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# Products of Toeplitz and Hankel operators on the Bergman space in the polydisk

Abstract. In this paper we obtain a condition for analytic square integrable functions $f, g$ which guarantees the boundedness of products of the Toeplitz operators $T_f T_{\bar{g}}$ densely defined on the Bergman space in the polydisk. An analogous condition for the products of the Hankel operators $H_f H_g^*$ is also given.

**1. Introduction.** Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For a fixed positive integer $n \geq 2$, the unit polydisk $\mathbb{D}^n$ is the Cartesian product of $n$ copies of $\mathbb{D}$. By $dA$ we will denote the Lebesgue volume measure on $\mathbb{D}^n$, normalized so that $A(\mathbb{D}^n) = 1$.

The Bergman space $A^2 = A^2(\mathbb{D}^n)$ is the space of all analytic functions on $\mathbb{D}^n$ such that

$$\|f\|^2 = \int_{\mathbb{D}^n} |f(z)|^2 dA(z) < \infty.$$

For $w = (w_1, w_2, \ldots, w_n) \in \mathbb{D}^n$ the reproducing kernel in $A^2$ is the function $K_w$ given by

$$K_w(z) = \prod_{j=1}^{n} \frac{1}{(1 - \bar{w}_j z_j)^2}, \quad z \in \mathbb{D}^n.$$

If $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{D}^n)$, then for every function $f \in A^2$ we have

$$\langle f, K_w \rangle = f(w), \quad w \in \mathbb{D}^n.$$

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In the special case when $f = K_w$, we obtain

$$\|K_w\|^2 = \langle K_w, K_w \rangle = K_w(w) = \prod_{j=1}^{n} \frac{1}{(1-|w_j|^2)^2}, \quad w \in \mathbb{D}^n.$$

So, the normalized reproducing kernel for $A^2$ is

$$k_w(z) = \prod_{j=1}^{n} \frac{1-|w_j|^2}{(1-\bar{w}_j z_j)^2}, \quad z \in \mathbb{D}^n.$$

Now we quote the definition of the Toeplitz operator. The orthogonal projection $P$ from $L^2(\mathbb{D}^n)$ onto $A^2$ is defined by

$$P(f)(w) = \langle f, K_w \rangle = \int_{\mathbb{D}^n} f(z) \prod_{j=1}^{n} \frac{1}{(1-\bar{z}_j w)^2} dA(z), \quad f \in L^2(\mathbb{D}^n), w \in \mathbb{D}^n.$$

For a function $f \in L^\infty$ and $h \in A^2$ the Toeplitz operator $T_f$ is given by

$$T_f h(w) = P(fh)(w), \quad w \in \mathbb{D}^n.$$

Similarly, the Hankel operator $H_f$ acting on $A^2$ is defined as

$$H_f h = fh - P(fh), \quad h \in A^2,$$

and $P$ is the projection mentioned above. It is clear that $H_f h \in A^{2\perp}$. Both operators $T_f$ and $H_f$ can be defined when the symbol $f$ belongs to the space $L^2(\mathbb{D}^n)$. In that case the Toeplitz and Hankel operators are densely defined on the Bergman space $A^2$, that is on $H^\infty$.

Let $w_i$, $i = 1, 2, \ldots, n$, belong to the unit disk $\mathbb{D}$. For each $w_i$ we define an automorphism $\varphi_{w_i}$ of $\mathbb{D}$ by

$$\varphi_{w_i}(z_i) = \frac{w_i - z_i}{1 - \bar{w}_i z_i}, \quad z_i \in \mathbb{D},\ i = 1, 2, \ldots, n.$$

Then the map

$$\varphi_w(z) = (\varphi_{w_1}(z_1), \varphi_{w_2}(z_2), \ldots, \varphi_{w_n}(z_n)), \quad z, w \in \mathbb{D}^n$$

is an automorphism of the polydisk $\mathbb{D}^n$, in fact, $\varphi_w^{-1} = \varphi_w$. The real Jacobian of $\varphi_w$ is equal to

$$|k_w|^2 = \prod_{j=1}^{n} \frac{(1-|w_j|^2)^2}{|1-\bar{w}_j z_j|^4},$$

thus we have change-of-variable formula

$$\int_{\mathbb{D}^n} (h \circ \varphi_w)(z) dA(z) = \int_{\mathbb{D}^n} h(z)|k_w(z)|^2 dA(z),$$

whenever such integrals make sense.

**2. Problem and results.** As we mentioned, the Toeplitz operator may be considered when the index $f$ belongs to the space $L^2(\mathbb{D}^n)$. If $f \in A^2$, then by the definition of the Toeplitz operator, we have

$$T_{\bar{f}}h(w) = P(\bar{f}h)(w) = \int_{\mathbb{D}^n} \overline{f(z)}h(z) \prod_{j=1}^{n} \frac{1}{(1-\bar{z}_j w)^2} dA(z), \quad w \in \mathbb{D}^n.$$

The main problem in this note is what conditions must be satisfied by functions $f, g \in A^2$ to guarantee that the product of the Toeplitz operators $T_f T_{\bar{g}}$ is bounded on the Bergman space $A^2$ in the polydisk $\mathbb{D}^n$. We provide a sufficient condition for boundedness of such products. Similarly, we give a sufficient condition to ensure that the product of the Hankel operators $H_f H_g^*$ is bounded on the space $(A^2)^\perp$, where $H^*$ is the adjoint of $H$.

For $u \in L^2(\mathbb{D}^n)$ we denote

$$\tilde{u}(w) = B[u](w) = \int_{\mathbb{D}^n} (u \circ \varphi_w)(z) dA(z), \quad w \in \mathbb{D}^n.$$

In [9] Stroethoff and Zheng established the following necessary condition for boundedness of the products $T_f T_{\bar{g}}$ on the unit disk $\mathbb{D}$.

**Theorem 1.** *Let $f$ and $g$ be in $A^2$. If $T_f T_{\bar{g}}$ is bounded, then*

$$\sup_{w \in \mathbb{D}} \widetilde{|f|^2}(w) \widetilde{|g|^2}(w) < \infty.$$

In the same paper the authors also gave a little stronger sufficient condition.

**Theorem 2.** *Let $f$ and $g$ be in $A^2$. If there is a positive constant $\varepsilon$ such that*

$$\sup_{w \in \mathbb{D}} \widetilde{|f|^{2+\varepsilon}}(w) \widetilde{|g|^{2+\varepsilon}}(w) < \infty,$$

*then $T_f T_{\bar{g}}$ is bounded.*

There is a conjecture that the necessary condition is also a sufficient condition for boundedness. But in view of a counter-example of Nazarov [6] for Toeplitz products on the Hardy space, it may not be possible to prove that this necessary condition is also sufficient.

Stroethoff and Zheng [12] showed the analogous results on the Bergman spaces of the polydisk [11], weighted Bergman space of the unit disk [13] and the unit ball [12]. Next, Miao in [4] gave an interesting way to transfer Theorem 1 and Theorem 2 to the space $A^p_\alpha$, $1 < p < \infty$, $\alpha > -1$, of the unit ball. Recently, Michalska and Sobolewski [5] improved a sufficient condition on boundedness of $T_f T_{\bar{g}}$ on $A^p_\alpha$.

A similar problem concerns the products of the Hankel operators $H_f H_g^*$. Such operators are densely defined on space $(A^2)^\perp$. The following condition for the Hankel products on the unit disk was established by Stroethoff and Zheng in [9].

**Theorem 3.** *Let $f$ and $g$ be in $L^2(\mathbb{D}, dA)$. If $H_f H_g^*$ is bounded on $(A^2)^\perp$, then*

$$\sup_{w\in\mathbb{D}} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_{L^2}\|g\circ\varphi_w - P(g\circ\varphi_w)\|_{L^2} < \infty.$$

The same authors showed that this necessary condition is, like for $T_f T_{\bar{g}}$, very close to being sufficient.

**Theorem 4.** *Let $f$ and $g$ be in $L^2(\mathbb{D}, dA)$. If there is a positive constant $\varepsilon$ such that*

$$\sup_{w\in\mathbb{D}} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_{L^{2+\varepsilon}}\|g\circ\varphi_w - P(g\circ\varphi_w)\|_{L^{2+\varepsilon}} < \infty,$$

*then the product $H_f H_g^*$ is bounded on $(A^2)^\perp$.*

Their theorems were extended to the weighted Bergman spaces of the unit ball by Lu and Liu [2] and for the Bergman space of the polydisk by Lu and Shang [3].

In this paper we provide a sufficient condition for the boundedness of the operators $T_f T_{\bar{g}}$ and $H_f H_g^*$.

For $u \in L^1$, $\varepsilon > 0$ and $w \in \mathbb{D}^n$ we define

$$B_\varepsilon[u](w) = \int_{\mathbb{D}^n} (u\circ\varphi_w)(z)\prod_{i=1}^{n}\log^{1+\varepsilon}\frac{1}{1-|z_i|}dA(z),$$

where $\varphi_w$ is the automorphism of $\mathbb{D}^n$ and $z = (z_1, z_2, \ldots, z_n)$. The following theorems are the main results in this paper.

**Theorem 5.** *Let $f, g \in A^2$. If there is a positive constant $\varepsilon > 0$ such that*

$$\sup_{w\in\mathbb{D}^n} B_\varepsilon[|f|^2](w)B_\varepsilon[|g|^2](w) < \infty,$$

*then the operator $T_f T_{\bar{g}}$ is bounded on $A^2$.*

**Theorem 6.** *Let $f, g \in L^2(\mathbb{D}^n)$. If there is a positive constant $\varepsilon > 0$ such that*

$$\sup_{w\in\mathbb{D}^n} \left\|(f\circ\varphi_w - P(f\circ\varphi_w))\prod_{j=1}^{n}\log^{(1+\varepsilon)/2}\frac{1}{1-|z_j|}\right\|_{L^2} \times \left\|(g\circ\varphi_w - P(g\circ\varphi_w))\prod_{j=1}^{n}\log^{(1+\varepsilon)/2}\frac{1}{1-|z_j|}\right\|_{L^2} < \infty,$$

*then the operator $H_f H_g^*$ is bounded on $(A^2)^\perp$.*

After sending this paper for publication we found that Theorem 5 is contained in a result obtained in [1].

**3. Proofs.** A very important role in our considerations is played by the formula for the inner product in $A^2$ introduced in [11]. Let $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_m\}$ be a nonempty subset of $\{1,2,\dots,n\}$ with $\alpha_1<\alpha_2<\dots<\alpha_m$. We define the measure on $\mathbb{D}^n$ by

$$d\mu_\alpha(z)=\frac{3^{n-m}}{6^m}(1-|z_1|^2)^2(1-|z_2|^2)^2\dots(1-|z_n|^2)^2 \times\prod_{j\in\alpha}(5-2|z_j|)^2 dA(z_1)dA(z_2)\dots dA(z_n)$$

and

$$d\mu_\emptyset(z)=3^n(1-|z_1|^2)^2(1-|z_2|^2)^2\dots(1-|z_n|^2)^2dA(z_1)dA(z_2)\dots dA(z_n),$$

where $m$ is the cardinality of $\alpha$. Let us set $D_jh=\partial h/\partial z_j$ and

$$D^\alpha h=D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_m}h,\quad D^\emptyset h=h.$$

For $f,g\in A^2$ we have

$$\int_{\mathbb{D}^n}f(z)\overline{g(z)}dA(z)=\sum_\alpha\int_{\mathbb{D}^n}D^\alpha f(z)\overline{D^\alpha g(z)}d\mu_\alpha(z), \tag{1}$$

where $\alpha$ runs over all subsets of $\{1,2,\dots,n\}$.

We start with some lemmas which we will apply to prove the main theorems.

**Lemma 1.** *Let $f\in A^2$, $h\in H^\infty$ and $\varepsilon>0$. If $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_m\}$ is a subset of $\{1,2,\dots,n\}$, then*

$$|D^\alpha T^\alpha_{\overline{f}}h(w)|\le C\prod_{i=1}^n\frac{1}{(1-|w_i|^2)}\left(B_\varepsilon[|f|^2](w)\right)^{\frac12} \times\left(\int_{\mathbb{D}^n}|h(z)|^2\prod_{i=1}^n\frac{1}{|1-\overline{w}_iz_i|^2}\prod_{i=1}^n\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(z)\right)^{\frac12}$$

*for all $w\in\mathbb{D}^n$.*

**Proof.** First we show the inequality for $\alpha = \emptyset$.

$$\begin{aligned}
|T_{\bar{f}}h(w)| &\le 2^n \int_{\mathbb{D}^n} |f(z)| \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^{n} \log^{\frac{1+\varepsilon}{2}} \frac{1}{1-|\varphi_{w_i}(z_i)|} \\
&\quad \times |h(z)| \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|} \prod_{i=1}^{n} \log^{-\frac{1+\varepsilon}{2}} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \\
&\le C \left( \int_{\mathbb{D}^n} \prod_{i=1}^{n} \frac{1}{(1-|w_i|^2)^2} |f(z)|^2 \prod_{i=1}^{n} \frac{(1-|w_i|^2)^2}{|1-\overline{w}_i z_i|^4} \prod_{i=1}^{n} \log^{1+\varepsilon} \frac{1}{1-|\varphi_{w_i}(z_i)|} \right)^{\frac{1}{2}} \\
&\quad \times \left( \int_{\mathbb{D}^n} |h(z)|^2 \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^{n} \log^{-(1+\varepsilon)} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \right)^{\frac{1}{2}} \\
&\le C \prod_{i=1}^{n} \frac{1}{(1-|w_i|^2)} \left\{ B_\varepsilon[|f|^2](w) \right\}^{\frac{1}{2}} \\
&\quad \times \left( \int_{\mathbb{D}^n} |h(z)|^2 \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^{n} \log^{-(1+\varepsilon)} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \right)^{\frac{1}{2}}.
\end{aligned}$$

In the case $\alpha = \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
|D^\alpha T_{\bar{f}}h(w)| &\le 2^n \int_{\mathbb{D}^n} |f(z)||h(z)| \prod_{i=1}^{n} \frac{|z_i|}{|1-\overline{w}_i z_i|^3} dA(z) \\
&\le \int_{\mathbb{D}^n} |f(z)| \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^{n} \log^{\frac{1+\varepsilon}{2}} \frac{1}{1-|\varphi_{w_i}(z_i)|} \\
&\quad \times |h(z)| \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|} \prod_{i=1}^{n} \log^{-\frac{1+\varepsilon}{2}} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z).
\end{aligned}$$

Following the previous calculations, we obtain the desired inequality. It remains to consider the case when $\alpha$ is a proper subset of $\{1, 2, \ldots, n\}$. Then

$$\begin{aligned}
|D^\alpha T_{\bar{f}}h(w)| &\le \int_{\mathbb{D}^n} |f(z)||h(z)| \prod_{i\in\alpha} \frac{2|z_i|}{|1-\overline{w}_i z_i|^3} \prod_{i\notin\alpha} \frac{1}{|1-\overline{w}_i z_i|^2} dA(z) \\
&\le C \int_{\mathbb{D}^n} |f(z)| \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^{n} \log^{\frac{1+\varepsilon}{2}} \frac{1}{1-|\varphi_{w_i}(z_i)|} \\
&\quad \times |h(z)| \prod_{i=1}^{n} \frac{1}{|1-\overline{w}_i z_i|} \prod_{i=1}^{n} \log^{-\frac{1+\varepsilon}{2}} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z),
\end{aligned}$$

where the last inequality follows from

$$\left|\prod_{j\in\alpha}\frac{2z_j}{(1-\bar{w}_j z_j)^3}\prod_{j\notin\alpha}\frac{1}{(1-\bar{w}_j z_j)^2}\right|\leq C\prod_{j=1}^{n}\frac{1}{|1-\bar{w}_j z_j|^3}.$$

□

**Lemma 2.** *Let* $\varepsilon>0$, $u\in(A^2)^{\perp}$, $f\in L^2(\mathbb{D}^n)$, $\alpha=\{\alpha_1,\alpha_2,\dots,\alpha_m\}\subset\{1,2,\dots,n\}$, $\alpha_1<\alpha_2<\dots<\alpha_m$. *Then*

$$|D^\alpha H_f^* u(w)|\leq C\prod_{j=1}^{n}\frac{1}{1-|w_j|^2}\left\|(f\circ\varphi_w-P(f\circ\varphi_w))\prod_{j=1}^{n}\log^{(1+\varepsilon)/2}\frac{1}{1-|z_j|}\right\|$$
$$\times\left\{\int_{\mathbb{D}^n}|u(z)|^2\prod_{j=1}^{n}\frac{1}{|1-\bar{z}_j w_j|^2}\prod_{j=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_j}(z_j)|}dA(z)\right\}^{\frac{1}{2}}.$$

**Proof.** The proof will proceed in three steps as above. Suppose first that $\alpha=\emptyset$. Then

$$\langle H_f^* u, K_w\rangle=\prod_{j=1}^{n}\frac{1}{1-|w_j|^2}\langle H_f^* u, k_w\rangle=\prod_{j=1}^{n}\frac{1}{1-|w_j|^2}\langle u, H_f k_w\rangle.$$

In view of [8, Proposition 1] we may write

$$H_f k_w=(f-P(f\circ\varphi_w)\circ\varphi_w)\,k_w$$

and

$$\langle H_f^* u, K_w\rangle=\prod_{j=1}^{n}\frac{1}{1-|w_j|^2}\langle u,(f-P(f\circ\varphi_w)\circ\varphi_w)\,k_w\rangle.$$

Thus, by Hölder's inequality, we obtain

$$|\langle u,(f-P(f\circ\varphi_w)\circ\varphi_w)\,k_w(z)\rangle|$$
$$=\left|\int_{\mathbb{D}^n}u(z)\prod_{j=1}^{n}\log^{-\frac{1+\varepsilon}{2}}\frac{1}{1-|\varphi_{w_j}(z_j)|}\overline{(f-P(f\circ\varphi_w)\circ\varphi_w)\,(z)k_w(z)}\right.$$
$$\left.\times\prod_{j=1}^{n}\log^{\frac{1+\varepsilon}{2}}\frac{1}{1-|\varphi_{w_j}(z_j)|}dA(z)\right|$$

$$\leq \left\{ \int_{\mathbb{D}^n} |\,(f - P(f\circ\varphi_w)\circ\varphi_w)\,(z)|^2 |k_w(z)|^2 \prod_{j=1}^n \log^{1+\varepsilon} \frac{1}{1-|\varphi_{w_j}(z_j)|} dA(z) \right\}^{\frac{1}{2}}$$

$$\times \left\{ \int_{\mathbb{D}^n} |u(z)|^2 \prod_{j=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_j}(z_j)|} dA(z) \right\}^{\frac{1}{2}}.$$

By the change-of-variable formula $z \mapsto \varphi_w(z)$ and using that $|1 - \bar{z}_j w_j| \leq 2$, we have

$$\begin{aligned} &|\langle u, (f - P(f\circ\varphi_w)\circ\varphi_w)\, k_w(z)\rangle| \\ &\quad \leq C \left\| (f\circ\varphi_w - P(f\circ\varphi_w)) \prod_{j=1}^n \log^{(1+\varepsilon)/2} \frac{1}{1-|z_j|} \right\| \\ &\qquad \times \left\{ \int_{\mathbb{D}^n} |u(z)|^2 \prod_{j=1}^n \frac{1}{|1-\bar{z}_j w_j|^2} \prod_{j=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_j}(z_j)|} dA(z) \right\}^{\frac{1}{2}}. \end{aligned}$$

This proves the first case. Now, let $\alpha = \{1, 2, \ldots, n\}$. Then

$$H_f^* u(w) = P(\bar{f}u)(w) = \int_{\mathbb{D}^n} \overline{f(z)} u(z) \prod_{j=1}^n \frac{1}{(1-w_j\bar{z}_j)^2} dA(z).$$

Hence

$$D^\alpha H_f^* u(w) = \int_{\mathbb{D}^n} \overline{f(z)} u(z) \prod_{j=1}^n \frac{2\bar{z}_j}{(1-w_j\bar{z}_j)^3} dA(z).$$

Let

$$F_w(z) = P(f\circ\varphi_w)\circ\varphi_w(z) \prod_{j=1}^n \frac{2z_j}{(1-\bar{w}_j z_j)^3}.$$

The function $F_w$ belongs to $\in A^2$, thus

$$\langle u, F_w\rangle = \int_{\mathbb{D}^n} u(z) \overline{P(f\circ\varphi_w)\circ\varphi_w(z) \prod_{j=1}^n \frac{2z_j}{(1-\bar{w}_j z_j)^3}} dA(z) \equiv 0.$$

So,

$$\begin{aligned} D^\alpha H_f^* u(w) &= D^\alpha H_f^* u(w) - \langle u, F_w\rangle \\ &= \int_{\mathbb{D}^n} u(z) \overline{(f(z) - P(f\circ\varphi_w)\circ\varphi_w(z)) \prod_{j=1}^n \frac{2z_j}{(1-\bar{w}_j z_j)^3}} dA(z). \end{aligned}$$

Using Hölder's inequality, we get

$$
\begin{aligned}
&|D^\alpha H_f^* u(w)| \\
&\le C\left\{\int_{\mathbb{D}^n} |u(z)|^2 \prod_{j=1}^n \frac{1}{|1-\bar z_j w_j|^2} \prod_{j=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_j}(z_j)|} dA(z)\right\}^{\frac12} \\
&\times \prod_{j=1}^n \frac{1}{1-|w_j|^2} \\
&\times \left\{\int_{\mathbb{D}^n} |\left(f - P(f\circ\varphi_w)\circ\varphi_w\right)(z)|^2 |k_w(z)|^2 \prod_{j=1}^n \log^{1+\varepsilon}\frac{1}{1-|\varphi_{w_j}(z_j)|} dA(z)\right\}^{\frac12} \\
&= C\prod_{j=1}^n \frac{1}{1-|w_j|^2} \\
&\times \left\{\int_{\mathbb{D}^n} |u(z)|^2 \prod_{j=1}^n \frac{1}{|1-\bar z_j w_j|^2} \prod_{j=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_j}(z_j)|} dA(z)\right\}^{\frac12} \\
&\times \left\| (f\circ\varphi_w - P(f\circ\varphi_w)) \prod_{j=1}^n \log^{(1+\varepsilon)/2} \frac{1}{1-|z_j|}\right\|_{L^2}.
\end{aligned}
$$

Suppose now that $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a nonempty subset of $\{1, 2, \ldots, n\}$. Then

$$
D^\alpha H_f^* u(w) = \int_{\mathbb{D}^n} \overline{f(z)} u(z) \prod_{j\in\beta} \frac{2\bar z_j}{(1-w_j\bar z_j)^3} \prod_{j\notin\beta} \frac{1}{(1-w_j\bar z_j)^2} dA(z).
$$

Putting

$$
F_w(z) = P(f\circ\varphi_w)\circ\varphi_w(z) \prod_{j\in\beta} \frac{2z_j}{(1-\bar w_j z_j)^3} \prod_{j\notin\beta} \frac{1}{(1-\bar w_j z_j)^2}
$$

and using the fact that

$$
\left|\prod_{j\in\beta} \frac{2z_j}{(1-\bar w_j z_j)^3} \prod_{j\notin\beta} \frac{1}{(1-\bar w_j z_j)^2}\right| \le C\prod_{j=1}^n \frac{1}{|1-\bar w_j z_j|^3},
$$

we obtain

$$
\begin{aligned}
&|D^\beta H_f^* u(w)| \\
&\le C\int_{\mathbb{D}^n} |u(z)| \prod_{j=1}^n \frac{1}{|1-\bar w_j z_j|} |f(z) - P(f\circ\varphi_w)\circ\varphi_w(z)| \prod_{j=1}^n \frac{1}{|1-\bar w_j z_j|^2} dA(z).
\end{aligned}
$$

Using the same arguments as in the proof of Lemma 1, the stated result follows. □

Now, we give the proofs of the main theorems.

**Proof of Theorem 5.** Let $u, v \in H^\infty$. We show that

$$|\langle T_f T_{\bar{g}} u, v\rangle| \leq C\|u\|\|v\|.$$

By (1), we get

$$\begin{aligned}\langle T_f T_{\bar{g}} u, v\rangle &= \langle T_{\bar{g}} u, T_{\bar{f}} v\rangle \\ &= \int_{\mathbb{D}^n} T_{\bar{g}} u(w)\overline{T_{\bar{f}} v(w)} dA(w) \\ &= \sum_\alpha \int_{\mathbb{D}^n} D^\alpha T_{\bar{g}} u(w)\overline{D^\alpha T_{\bar{f}} v(w)} d\mu_\alpha(w).\end{aligned}$$

Using Lemma 1, we obtain

$$\begin{aligned}|\langle T_f T_{\bar{g}} u, v\rangle| &\leq C \sum_\alpha \int_{\mathbb{D}^n} \Bigg( \int_{\mathbb{D}^n} \prod_{i=1}^n \frac{1}{(1-|w_i|^2)} \left(B_\varepsilon[|f|^2](w)\right)^{\frac{1}{2}} \\ &\times \left( \int_{\mathbb{D}^n} |u(z)|^2 \prod_{i=1}^n \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \right)^{\frac{1}{2}} \\ &\times \int_{\mathbb{D}^n} \prod_{i=1}^n \frac{1}{(1-|w_i|^2)} \left(B_\varepsilon[|g|^2](w)\right)^{\frac{1}{2}} \\ &\times \left( \int_{\mathbb{D}^n} |v(z)|^2 \prod_{i=1}^n \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \right)^{\frac{1}{2}} \Bigg) d\mu_\alpha(z) \\ &\leq C \sup_{w\in D^n} \left\{B_\varepsilon[|f|^2](w) B_\varepsilon[|g|^2](w)\right\}^{\frac{1}{2}} \sum_\alpha \int_{\mathbb{D}^n} \prod_{i=1}^n \frac{1}{(1-|w_i|^2)^2} \\ &\times \left( \int_{\mathbb{D}^n} |u(z)|^2 \prod_{i=1}^n \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \right)^{\frac{1}{2}} \\ &\times \left( \int_{\mathbb{D}^n} |v(z)|^2 \prod_{i=1}^n \frac{1}{|1-\overline{w}_i z_i|^2} \prod_{i=1}^n \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{w_i}(z_i)|} dA(z) \right)^{\frac{1}{2}} d\mu_\alpha(w).\end{aligned}$$

Since

$$d\mu_\alpha(z) = \frac{3^{n-m}}{6^m}\prod_{j=1}^{n}(1-|z_j|^2)^2\prod_{j\in\alpha}(5-2|z_j|)^2 dA(z_1)dA(z_2)\dots dA(z_n)$$
$$\le 3^n\prod_{j=1}^{n}(1-|z_j|^2)^2 dA(z_1)dA(z_2)\dots dA(z_n),$$

we get

$$|\langle T_f T_{\overline{g}}u,\upsilon\rangle| \le C\sup_{w\in D^n}\left\{B_\varepsilon[|f|^2](w)B_\varepsilon[|g|^2](w)\right\}^{\frac{1}{2}}$$
$$\times\int_{\mathbb{D}^n}\left(\int_{\mathbb{D}^n}|u(z)|^2\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(z)\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{D}^n}|\upsilon(z)|^2\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(z)\right)^{\frac{1}{2}}dA(w).$$

Now, applying Hölder's inequality and Fubini's theorem, we have

$$|\langle T_f T_{\overline{g}}u,\upsilon\rangle| \le C\sup_{w\in D^n}\left\{B_\varepsilon[|f|^2](w)B_\varepsilon[|g|^2](w)\right\}^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}|u(z)|^2\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(z)dA(w)\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{D}^n}\int_{\mathbb{D}^n}|\upsilon(z)|^2\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(z)dA(w)\right)^{\frac{1}{2}}$$
$$= C\sup_{w\in D^n}\left\{B_\varepsilon[|f|^2](w)B_\varepsilon[|g|^2](w)\right\}^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{D}^n}|u(z)|^2\int_{\mathbb{D}^n}\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(w)dA(z)\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{D}^n}|\upsilon(z)|^2\int_{\mathbb{D}^n}\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(w)dA(z)\right)^{\frac{1}{2}}.$$

It remains to prove that the integral

$$I = \int_{\mathbb{D}^n}\prod_{i=1}^{n}\frac{1}{|1-\overline{w}_i z_i|^2}\prod_{i=1}^{n}\log^{-1-\varepsilon}\frac{1}{1-|\varphi_{w_i}(z_i)|}dA(w)$$

is convergent independently of $z$. Indeed, the change-of-variable formula $\zeta = \varphi_z(w)$ and the fact that $|\varphi_{w_i}(z_i)| = |\varphi_{z_i}(w_i)|$ imply

$$\begin{aligned}
I &= \int_{\mathbb{D}^n} \prod_{i=1}^{n} \frac{|1-\overline{z}_i w_i|^2}{(1-|z_i|^2)^2} \prod_{i=1}^{n} \log^{-1-\varepsilon} \frac{1}{1-|\varphi_{z_i}(w_i)|} \prod_{i=1}^{n} \frac{(1-|z_i|^2)^2}{|1-\overline{z}_i w_i|^4} dA(w) \\
&= \int_{\mathbb{D}^n} \prod_{i=1}^{n} \frac{|1-\overline{z}_i \varphi_{z_i}(\zeta_i)|^2}{(1-|z_i|^2)^2} \prod_{i=1}^{n} \log^{-1-\varepsilon} \frac{1}{1-|\zeta_i|} dA(\zeta) \\
&= \int_{\mathbb{D}^n} \prod_{i=1}^{n} \frac{\frac{(1-|z_i|^2)^2}{|1-\overline{z}_i\zeta_i|^2}}{(1-|z_i|^2)^2} \prod_{i=1}^{n} \log^{-1-\varepsilon} \frac{1}{1-|\zeta_i|} dA(\zeta) \\
&= \prod_{i=1}^{n} \int_{\mathbb{D}} \frac{1}{|1-\overline{z}_i\zeta_i|^2} \log^{-1-\varepsilon} \frac{1}{1-|\zeta_i|} dA(\zeta_i).
\end{aligned}$$

We need only to show that

$$I_j = \int_{\mathbb{D}} \frac{1}{|1-\overline{z}_j\zeta_j|^2} \log^{-1-\varepsilon} \frac{1}{1-|\zeta_j|} dA(\zeta_j) \le C$$

for $j = 1, 2, \ldots, n$. Let $\zeta_j = re^{i\theta}$.

According to Theorem 1.7 in [14], we have

$$\int_0^{2\pi} \frac{1}{|1-\overline{z}_j re^{i\theta}|^2} d\theta \le \frac{C}{1-|z|r} \le \frac{C}{1-r}.$$

Therefore

$$I_j \le C\frac{1}{\pi} \int_0^1 \frac{r}{1-r} \log^{-1-\varepsilon} \frac{1}{1-r} dr.$$

By the change-of-variable formula,

$$\begin{aligned}
I_j &\le C \int_0^{+\infty} t^{-1-\varepsilon}(1-e^{-t})dt \\
&= C \int_0^1 t^{-1-\varepsilon}(1-e^{-t})dt + \int_1^{+\infty} t^{-1-\varepsilon}(1-e^{-t})dt \\
&\le C \int_0^1 t^{-\varepsilon}dt + \int_1^{+\infty} t^{-1-\varepsilon}dt.
\end{aligned}$$

Clearly, for $\varepsilon \in (0,1)$ the integrals $I_i$ are bounded by a constant which is independent of $z$. Finally, we conclude that

$$|\langle T_f T_{\overline{g}} u, v\rangle| \le C\|u\|\|v\|,$$

which proves the theorem. □

**Proof of Theorem 6.** To prove the theorem we need to use Lemma 2 and the method used in the proof of Theorem 5. The details are left to the reader. □

Now, we propose one additional theorem concerning products of Toeplitz and Hankel operators $T_f H_g^*$. The following result can be proved in much the same way as Theorem 5 and Theorem 6.

**Theorem 7.** *Let* $f \in A^2, g \in L^2(\mathbb{D}^n)$. *If*

$$\sup_{\mathbb{D}^n} B_\varepsilon[|f|^2](w) \left\| (g \circ \varphi_w - P(g \circ \varphi_w)) \prod_{j=1}^n \log^{(1+\varepsilon)/2} \frac{1}{1-|z_j|} \right\|_{L^2} < \infty,$$

*then the operator* $T_f H_g^*$ *is bounded on* $(A^2)^\perp$.

It is clear that the above condition also gives the boundedness of $H_g T_{\bar{f}}$. The next proposition reveals that Theorem 5 extends Theorem 2.

**Proposition 1.** *Let* $f, g \in A^2$ *and* $\varepsilon > 0$. *Then for all* $w \in \mathbb{D}^n$,

$$B_\varepsilon[|f|^2] B_\varepsilon[|g|^2] \leq C \left\{ B[|f|^{2+\varepsilon}] B_\varepsilon[|g|^{2+\varepsilon}] \right\}^{2/(2+\varepsilon)}.$$

**Proof.** Let $w \in \mathbb{D}^n$. Then by the change-of-variable formula and Hölder's inequality we have

$$B_\varepsilon[|f|^2](w) = \int_{\mathbb{D}^n} |f(z)|^2 \prod_{i=1}^n \log^{1+\varepsilon} \frac{1}{1-|\varphi_{w_i}(z_i)|} \prod_{j=1}^n \frac{(1-|w_j|^2)^2}{|1-\bar{w}_j z_j|^4} dA(z)$$

$$\leq \left\{ \int_{\mathbb{D}^n} |f(z)|^{2+\varepsilon}(z) \prod_{j=1}^n \frac{(1-|w_j|^2)^2}{|1-\bar{w}_j z_j|^4} dA(z) \right\}^{\frac{2}{2+\varepsilon}}$$

$$\times \left\{ \int_{\mathbb{D}^n} \prod_{j=1}^n \log^{\frac{(1+\varepsilon)(2+\varepsilon)}{\varepsilon}} \left( \frac{1}{1-|\varphi_{w_i}(z_i)|} \right) \prod_{j=1}^n \frac{(1-|w_j|^2)^2}{|1-\bar{w}_j z_j|^4} dA(z) \right\}^{\frac{\varepsilon}{2+\varepsilon}}$$

$$= \{B[|f|^{2+\varepsilon}](w)\}^{\frac{2}{2+\varepsilon}} \left\{ \int_{\mathbb{D}^n} \prod_{j=1}^n \log^{\frac{(1+\varepsilon)(2+\varepsilon)}{\varepsilon}} \left( \frac{1}{1-|z_i|} \right) dA(z) \right\}^{\frac{\varepsilon}{2+\varepsilon}}.$$

Since the last integral is convergent, our claim follows. □

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