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On a Certain Extension of Epstein's Univalence Criterion

O pewnym uogólnieniu kryterium jednolistości Epsteina

Abstract. In this paper a sufficient univalence condition for meromorphic and locally univalent functions in the unit disk is given (Theorem 2).

This condition is an essential generalisation of the Epstein's univalence criterion [7]. As particular cases the well-known univalence criteria of Ahlfors [1] and Nehari [6] are obtained.

Moreover, a sufficient univalence criterion for meromorphic and locally univalent functions in the upper half plane is given (Theorem 3).

1. Ch. Pommerenke has recently given a simplified proof of a univalence criterion obtained earlier by Epstein in another way. In his proof an additional assumption made by Epstein was dropped (see e.g. [7, p.143]).

Let $D = \{z : |z| < 1\}$ and let S_f denote the Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

Theorem 1. (C.L. Epstein, see e.g. [7]) *Let f be meromorphic and g analytic in D . If both functions are locally univalent in D and if*

$$(1.1) \quad \left| \frac{1}{2}(1 - |z|^2)^2 (S_f(z) - S_g(z)) + (1 - |z|^2)^2 \frac{g''(z)}{g'(z)} \right| \leq 1$$

for $z \in D$ then f is univalent in D .

If $g(z) \equiv z$ then (1.1) gives

$$(1.2) \quad |(1 - |z|^2)^2 S_f(z)| \leq 2$$

and this is the well-known univalence criterion of Nehari [6].

If $g = f$ then (1.1) implies

$$(1.3) \quad (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

and this is also a well-known univalence criterion (see e.g. [9, p.172]).

2. The following theorem is a generalization of the univalence criterion given by Epstein.

Theorem 2. *Let f be meromorphic and g analytic in D . If both functions are locally univalent in D and if there exists a function h , analytic in D , satisfying $\operatorname{Re} h(z) \geq \frac{1}{2}$ and such that*

$$(2.1) \quad \left| \frac{h(z)-1}{h(z)} |z|^2 - (1-|z|^2) \left(\frac{zh'(z)}{h(z)} + \frac{zg''(z)}{g'(z)} \right) - \frac{1}{2}(1-|z|^2)^2 \frac{z}{\bar{z}} h(z) (S_f(z) - S_g(z)) \right| \leq 1, \quad z \in D$$

then f is univalent in D .

Proof. Let $f(z) = a_0 + a_1 z + \dots$, $g(z) = b_0 + b_1 z + \dots$, $a_1 \neq 0$, $b_1 \neq 0$. Then it is possible to consider the functions:

$$f^*(z) = \frac{f - a_0}{a_1 + \left(\frac{a_2}{a_1} - \frac{b_2}{b_1}\right)(f - a_0)}, \quad g^*(z) = \frac{g - b_0}{b_1}$$

instead of f and g .

These normalizations don't impair the generality of our considerations and we may also assume that

$$f(z) = g(z) + O(z^3) \quad \text{and} \quad \frac{f'}{g'} = 1 + O(z^2) \quad \text{as} \quad z \rightarrow 0.$$

Let us introduce now

$$(2.2) \quad v(z) = \sqrt{\frac{g'(z)}{f'(z)}} = 1 + \beta z^2 + O(z^3),$$

$$(2.3) \quad u(z) = f(z) \cdot v(z) = z + \alpha z^2 + O(z^3).$$

Both functions are analytic in D because f cannot have multiple poles and because f' and g' do not vanish in D .

For $t \in I =]-\infty, \infty[$ we consider [3, p.38]

$$(2.4) \quad f(z, t) = \frac{u(ze^{-t}) + (e^t - e^{-t})zh(ze^{-t})v'(ze^{-t})}{v(ze^{-t}) + (e^t - e^{-t})zh(ze^{-t})v'(ze^{-t})}.$$

The function $f(z, t)$ is for each fixed $t \in I$ meromorphic in D . From (2.2) and from the assumption on the function h given in Theorem 1 it follows that the denominator in (2.4) has the form $1 + O(z^2)$ as $z \rightarrow 0$, uniformly with respect to t .

It is easy to show, that $\frac{f(z, t)}{a_1(t)} = z + \dots$, $t \in I$, is a normal family in D , where

$$(2.5) \quad a_1(t) = e^{-t} + (e^t - e^{-t})h(0) \quad \text{and} \quad |a_1(t)| \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty.$$

From (2.3) and (2.4) we have

$$(2.6) \quad f(z, t) = a_1(t)z + O(z^2) \quad \text{as } z \rightarrow \infty.$$

where $a_1(t)$ is defined by (2.5).

Let us denote $f'(z, t) = \frac{\partial f(z, t)}{\partial z}$, $\dot{f}(z, t) = \frac{\partial f(z, t)}{\partial t}$. After some calculations we obtain from (2.4)

$$(2.7) \quad w = \frac{\dot{f}(z, t) - zf'(z, t)}{f(z, t) + zf'(z, t)} = \frac{h-1}{h} e^{-2t} - (1-e^{-2t}) \frac{ze^{-t}h'}{h} - \frac{(1-e^{-2t})ze^{-t}(u''v - uv'') + (1-e^{-2t})^2 z^2 h(u''v' - u'v'')}{u'v - uv'}$$

where $u, v, h, u', v', h', u'', v''$ are evaluated at the point ze^{-t}

From (2.3) and the assumption (2.2) we have

$$\begin{aligned} u'v - uv' &= f'v^2 \\ u''v - uv'' &= f''v^2 + 2f'v'v = f'v^2 \frac{g''}{g'} \\ u''v' - u'v'' &= f''v'v - f'v''v + 2f'(v')^2 = \frac{1}{2}f'v^2(S_f - S_g). \end{aligned}$$

Taking this and (2.7) into account we have for $z \in D$

$$(2.8) \quad w = \frac{h(ze^{-t})-1}{h(ze^{-t})} e^{-2t} - (1-e^{-2t}) \left(\frac{ze^{-t}h'(ze^{-t})}{h(ze^{-t})} + \frac{ze^{-t}g''(ze^{-t})}{g'(ze^{-t})} \right) - \frac{1}{2}(1-e^{-2t})^2 z^2 h(ze^{-t})(S_f(ze^{-t}) - S_g(ze^{-t})).$$

The right-hand side is equal $\frac{h(z)-1}{h(z)}$ for $t = 0$ and is analytic in $\bar{D} = \{z : |z| \leq 1\}$ if $t > 0$. Then putting $ze^{-t} = \zeta$, $e^{-t} = |\zeta|$ and replacing ζ through z we have from (2.8) and the assumption (2.1) that

$$\left| \frac{\dot{f}(z, t) - zf'(z, t)}{f(z, t) + zf'(z, t)} \right| \leq 1,$$

so $\dot{f}(z, t) = zf'(z, t) \cdot p(z, t)$, $\text{Re } p(z, t) > 0$ for $z \in D$, $t \in I$.

Since $\frac{f(z, t)}{a_1(t)}$, $t \in I$, is a normal family it follows from (2.5) that $f(z, t)$ is a Löwner chain and $f(z, t)$ is univalent in D [8, Corollary 3].

In particular we conclude from (2.3) and (2.4) that $f(z) = f(z, 0) = \frac{u(z)}{v(z)}$ is univalent in D and this ends the proof.

The proof given here is analogous to the proof given by Pommerenke in [7].

3. Corollary 1. If $h(z) \equiv 1$, $z \in D$, then (2.1) gives (1.1).

Corollary 2. If we put $h(z) = \frac{1}{1-\omega(z)}$, where $|\omega(z)| \leq 1$, $\omega(z) \neq 1$ for $z \in D$, then we obtain from (2.1)

$$(3.1) \quad \left| \omega(z)|z|^2 - (1-|z|^2) \left(\frac{z\omega'(z)}{1-\omega(z)} + \frac{zg''(z)}{g'(z)} - \frac{1}{2}(1-|z|^2)^2 \frac{z}{\bar{z}} \cdot \frac{1}{1-\omega(z)} (S_f(z) - S_g(z)) \right) \right| \leq 1.$$

The univalence criterion obtained by Z. Lewandowski and J. Stankiewicz [4] shows to be a particular case of (3.1) as $g(z) \equiv z$.

Corollary 3. On putting $\omega(z) = c = \text{const.}$, $|c| \leq 1$, $c \neq 1$ and $g = f$, or $g(z) \equiv z$, respectively, in (3.1) we obtain

$$\left| c|z|^2 - (1-|z|^2) \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in D$$

or

$$\left| (1-|z|^2)^2 S_f(z) - 2c(1-c)\bar{z}^2 \right| \leq 2|1-c|, \quad z \in D$$

These are the well-known univalence criteria given by Ahlfors [1], which for $c = 0$ give (1.3) and (1.2), respectively.

Remark. The univalence criterion given here is a generalization of the criterion obtained by Epstein as the following example shows.

Let $f(z) = (1+z)^2$, $g(z) = z + \frac{1}{2}z^2$.

For $z \in D$ we have

$$\left| \frac{1}{2}(1-|z|^2)^2 (S_f - S_g) + (1-|z|^2) \frac{z}{\bar{z}} \frac{g''}{g'} \right| = (1-|z|^2) \left| \frac{\bar{z}}{1+z} \right| \leq |z| + |z|^2,$$

thus the inequality (1.1) isn't satisfied in D .

The same pair of functions f and g and $h(z) = \frac{1}{1+z}$, gives

$$\left| \frac{h-1}{h}|z|^2 - (1-|z|^2) \left(\frac{zh'}{h} + \frac{zg''}{g'} \right) - \frac{1}{2}(1-|z|^2)^2 \frac{z}{\bar{z}} h(S_f - S_g) \right| = |z|^3 < 1.$$

Also the criteria obtained by Ahlfors [1], Lewandowski and Stankiewicz [4] are not satisfied by the function $f(z) = (1+z)^2$.

The criteria of univalence in the unit disk D can be transferred on the upper half plane $U = \{z \in \mathbb{C}, \text{Im } z > 0\}$. The mapping $w = t \frac{1+z}{1-z} i$, $t > 0$, $z \in D$, together with Theorem 2, gives

Theorem 3. Let f be meromorphic and g analytic in U . If both functions are locally univalent in U and if there exists a function h , $\operatorname{Re} h \geq \frac{1}{2}$, analytic in U such that

$$(3.2) \left| \frac{h(z) - 1}{h(z)} \left| \frac{z - it}{z + it} \right|^2 + 2iy \frac{z - it}{z - it} \left(\frac{h'(z)}{h(z)} + \frac{g''(z)}{g'(z)} + \frac{2}{z + it} \right) + 2y^2 \frac{z^2 + t^2}{z^2 - t^2} h(z) (S_f(z) - S_g(z)) \right| \leq 1, \quad z \in U, \quad t > 0, \quad y = \operatorname{Im} z,$$

then f is univalent in U .

Putting in (3.2) $g(z) = \frac{z - it}{z + it}$, $t > 0$, $z \in U$, $h(z) = \frac{1 + p(z)}{2}$, $\operatorname{Re} p(z) \geq 0$ in U we obtain a theorem due to Lewandowski and Stankiewicz [5].

Putting in turn $h(z) = \frac{c}{z}$, where c is a constant satisfying $|c - 1| \leq 1$, $g(z) = \frac{z - it}{z + it}$, $t > 0$, $z \in U$ we obtain the inequality of Ahlfors [1] for $t \rightarrow \infty$ from (3.2):

$$|2y^2 S_f(z) + c(1 - c)| \leq |c|, \quad z \in U, \quad \operatorname{Im} z = y.$$

If in the last inequality the right hand side is replaced by $k|c|$, $0 \leq k < 1$, then the inequality implies the possibility of a K -quasiconformal extension of f on \bar{U} , $K = (1 + k)/(1 - k)$. This was proved in [2] and was a positive answer to the conjecture put forward by Ahlfors [1]. A similar question arises in the case of the inequality (3.1) for $t \rightarrow \infty$. This problem is in the course of study.

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STRESZCZENIE

W pracy podano warunek dostateczny jednolistości funkcji meromorficznej i lokalnie jednolistej w kole jednostkowym (tw.2). Warunek ten jest istotnym uogólnieniem warunku podanego przez Epsteina [7]. Przy odpowiednich założeniach otrzymuje się znane kryteria jednolistości Ahlforsa [1] lub Nehariego [6].

W twierdzeniu 3 podano dostateczny warunek jednolistości funkcji meromorficznej i lokalnie jednolistej w górnej półpłaszczyźnie.

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