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On a Certain Extension of Epstein's Univalence Criterion

O pewnym uogólnieniu kryterim jednolistności Epsteina

Abstract. In this paper a sufficient univalence condition for meromorphic and locally univalent functions in the unit disk is given (Theorem 2).

This condition is an essential generalization of the Epstein's univalence criterion [7]. As particular cases the well-known univalence criteria of Ahlfors [1] and Nehari [6] are obtained.

Moreover, a sufficient univalence criterion for meromorphic and locally univalent functions in the upper half plane is given (Theorem 3).

1. Ch. Pommerenke has recently given a simplified proof of a univalence criterion obtained earlier by Epstein in another way. In his proof an additional assumption made by Epstein was dropped (see e.g. [7, p.143]).

Let $D = \{z : |z| < 1\}$ and let S_f denote the Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

Theorem 1. (C.L. Epstein, see e.g. [7]) *Let f be meromorphic and g analytic in D. If both functions are locally univalent in D and if*

$$\left|\tfrac{1}{2}(1-|z|^2)^2\big(S_f(z)-S_g(z)\big) + (1-|z|^2)\bar{z}\,\frac{g''(z)}{g'(z)}\right| \le 1 \tag{1.1}$$

for $z \in D$ then f is univalent in D.

If $g(z) \equiv z$ then (1.1) gives

$$|(1-|z|^2)^2 S_f(z)| \le 2 \tag{1.2}$$

and this is the well-known univalence criterion of Nehari [6].

If $g = f$ then (1.1) implies

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 , \tag{1.3}$$

and this is also a well-known univalence criterion (see e.g. [9, p.172]).

2. The following theorem is a generalization of the univalence criterion given by Epstein.

Theorem 2. *Let f be meromorphic and g analytic in D. If both functions are locally univalent in D and if there exists a function h, analytic in D, satisfying* $\operatorname{Re} h(z) \geq \frac{1}{2}$ *and such that*

$$\begin{aligned}(2.1)\qquad &\Big|\frac{h(z)-1}{h(z)}|z|^2-(1-|z|^2)\Big(\frac{zh'(z)}{h(z)}+\frac{zg''(z)}{g'(z)}\Big)-\\ &-\tfrac{1}{2}(1-|z|^2)^2\frac{z}{\bar z}h(z)\big(S_f(z)-S_g(z)\big)\Big|\leq 1\,,\quad z\in D\end{aligned}$$

then f is univalent in D.

Proof. Let $f(z)=a_0+a_1z+\cdots$, $g(z)=b_0+b_1z+\cdots$, $a_1\neq 0$, $b_1\neq 0$. Then it is possible to consider the functions :

$$f^*(z)=\frac{f-a_0}{a_1+\left(\frac{a_2}{a_1}-\frac{b_2}{b_1}\right)(f-a_0)}\,,\qquad g^*(z)=\frac{g-b_0}{b_1}$$

instead of f and g.

These normalizations don't impair the generality of our considerations and we may also assume that

$$f(z)=g(z)+O(z^3)\quad\text{and}\quad \frac{f'}{g'}=1+O(z^2)\quad\text{as}\quad z\to 0\,.$$

Let us introduce now

$$(2.2)\qquad v(z)=\sqrt{\frac{g'(z)}{f'(z)}}=1+\beta z^2+O(z^3)\,,$$

$$(2.3)\qquad u(z)=f(z)\cdot v(z)=z+\alpha z^2+O(z^3)\,.$$

Both functions are analytic in D because f cannot have multiple poles and because f' and g' do not vanish in D.

For $t\in I=<0,\infty)$ we consider [3, p.38]

$$(2.4)\qquad f(z,t)=\frac{u(ze^{-t})+(e^t-e^{-t})zh(ze^{-t})u'(ze^{-t})}{v(ze^{-t})+(e^t-e^{-t})zh(ze^{-t})v'(ze^{-t})}\,.$$

The function $f(z,t)$ is for each fixed $t\in I$ meromorphic in D. From (2.2) and from the assumption on the function h given in Theorem 1 it follows that the denominator in (2.4) has the form $1+O(z^2)$ as $z\to 0$, uniformly with respect to t.

It is easy to show, that $\dfrac{f(z,t)}{a_1(t)}=z+\cdots$, $t\in I$, is a normal family in D, where

$$(2.5)\qquad a_1(t)=e^{-t}+(e^t-e^{-t})h(0)\quad\text{and}\quad |a_1(t)|\to\infty\quad\text{as}\quad t\to\infty\,.$$

From (2.3) and (2.4) we have

$$f(z,t) = a_1(t)z + O(z^2) \quad \text{as} \quad z \to \infty \,, \tag{2.6}$$

where $a_1(t)$ is defined by (2.5).

Let us denote $f'(z,t) = \dfrac{\partial f(z,t)}{\partial z}$, $\dot{f}(z,t) = \dfrac{\partial f(z,t)}{\partial t}$. After some calculations we obtain from (2.4)

$$w = \frac{\dot{f}(z,t) - zf'(z,t)}{\dot{f}(z,t) + zf'(z,t)} = \frac{h-1}{h} e^{-2t} - (1-e^{-2t})\frac{ze^{-t}h'}{h} - \tag{2.7}$$
$$- \frac{(1-e^{-2t})ze^{-t}(u''v - uv'') + (1-e^{-2t})^2 z^2 h(u''v' - u'v'')}{u'v - uv'}$$

where $u, v, h, u', v', h', u'', v''$ are evaluated at the point ze^{-t}

From (2.3) and the assumption (2.2) we have

$$u'v - uv' = f'v^2$$
$$u''v - uv'' = f''v^2 + 2f'v'v = f'v^2\frac{g''}{g'}$$
$$u''v' - u'v'' = f''v'v - f'v''v + 2f'(v')^2 = \tfrac{1}{2}f'v^2(S_f - S_g)\,.$$

Taking this and (2.7) into account we have for $z \in D$

$$w = \frac{h(ze^{-t}) - 1}{h(ze^{-t})} e^{-2t} - (1-e^{-2t})\left(\frac{ze^{-t}h'(ze^{-t})}{h(ze^{-t})} + \frac{ze^{-t}g''(ze^{-t})}{g'(ze^{-t})}\right) - \tag{2.8}$$
$$- \tfrac{1}{2}(1-e^{-2t})^2 z^2 h(ze^{-t})\big(S_f(ze^{-t}) - S_g(ze^{-t})\big)\,.$$

The right-hand side is equal $\dfrac{h(z)-1}{h(z)}$ for $t = 0$ and is analytic in $\overline{D} = \{z : |z| \le 1\}$ if $t > 0$. Then putting $ze^{-t} = \varsigma$, $e^{-t} = |\varsigma|$ and replacing ς through z we have from (2.8) and the assumption (2.1) that

$$\left|\frac{\dot{f}(z,t) - zf'(z,t)}{\dot{f}(z,t) + zf'(z,t)}\right| \le 1\,,$$

so $\dot{f}(z,t) = zf'(z,t)\cdot p(z,t)$, $\operatorname{Re} p(z,t) > 0$ for $z \in D$, $t \in I$.

Since $\dfrac{f(z,t)}{a_1(t)}$, $t \in I$, is a normal family it follows from (2.5) that $f(z,t)$ is a Löwner chain and $f(z,t)$ is univalent in D [8, Corollary 3].

In particular we conclude from (2.3) and (2.4) that $f(z) = f(z,0) = \dfrac{u(z)}{v(z)}$ is univalent in D and this ends the proof.

The proof given here is analogous to the proof given by Pommerenke in [7].

3. Corollary 1. *If $h(z) \equiv 1$, $z \in D$, then (2.1) gives (1.1).*

Corollary 2. *If we put* $h(z) = \dfrac{1}{1-\omega(z)}$, *where* $|\omega(z)| \le 1$, $\omega(z) \ne 1$ *for* $z \in D$, *then we obtain from (2.1)*

$$(3.1)\qquad \Big|\omega(z)|z|^2 - (1-|z|^2)\Big(\frac{z\omega'(z)}{1-\omega(z)} + \frac{zg''(z)}{g'(z)} - \\ - \tfrac{1}{2}(1-|z|^2)^2\frac{z}{\bar z}\cdot\frac{1}{1-\omega(z)}\,(S_f(z) - S_g(z))\Big| \le 1\,.$$

The univalence criterion obtained by Z. Lewandowski and J. Stankiewicz [4] shows to be a particular case of (3.1) as $g(z) \equiv z$.

Corollary 3. *On putting* $\omega(z) = c = \text{const.}$, $|c| \le 1$, $c \ne 1$ *and* $g = f$, *or* $g(z) \equiv z$, *respectively, in (3.1) we obtain*

$$\Big|c|z|^2 - (1-|z|^2)\frac{zf''(z)}{f'(z)}\Big| \le 1\,, \quad z \in D$$

or

$$|(1-|z|^2)^2 S_f(z) - 2c(1-c)\bar z^2| \le 2|1-c|\,, \quad z \in D$$

These are the well-known univalence criteria given by Ahlfors [1], which for $c = 0$ give (1.3) and (1.2), respectively.

Remark. The univalence criterion given here is a generalization of the criterion obtained by Epstein as the following example shows.

Let $f(z) = (1+z)^2$, $g(z) = z + \frac{1}{2}z^2$.
For $z \in D$ we have

$$\Big|\tfrac{1}{2}(1-|z|^2)^2(S_f - S_g) + (1-|z|^2)\bar z\,\frac{g''}{g'}\Big| = (1-|z|^2)\Big|\frac{\bar z}{1+z}\Big| \le |z| + |z|^2\,,$$

thus the inequality (1.1) isn't satisfied in D.

The same pair of functions f and g and $h(z) = \dfrac{1}{1+z}$, gives

$$\Big|\frac{h-1}{h}|z|^2 - (1-|z|^2)\Big(\frac{zh'}{h} + \frac{zg''}{g'}\Big) - \tfrac{1}{2}(1-|z|^2)^2\frac{z}{\bar z}\,h(S_f - S_g)\Big| = |z|^3 < 1\,.$$

Also the criteria obtained by Ahlfors [1], Lewandowski and Stankiewicz [4] are not satisfied by the function $f(z) = (1+z)^2$.

The criteria of univalence in the unit disk D can be transferred on the upper half plane $U = \{z \in C\,,\ \mathrm{Im}\, z > 0\}$. The mapping $w = t\,\dfrac{1+z}{1-z}\,i$, $t > 0$, $z \in D$, together with Theorem 2, gives

Theorem 3. *Let f be meromorphic and g analytic in U. If both functions are locally univalent in U and if there exists a function h, $\operatorname{Re} h \geq \frac{1}{2}$, analytic in U such that*

$$(3.2)\quad \Big|\frac{h(z)-1}{h(z)}\Big|\frac{z-it}{z+it}\Big|^2 + 2iy\frac{z-it}{\bar{z}-it}\Big(\frac{h'(z)}{h(z)}+\frac{g''(z)}{g'(z)}+\frac{2}{z+it}\Big)+ + 2y^2\frac{z^2+t^2}{\bar{z}^2+t^2}h(z)\big(S_f(z)-S_g(z)\big)\Big| \leq 1\,,\quad z\in U\,,\quad t>0\,,\quad y=\operatorname{Im} z\,,$$

then f is univalent in U.

Putting in (3.2) $g(z) = \dfrac{z-it}{z+it}$, $t>0$, $z\in U$, $h(z) = \dfrac{1+p(z)}{2}$, $\operatorname{Re} p(z) \geq 0$ in U we obtain a theorem due to Lewandowski and Stankiewicz [5].

Putting in turn $h(z) = \frac{1}{c}$, where c is a constant satisfuing $|c-1| \leq 1$, $g(z) = \dfrac{z-it}{z+it}$, $t>0$, $z\in U$ we obtain the inequality of Ahlfors [1] for $t\to\infty$ from (3.2):

$$|2y^2 S_f(z) + c(1-c)| \leq |c|\,,\quad z\in U\,,\quad \operatorname{Im} z = y\,.$$

If in the last inequality the right hand side is replaced by $k|c|$, $0\leq k<1$, then the inequality implies the possibility of a K-quasiconformal extension of f on $\overline{C}$, $K=(1+k)/(1-k)$. This was proved in [2] and was a positive answer to the conjecture put forward by Ahlfors [1]. A similiar question arises in the case of the inequality (3.1) for $t\to\infty$. This problem is in the course of study.

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STRESZCZENIE

W pracy podano warunek dostateczny jednolistności funkcji meromorficznej i lokalnie jednolistnej w kole jednostkowym (tw.2). Warunek ten jest istotnym uogólnieniem warunku podanego przez Epsteina [7]. Przy odpowiednich założeniach otrzymuje się znane kryteria jednolistności Ahlforsa [1] lub Nehariego [6].

W twierdzeniu 3 podano dostateczny warunek jednolistności funkcji meromorficznej i lokalnie jednolistnej w górnej półpłaszczyźnie.

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