excited levels  $3/2^+$  and  $7/2^-$ . But the extension of the theoretical calculation presented here is difficult because of the classification of nuclear states. It was shown by Flowers [3] that the classification of the  $j^n$  configuration can be performed with the help of the unimodular  $SU_{2j+1}$  and symplectic  $Sp_{2j+1}$  groups whose dimensions vary from shell to shell. Thus it was difficult, from the point of view of group theory, to deal even with two shells. But Helmers [6] as well as Flowers and Szpikowski [10] demonstrated that the extension of the quasi-spin formalism to two kinds of particles led to the classification of states in the  $j-j$  coupling under the orthogonal  $R_5$  group. The dimension of the group space independent of the angular momentum of the nuclear shell offers the possibility of extending the calculation presented here to the case of two kinds of particles. Such calculations were recently made [11] with the help of the matrix elements of generators in the  $R_5$  group

given by Hecht [12]. Although the interpretation of the results has not been published yet, some remarks can be given here.

Pairing interaction in the  $j-j$  coupling, due to the antisymmetric condition of the total wave function, can act only in the two-particle state with isospin  $T = 1$ . It is just the case for one kind of particles. But for both neutrons and protons the contribution to the total interaction resulting from the  $T = 0$  state is not of small value. This interaction cannot be taken into account in the quasi-spin formalism for the  $j-j$  coupling and then, the final result is supposed to be not so good as that for one kind of particles. Let us finally notice that pairing interaction in  $LS$  coupling occurs both for  $T=1$  and  $T=0$ . It was recently found [13] that the symmetry group of the pairing Hamiltonian in  $LS$  coupling is the orthogonal group  $R_8$  with the help of which similar calculations as those for  $R_5$  group in  $j-j$  coupling are, in principle, possible.

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#### REFERENCES

1. Kerman A. K., Lawson R. D., Macfarlane M. H.: Phys. Rev., **124**, 162 (1961).
2. Racah G.: Phys. Rev., **63**, 367 (1943).
3. Flowers B. H.: Proc. Roy. Soc., A **212**, 248 (1952).
4. Anderson P. W.: Phys. Rev., **112**, 1900 (1958).
5. Wada Y., Takano F., Fokuda N.: Prog. Theor. Phys. (Japan), **19**, 597 (1958).
6. Helmers K.: Nuclear Phys., **23**, 594 (1961).
7. Flowers B. H., Irvine J. M.: "Pairing Correlations in Finite System" — preprint.
8. Högassen-Feldman J.: Nuclear Phys., **28**, 258 (1961).
9. Endt P. M., Van Der Leum C.: Nuclear Phys., **34**, 1 (1962).
10. Flowers B. H., Szpikowski S.: Proc. Phys. Soc., **84**, 193 (1964).
11. Johnston J. private communication.
12. Hecht K. T.: Nuclear Phys., **63**, 177 (1965).
13. Flowers B. H., Szpikowski S.: Proc. Phys. Soc., **84**, 673 (1964).

#### STRESZCZENIE

Zastosowano formalizm quasi-spinu do oddziaływania typu *pairing* między identycznymi nukleonami na powłokach  $p_{3/2}^+$  i  $f_{7/2}^-$ . Elementy macierzowe oddziaływania mogły być następnie dokładnie wyliczone, co

pozwoili na diagonalizacj hamiltonianu. Pokazano dalej, jak zmienia si energia okrealonego poziomu w zaleznosci od liczby czstek na niezamknietej powloce. Dokonano takze porownania otrzymanych wynikow z odpowiednimi danymi dla jader nieparzystych z rozwanego regionu.

### РЕЗЮМЕ

Формализм квази-спина применен к парному взаимодействию идентичных нуклонов на оболочках  $p_{3/2}^+$  и  $f_{7/2}$ . Это позволило точно вычислить матричные элементы взаимодействия, а в следующем диагонализировать гамилтониан. В дальнейшем указано, как меняется энергия определенного уровня в зависимости от числа частиц в незаполненной оболочке. Сравнено также полученные результаты с известными данными для нечетных ядер рассматриваемой области.

Suspensionen Polymerisation Styrols, Methylmetacrylats und ihrer Mischungen in wässrigen Lösungen von Natrium Silicat

Suspension Polymerization of Styrene, Methylmetacrylate and Their Mixtures in Aqueous Solutions of Sodium Silicate

Przedmiotem niniejszego artykułu jest opis wyników badań nad polimerizacją w postaci suspenzji styrolu (St) i metakrylanu metylnego (MAM) w wodnych roztworach tlenku krzemianu sodu (Na<sub>2</sub>SiO<sub>3</sub>) z dodatkiem do mieszaniny reakcyjnej niewielkich ilości różnych substancji (np. kwasów organicznych lub nieorganicznych). Przy doświadczeniach z tymi substancjami do mieszaniny reakcyjnej wydzielają się koloidalne kwasy krzemowe, stabilizujące krople monomeru. Przypuszczamy, że w przypadku kwasów organicznych (akrylowy, metakrylowy) i metakrylanu metylnego w polimerizacyjnych układach mogą tworzyć się ich pochodne, które powiększają stabilizację suspenzji.

W niniejszym artykule omawiamy wyniki badań polimerizacji styrolu (St), metakrylanu metylnego (MAM) i ich mieszanin w wodnych roztworach tlenku krzemianu sodu. Metodą badania opisano acyde poprzednio (1). Przy wyborze warunków przeprowadzenia reakcji polimerizacji zostały wzięte pod uwagę parametry parametrów (gramulacja) i z własności procent. Okazuje się, że optymalne warunki są następujące: stosunek fazy węglowodorowej do wodnej — 1 : 4, stężenie tlenku krzemianu sodu (w fazie wodnej) — 4% stężenie inicjatora (undifenil benzoyl) w fazie węglowodorowej od 1,0 do 1,5% temp. — 50°C. W celu zabrania własności

