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The Structures on Certain Submanifolds of the Riemannian Manifold with a 3-structure

Struktury na podrozmaitościach rozmaitości Riemanna z 3-strukturą

Abstract. This paper deals with 3-structures on some 3- and $(4n-4)$-dimensional Riemannian manifolds M^{4n} with a given 3-stucture $\{\underset{\alpha}{\bar{F}}\}$. Both manifolds are submanifolds of a hypersurface embedded in M^{4n} with an induced 3-structure. Also connections induced on these manifolds and integrability conditions of distributions determining the above mentioned submanifolds are considered.

Introduction. In this paper we will study structures on certain submanifolds (of the dimension 3 and $4n-4$, respectively) of a Riemannain manifold with a 3-structure. Both submanifolds are submanifolds of a hypersurface in the given manifold and they are defined by the given 3-structure. We will prove that the structure on one of these submanifolds is generated by the original 3-structure given on the manifold ; but on the other ones the 3-structure is that induced on the hypersurface. Finally, we will consider connections induced on these submanifolds.

1. Induced structures on submanifolds. Let be given the 3-structure $\{\underset{\alpha}{\bar{F}}\}$, $\alpha = 1,2,3$, on a Riemannian manifold M^{4n} with the metric $\bar{g}$ such that

$$\underset{\alpha}{\bar{F}}^2 = \underset{\alpha}{\varepsilon}\,\bar{I} \quad , \quad \underset{\alpha}{\bar{F}} \circ \underset{\beta}{\bar{F}} = \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\bar{F}} \quad , \tag{1.1}$$

$\underset{\alpha}{\varepsilon} = \pm 1$, $\underset{\alpha\beta}{\varepsilon} = \pm 1$, $\alpha \neq \beta \neq \gamma \neq \alpha$ ([1]), where $\underset{\alpha}{\bar{F}}$ is a tensor field of the type $(1,1)$ on M^{4n} , $\bar{I}$ – the identity mapping on TM^{4n}. The coefficients $\underset{\alpha}{\varepsilon}$, $\underset{\alpha\beta}{\varepsilon}$ satisfy the following identities

$$\begin{gathered} \underset{\alpha}{\varepsilon}\,\underset{\beta\gamma}{\varepsilon} = \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\varepsilon} \\ \underset{\alpha\beta}{\varepsilon}\,\underset{\alpha\gamma}{\varepsilon} = \underset{\beta\alpha}{\varepsilon}\,\underset{\gamma\alpha}{\varepsilon} = \underset{\alpha}{\varepsilon} \end{gathered} \tag{1.2}$$

for $\alpha \neq \beta \neq \gamma \neq \alpha$ ([1]).

Let M^{4n-1} be smooth, oriented hypersurface immersed in M^{4n}. We assume that there exists a smooth vector field N normal to M^{4n-1} with respect to metric $\tilde{g}$ and $\tilde{g}(N,N) = 1$. Then for an arbitrary vector field $\tilde{X} \in TM^{4n}$ we have the decomposition

$$\underset{\alpha}{\tilde{F}}\tilde{X} = \underset{\alpha}{F}\tilde{X} + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega}(\tilde{X})N\,, \qquad \alpha = 1,2,3, \tag{1.3}$$

where $\underset{\alpha}{F}$ denotes the tensor field of the type $(1,1)$, $\underset{\alpha}{F}\tilde{X} \in TM^{4n-1}$, $\underset{\alpha}{\omega}$ – tensor field of the type $(0,1)$, ([1]).

We introduce the notations

$$\begin{aligned} \underset{\alpha}{\eta} &= \underset{\alpha}{\tilde{F}}N \in TM^{4n-1} \\ \underset{\alpha}{\lambda} &= \underset{\alpha}{\omega}(N) \in R\,. \end{aligned} \tag{1.4}$$

In particular we have

$$\underset{\alpha}{\tilde{F}}N = \underset{\alpha}{\eta} + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,N\,. \tag{1.5}$$

With respect to (1.3) we get

$$\underset{\alpha}{\tilde{F}}X = \underset{\alpha}{F}X + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega}(X)N \tag{1.6}$$

for $X \in TM^{4n-1}$.

In this way the 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ on the manifold M^{4n} induces a 3-structure $\{\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\}$ on an oriented hypersurface M^{4n-1} satisfying the following conditions (Theorem 2, [1]) :

$$\begin{aligned} \underset{\alpha}{F}^2 &= \underset{\alpha}{\varepsilon}(I - \underset{\alpha}{\omega} \otimes \underset{\alpha}{\eta}) \\ \underset{\alpha}{\omega} \circ \underset{\alpha}{F} &= -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\omega} \\ \underset{\alpha}{F}\,\underset{\alpha}{\eta} &= -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\eta} \\ \underset{\alpha}{\omega}(\underset{\alpha}{\eta}) &= 1 - \underset{\alpha}{\varepsilon}(\underset{\alpha}{\lambda})^2 \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \underset{\alpha}{F} \circ \underset{\beta}{F} &= \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{F} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\omega} \otimes \underset{\alpha}{\eta} \\ \underset{\alpha}{\omega} \circ \underset{\beta}{F} &= \underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\omega} - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\beta}{\omega} \\ \underset{\alpha}{F}\,\underset{\beta}{\eta} &= \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\eta} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}\,\underset{\alpha}{\eta} \\ \underset{\alpha}{\omega}(\underset{\beta}{\eta}) &= \underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\lambda} - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\beta}{\lambda} \end{aligned} \tag{1.8}$$

We will assume that the metric $\tilde{g}$ on M^{4n} satisfies the condition

$$\tilde{g}(\underset{\alpha}{\tilde{F}}\underset{1}{\tilde{X}}, \underset{\alpha}{\tilde{F}}\underset{2}{\tilde{X}}) = \tilde{g}(\underset{1}{\tilde{X}}, \underset{2}{\tilde{X}}) \tag{1,9}$$

for $\alpha = 1,2,3$ and for an arbitrary $\underset{1}{\bar{X}}, \underset{2}{\bar{X}} \in TM^{4n}$. The existence of this metric was proved in Theorem 1, [1].

On the hypersurface M^{4n-1} we introduce the metric g induced by $\bar{g}$ as follows

$$(1.10) \qquad g(\underset{1}{X}, \underset{2}{X}) = \bar{g}(\underset{1}{X}, \underset{2}{X}) \qquad \text{for } \underset{1}{X}, \underset{2}{X} \in TM^{4n-1}$$

For the metric g and the fields $\underset{\alpha}{\omega}, \underset{\alpha}{\eta}$ we have

$$(1.11) \qquad g(X, \underset{\alpha}{\eta}) = \underset{\alpha}{\omega}(X)$$

for arbitrary $X \in TM^{4n-1}$ (Theorem 4, [1]).

Moreover, we have

$$(1.12) \qquad g(\underset{\alpha}{F}\underset{1}{X}, \underset{\alpha}{F}\underset{2}{X}) = g(\underset{1}{X}, \underset{2}{X}) - \underset{\alpha}{\omega}(\underset{1}{X})\underset{\alpha}{\omega}(\underset{2}{X}) .$$

We will assume that the vector fields $\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta} \in TM^{4n-1}$ are linearly independent. The conditions of existence of these fields are given in Theorem 3, [1]. At each point $p \in M^{4n-1}$ we define a 3-dimensional linear subspace generated by the vectors $\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta}$:

$$W_p = \left(\mathrm{Lin}\, \underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta}\right)_p .$$

Then we have $W_p \subset (TM^{4n-1})_p$. Let $W_p^{\perp}$ denote the orthogonal complement with respect to g. Therefore, we get

$$(TM^{4n-1})_p = W_p \oplus W_p^{\perp} .$$

Then

$$W_3 = \bigcup_{p \in M^{4n-1}} W_p \ , \quad W_{4n-4}^{\perp} = \bigcup_{p \in M^{4n-1}} W_p^{\perp}$$

are smooth distributions. We will assume the integrability of these distributions. Let M^3, M^{4n-4} be integral manifolds of the distributions W_3, $W_{4n-4}^{\perp}$, respectively (at least locally). The conditions of the integrability of these distributions will be studied later.

For each vector field $X \in TM^{4n-1}$ we have the following decomposition

$$(1.13) \qquad \underset{\alpha}{F}X = \underset{\alpha}{\varphi}X + \underset{\alpha}{\psi}X , \qquad \alpha = 1,2,3,$$

where $\underset{\alpha}{\varphi}, \underset{\alpha}{\psi}$ are the tensor fields of the type (1,1) on M^{4n-1}, $\underset{\alpha}{\varphi}X \in W_3$, $\underset{\alpha}{\psi}X \in W_{4n-4}$, $g(\underset{\alpha}{\varphi}X, \underset{\alpha}{\psi}X) = 0$.

Lemma 1.1. *If $Y \in W_3$, then $\underset{\alpha}{F}Y \in W_3$.*

Proof. From the definition of the distribution W_3 it sufficies to demonstrate that $\underset{\alpha}{F}\underset{\alpha}{\eta}, \underset{\alpha}{F}\underset{\beta}{\eta} \in W_3$, $\alpha \neq \beta$. Namely, from (1.7) and (1.8) we get

$$\underset{\alpha}{F}\underset{\alpha}{\eta} = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\eta} \in W_3 \,,$$
$$\underset{\alpha}{F}\underset{\beta}{\eta} = \underset{\alpha\beta\gamma}{\varepsilon}\,\underset{\gamma}{\eta} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}\,\underset{\alpha}{\eta} \in W_3 \,.$$

It ends our proof.

Lemma 1.2. *If $Z \in W^{\perp}_{4n-4}$, then $\underset{\alpha}{F}Z \in W^{\perp}_{4n-4}$.*

Proof. Let $Z \in W^{\perp}_{4n-4}$. For any $\alpha = 1,2,3$ we have $g(Z, \underset{\alpha}{\eta}) = 0$. From the formula (1.11) we have

$$\underset{\alpha}{\omega}(Z) = g(Z, \underset{\alpha}{\eta}) = 0\,, \qquad \alpha = 1,2,3 \tag{1.14}$$

for each $Z \in W^{\perp}_{4n-4}$. Making use of (1.14) and (1.7), (1.8) we obtain

$$g(\underset{\alpha}{F}Z, \underset{\alpha}{\eta}) = (\underset{\alpha}{\omega} \circ \underset{\alpha}{F})(Z) = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\omega}(Z) = 0\,,$$
$$g(\underset{\alpha}{F}Z, \underset{\beta}{\eta}) = (\underset{\beta}{\omega} \circ \underset{\alpha}{F})(Z) = \underset{\alpha\gamma\gamma}{\varepsilon}\,\underset{\gamma}{\omega}(Z) - \underset{\alpha}{\varepsilon}\,\underset{\beta}{\lambda}\,\underset{\alpha}{\omega}(Z) = 0\,, \qquad \alpha \neq \beta\,.$$

The above conditions imply $\underset{\alpha}{F}Z \in W^{\perp}_{4n-4}$.

The lemmata 1.1 and 1.2 imply

Theorem 1.1. *The distributions W_3 and $W^{\perp}_{4n-4}$ are invariant with respect to the mappings $\underset{\alpha}{F}$.*

Theorem 1.2. *The mappings $\underset{\alpha}{\varphi}, \underset{\alpha}{\psi}$ satisfy the following conditions*

$$\underset{\alpha}{\psi}Y = 0 \qquad \textit{for } Y \in W_3\,, \tag{1.15}$$
$$\underset{\alpha}{\varphi}Z = 0 \qquad \textit{for } Z \in W_{4n-4}\,, \tag{1.16}$$
$$\underset{\alpha}{\varphi} \circ \underset{\beta}{\psi} = 0\,, \quad \underset{\alpha}{\psi} \circ \underset{\beta}{\varphi} = 0 \qquad \textit{for any } \alpha = 1,2,3. \tag{1.17}$$

Proof. Namely, from (1.13) and (1.7), (1.8) we obtain

$$\underset{\alpha}{F}\underset{\alpha}{\eta} = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\underset{\alpha}{\eta} = \underset{\alpha}{\varphi}\,\underset{\alpha}{\eta} + \underset{\alpha}{\psi}\,\underset{\alpha}{\eta}$$
$$\underset{\alpha}{F}\underset{\beta}{\eta} = \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\eta} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}\,\underset{\alpha}{\eta} = \underset{\alpha}{\varphi}\,\underset{\beta}{\eta} + \underset{\alpha}{\psi}\,\underset{\beta}{\eta}\,.$$

Then we have

$$\underset{\alpha}{\psi}\,\underset{\alpha}{\eta} = \underset{\alpha}{\psi}\,\underset{\beta}{\eta} = 0\,.$$

From the above equality and from the definition of the subspace W_3 we obtain (1.15).

However, the condition (1.16) follows from the decomposition (1.13) and Lemma 1.2. For $Z \in W_{4n-4}^{\perp}$ we have

$$0 = g(\underset{\alpha}{F}Z, \underset{\beta}{\eta}) = g(\underset{\alpha}{\varphi}Z, \underset{\beta}{\eta}) + g(\underset{\alpha}{\psi}Z, \underset{\beta}{\eta}) , \qquad \text{where } \alpha, \beta = 1,2,3.$$

Therefore

$$\underset{\alpha}{\varphi}Z = 0 \qquad \text{for each } Z \in W_{4n-4}^{\perp} .$$

The equalities (1.17) are consequences of (1.15) and (1.16)

Corollary . *The restrictions* $\underset{\alpha}{F}|_{W_3}$, $\underset{\alpha}{F}|_{W_{4n-4}^{\perp}}$ *of* $\underset{\alpha}{F}$ *coincide with* $\underset{\alpha}{\varphi}$ $(\underset{\alpha}{\psi})$,

$$\underset{\alpha}{F}|_{W_3} = \underset{\alpha}{\varphi} \quad , \quad \underset{\alpha}{F}|_{W_{4n-4}^{\perp}} = \underset{\alpha}{\psi} .$$

We will investigate the structures on integral submanifolds M^3 and M^{4n-4} generated by the structure $\{\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\}$ on the manifold M^{4n-1}. From (1.13) and (1.17) we get

$$\underset{\alpha}{F}^2 = \underset{\alpha}{\varphi}^2 + \underset{\alpha}{\psi}^2 , \tag{1.18}$$

$$\underset{\alpha}{F} \circ \underset{\beta}{F} = \underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi} + \underset{\alpha}{\psi} \circ \underset{\beta}{\psi} . \tag{1.19}$$

These conditions and (1.7), (1.8), (1.13) imply

$$\underset{\alpha}{\varepsilon}(I - \underset{\alpha}{\omega} \otimes \underset{\alpha}{\eta}) = \underset{\alpha}{\varphi}^2 + \underset{\alpha}{\psi}^2 \tag{1.20}$$

$$\underset{\alpha\beta}{\varepsilon}(\underset{\alpha}{\varphi} + \underset{\alpha}{\psi}) - \underset{\beta}{\varepsilon}\underset{\beta}{\omega} \otimes \underset{\alpha}{\eta} = \underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi} + \underset{\alpha}{\psi} \circ \underset{\beta}{\psi} \tag{1.21}$$

From the above equalities and (1.15), (1.16) we get for the subspace W_3 :

$$\underset{\alpha}{\varphi}^2 = \underset{\alpha}{\varepsilon}(I - \underset{\alpha}{\omega} \otimes \underset{\alpha}{\eta})$$

$$\underset{\alpha}{\varphi} \circ \underset{\beta}{\varphi} = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varphi} - \underset{\beta}{\varepsilon}\underset{\beta}{\omega} \otimes \underset{\alpha}{\eta}$$

and for the subspace $W_{4n-4}^{\perp}$:

$$\underset{\alpha}{\psi}^2 = \underset{\alpha}{\varepsilon} I$$

$$\underset{\alpha}{\psi} \circ \underset{\beta}{\psi} = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\psi}$$

We will say that the 3-structures $\{\underset{\alpha}{F}\}$ and $\{\underset{\alpha}{\psi}\}$ (or $\{\underset{\alpha}{F}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\}$ and $\{\underset{\alpha}{\varphi}, \underset{\alpha}{\omega}, \underset{\alpha}{\eta}\}$) are of the same type if they satisfy conditions of the same form (1.1) (or of the forms (1.7) and (1.8)). Thus we obtain

Theorem 1.3. *The 3-structure $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ on the submanifold M^{4n-1} immersed in M^{4n}, generated by 3-structure $\{\underset{\alpha}{\tilde{F}}\}$ on M^{4n} induces 3-structure $\{\underset{\alpha}{\varphi},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ on integral submanifold M^3 and 3-structure $\{\underset{\alpha}{\psi}\}$ on submanifold M^{4n-4}. Moreover, 3-structure $\{\underset{\alpha}{F}\}$, $\{\underset{\alpha}{\psi}\}$ and $\{\underset{\alpha}{F},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$, $\{\underset{\alpha}{\varphi},\underset{\alpha}{\omega},\underset{\alpha}{\eta}\}$ are of the same type.*

2. Induced connections. The equations of Gauss and Codazzi for the hypersurface M^{4n-1} in M^{4n} are of the form

$$(2.1)\qquad \begin{cases} \tilde{\nabla}_{\underset{1}{X}} \underset{2}{X} = \nabla_{\underset{1}{X}} \underset{2}{X} + h(\underset{1}{X},\underset{2}{X})N \\ \tilde{\nabla}_{\underset{1}{X}} N = -K \underset{1}{X} + k(\underset{1}{X})N \end{cases}$$

where $\underset{1}{X},\underset{2}{X} \in TM^{4n-1}$, ∇ — the operator of the Levi–Civita induced connection on M^{4n-1}, h — the second fundamental form on M^{4n-1}, $K : TM^{4n-1} \to TM^{4n-1}$ — the fundamental Weingarten tensor with respect to the normal vector N, $k : TM^{4n-1} \to R$ — the tensor of the type $(0,1)$ on M^{4n-1}.

If $\overset{0}{\tilde{\nabla}}$ is an operator of $\{\underset{\alpha}{\tilde{F}}\}$ — connection on the Riemannian manifold M^{4n}, i.e. $\overset{0}{\tilde{\nabla}}\underset{\alpha}{\tilde{F}} = 0$ for $\alpha = 1,2,3$, then the induced connection $\overset{0}{\nabla}$ on the submanifold M^{4n-1} satisfies the relations :

$$(2.2)\qquad \begin{cases} (\overset{0}{\nabla}_{\underset{1}{X}} \underset{\alpha}{F})(\underset{2}{X}) = \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\omega}(\underset{2}{X})\overset{0}{K} \underset{1}{X} + \overset{0}{h}(\underset{1}{X},\underset{2}{X})\underset{\alpha}{\eta} \\ (\overset{0}{\nabla}_{\underset{1}{X}} \underset{\alpha}{\omega})(\underset{2}{X}) = \underset{\alpha}{\lambda}\, \overset{0}{h}(\underset{1}{X},\underset{2}{X}) - \underset{\alpha}{\varepsilon}\, h(\underset{1}{X},\underset{\alpha}{F}\underset{2}{X}) - \underset{\alpha}{\omega}(\underset{2}{X})\overset{0}{k}(\underset{1}{X}) \\ \overset{0}{\nabla}_{\underset{1}{X}} \underset{\alpha}{\eta} = \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\lambda}\, \overset{0}{K} \underset{1}{X} + \overset{0}{k}(\underset{1}{X})\underset{\alpha}{\eta} - (\underset{\alpha}{F} \circ \overset{0}{K})(\underset{1}{X}) \\ \partial_{\underset{1}{X}} \underset{\alpha}{\lambda} = -\underset{\alpha}{\varepsilon}\, \overset{0}{h}(\underset{1}{X},\underset{\alpha}{\eta}) - (\underset{\alpha}{\omega} \circ \overset{0}{K})(\underset{1}{X}) \end{cases}$$

for $\underset{1}{X},\underset{2}{X} \in TM^{4n-1}$ ([2]).

From the above considerations we have the following :

Theorem 2.1. *The distribution W_3 is integrable if and only if*

$$\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}\,\overset{0}{K}\,\underset{\beta}{\eta} - (\underset{\alpha}{F} \circ \overset{0}{K})(\underset{\beta}{\eta}) - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}\,\overset{0}{K}\,\underset{\alpha}{\eta} + (\underset{\beta}{F} \circ \overset{0}{K})(\underset{\alpha}{\eta}) \in W_3$$

for arbitrary α,β.

This result immediately follows from (2.2).

Theorem 2.2. *The distribution $W^{\perp}_{4n-4}$ is integrable if and only if*

$$(2.3)\qquad \overset{0}{h}(\underset{\alpha}{F}\underset{1}{Z},\underset{\alpha}{F}\underset{2}{Z}) = \underset{\alpha}{\varepsilon}\,\overset{0}{h}(\underset{1}{Z},\underset{2}{Z})$$

for any $\underset{1}{Z}, \underset{2}{Z} \in W^{\perp}_{4n-4}$, $\alpha = 1,2,3$.

Proof. The distribution $W^{\perp}_{4n-4}$ is characterized by the condition (1.14) :

$$\underset{\alpha}{\omega}(Z) = 0 \quad \text{for arbitrary } Z \in W^{\perp}_{4n-4} .$$

Hence we have

$$\overset{0}{\nabla}_{\underset{1}{Z}} \underset{\alpha}{\omega}(\underset{2}{Z}) = 0 , \qquad \underset{1}{Z}, \underset{2}{Z} \in W^{\perp}_{4n-4}$$

and

$$(\overset{0}{\nabla}_{\underset{1}{Z}} \underset{\alpha}{\omega})(\underset{2}{Z}) + \underset{\alpha}{\omega}(\overset{0}{\nabla}_{\underset{1}{Z}} \underset{2}{Z}) = 0 .$$

Hence and from the formula (2.2) there arises

$$\underset{\alpha}{\omega}(\overset{0}{\nabla}_{\underset{1}{Z}} \underset{2}{Z}) = -(\overset{0}{\nabla}_{\underset{1}{Z}} \underset{\alpha}{\omega})(\underset{2}{Z}) = -\underset{\alpha}{\lambda} \overset{0}{h}(\underset{1}{Z}, \underset{2}{Z}) + \underset{\alpha}{\varepsilon} \overset{0}{h}(\underset{1}{Z}, \underset{\alpha}{F} \underset{2}{Z})$$

Since $\overset{0}{h}(\underset{1}{Z}, \underset{2}{Z}) = \overset{0}{h}(\underset{2}{Z}, \underset{1}{Z})$, so we have

$$\underset{\alpha}{\omega}([\underset{1}{Z}, \underset{2}{Z}]) = \underset{\alpha}{\omega}(\overset{0}{\nabla}_{\underset{1}{Z}} \underset{2}{Z} - \overset{0}{\nabla}_{\underset{2}{Z}} \underset{1}{Z}) = \underset{\alpha}{\omega}(\overset{0}{\nabla}_{\underset{1}{Z}} \underset{2}{Z}) - \underset{\alpha}{\omega}(\overset{0}{\nabla}_{\underset{2}{Z}} \underset{1}{Z}) = \underset{\alpha}{\varepsilon}(\overset{0}{h}(\underset{1}{Z}, \underset{\alpha}{F} \underset{2}{Z}) - \overset{0}{h}(\underset{\alpha}{F} \underset{1}{Z}, \underset{2}{Z})) .$$

Let $\underset{\alpha}{\omega}(\underset{1}{Z}) = \underset{\alpha}{\omega}(\underset{2}{Z}) = 0$. Then $\underset{\alpha}{\omega}([\underset{1}{Z}, \underset{2}{Z}]) = 0$ if and only if

$$\overset{0}{h}(\underset{1}{Z}, \underset{\alpha}{F} \underset{2}{Z}) = \overset{0}{h}(\underset{\alpha}{F} \underset{1}{Z}, \underset{2}{Z}) .$$

Replacing in the above equality $\underset{1}{Z}$ by $F\underset{1}{Z}$ we obtain (2.3).

From the formulas (1.3) and (1.13) we have

$$\underset{\alpha}{\tilde{F}} X = \underset{\alpha}{\varphi} X + \underset{\alpha}{\psi} X + \underset{\alpha}{\varepsilon} \underset{\alpha}{\omega}(X) N \qquad \text{for } X \in TM^{4n-1} ;$$

since $\underset{\alpha}{\varphi} X \in W_3$, so we can take

$$\underset{\alpha}{\varphi} X = \underset{\alpha}{A}^{\beta}(X) \underset{\beta}{\eta} , \qquad \beta = 1,2,3.$$

Then we have

$$\underset{\alpha}{\tilde{F}} X = \underset{\alpha}{\psi} X + \underset{\alpha}{A}^{\beta}(X) \underset{\beta}{\eta} + \underset{\alpha}{\varepsilon} \underset{\alpha}{\omega}(X) N .$$

Using the decomposition $TM^{4n-1} = W_3 \oplus W^{\perp}_{4n-4}$, we rewrite the formulas of Gauss and Codazzi (2.1) for a hypersurface M^{4n-1} in the following form :

$$\text{(2.5)} \qquad \begin{cases} \tilde{\nabla}_X Z = \nabla_X Z + h^{\alpha}(X, Z) + h(X, Z) N \\ \tilde{\nabla}_X \underset{\beta}{\eta} = -\underset{\beta}{L} X + \underset{\beta}{l}^{\alpha}(X) \underset{\alpha}{\eta} + \underset{\beta}{l}(X) N \\ \tilde{\nabla}_X N = -K X + k^{\alpha}(X) \underset{\alpha}{\eta} + k(X) N \end{cases}$$

where $X \in TM^{4n-1}$, $Z \in W^{\perp}_{4n-4}$, $\nabla_X Z$, $\underset{\beta}{L}(X)$, $K(X) \in W^{\perp}_{4n-4}$, $h^{\alpha}(X,Z)$, $\underset{\beta}{l}^{\alpha}(X)$, $k^{\alpha}(X) \in R$.

Let $\overset{\tilde{0}}{\nabla} \underset{\alpha}{\tilde{F}} = 0$ for $\alpha = 1,2,3$. Then from the formulas (2.5) and the decomposition (2.4) we get

$$
\begin{aligned}
&\overset{\tilde{0}}{\nabla}_{\underset{1}{X}}(\underset{\alpha}{\psi}\underset{2}{X}) + \partial_{\underset{1}{X}} \underset{\alpha}{A}^{\beta}(\underset{2}{X})\underset{\beta}{\eta} + \underset{\alpha}{A}^{\beta}(\underset{2}{X})(-\underset{\beta}{\overset{0}{L}}\underset{1}{X} + \underset{\beta}{\overset{0}{l}}{}^{\gamma}(\underset{1}{X})\underset{\gamma}{\eta} + \underset{\beta}{\overset{0}{l}}(\underset{1}{X})N) + \\
(2.6) \qquad &+ \underset{\alpha}{\varepsilon}\, \partial_{\underset{1}{X}} \underset{\alpha}{\omega}(\underset{2}{X})N + \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\omega}(\underset{2}{X})(-\overset{0}{K}\, \underset{1}{X} + k^{\alpha}(\underset{1}{X})\underset{\alpha}{\eta} + \overset{0}{k}(\underset{1}{X})N) = \\
&= \underset{\alpha}{\psi}(\overset{\tilde{0}}{\nabla}_{\underset{1}{X}} \underset{2}{X}) + \underset{\alpha}{A}^{\beta}(\overset{\tilde{0}}{\nabla}_{\underset{1}{X}} \underset{2}{X})\underset{\beta}{\eta} + \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\omega}(\overset{\tilde{0}}{\nabla}_{\underset{1}{X}} \underset{2}{X})N .
\end{aligned}
$$

1. If $\underset{1}{X} = X \in TM^{4n-1}$, $\underset{2}{X} = Z \in W^{\perp}_{4n-4}$ then $\underset{\alpha}{A}^{\beta}(Z) = 0$, $\underset{\alpha}{\omega}(Z) = 0$, $\underset{\alpha}{\psi}(\underset{\beta}{\eta}) = 0$, $\underset{\alpha}{\psi}(N) = 0$ and from the formulas (2.5) we obtain

$$
\begin{aligned}
&\overset{0}{\nabla}_X(\underset{\alpha}{\psi}Z) + \overset{0}{h}{}^{\beta}(X,\underset{\alpha}{\psi}Z)\underset{\beta}{\eta} + \overset{0}{h}(X,\underset{\alpha}{\psi}Z)N = \\
&= \underset{\alpha}{\psi}(\overset{0}{\nabla}_X Z) + \underset{\alpha}{A}^{\beta}(\overset{0}{h}{}^{\gamma}(X,Z)\underset{\gamma}{\eta} + \overset{0}{h}(X,Z)N)\underset{\beta}{\eta} + \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\omega}(\overset{0}{h}{}^{\gamma}(X,Z)\underset{\gamma}{\eta} + \overset{0}{h}(X,Z)N)N .
\end{aligned}
$$

Since $\underset{\alpha}{A}^{\beta}(N) = \delta^{\beta}_{\alpha}$, then

$$
(2.7) \qquad \begin{cases}
(\overset{0}{\nabla}_X \underset{\alpha}{\psi})(Z) = 0 \\
\overset{0}{h}{}^{\beta}(X,\underset{\alpha}{\psi}Z) - \overset{0}{h}{}^{\gamma}(X,Z)\underset{\alpha}{A}^{\beta}(\underset{\gamma}{\eta}) - \overset{0}{h}(X,Z)\delta^{\beta}_{\alpha} = 0 \\
\overset{0}{h}(X,\underset{\alpha}{\psi}Z) - \overset{0}{h}{}^{\gamma}(X,Z)\, \underset{\alpha}{\omega}(\underset{\gamma}{\eta}) - \overset{0}{h}(X,Z)\underset{\alpha}{\lambda} = 0
\end{cases}
$$

where $\underset{\alpha}{A}^{\beta}(\underset{\gamma}{\eta})$ is derived from the relation

$$
\underset{\alpha}{\varphi}\, \underset{\gamma}{\eta} = \underset{\alpha}{F}\, \underset{\gamma}{\eta} = \underset{\alpha}{A}^{\beta}(\underset{\gamma}{\eta})\underset{\beta}{\eta}
$$

and from the formulas (1.7) or (1.8).

Now we obtain

Corollary . The following formula holds

$$
(\overset{0}{\nabla}_X \underset{\alpha}{\psi})(Z) = 0
$$

for $X \in TM^{4n-1}$, $Z \in W^{\perp}_{4n-4}$. Thus $\{\underset{\alpha}{\tilde{F}}\}$ – connection on the manifold M^{4n}, $(\overset{\tilde{0}}{\nabla} \underset{\alpha}{\tilde{F}} = 0)$ induces on the integral manifold M^{4n-4} $\{\underset{\alpha}{\psi}\}$ – connection $(\nabla \underset{\alpha}{\psi} = 0)$.

2. If $\underset{1}{X} = X \in TM^{4n-1}$, $\underset{2}{X} = Y \in W_3$, then $\underset{\alpha}{\psi}(Y) = 0$, $\underset{\alpha}{\psi}(N) = 0$ and we have from (2.6)

$$(\overset{0}{\nabla}_X \underset{\alpha}{A^\beta})(Y)\underset{\beta}{\eta} + \underset{\alpha}{A^\beta}(\overset{0}{\nabla}_X Y)\underset{\beta}{\eta} + \underset{\alpha}{A^\beta}(Y)(-\underset{\beta}{\overset{0}{L}}X + \underset{\beta}{\overset{0}{l^\gamma}}(X) + \underset{\beta}{\overset{0}{l}}(X)N) +$$

$$+ \underset{\alpha}{\varepsilon}(\overset{0}{\nabla}_X \underset{\alpha}{\omega})(Y)N + \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(\overset{0}{\nabla}_X Y)N + \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(Y)(-\overset{0}{K}X + \overset{0}{k^\beta}(X)\underset{\beta}{\eta} + \overset{0}{k}(X)N) =$$

$$= \underset{\alpha}{\psi}(\overset{\tilde{0}}{\nabla}_X Y) + \underset{\alpha}{A^\beta}(\overset{\tilde{0}}{\nabla}_X Y)\underset{\beta}{\eta} + \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(\overset{\tilde{0}}{\nabla}_X Y)N .$$

Putting $Y = \underset{\gamma}{\eta}$ we obtain

$$(\overset{0}{\nabla}_X \underset{\alpha}{A^\beta})(\underset{\gamma}{\eta}) = -\underset{\alpha}{A^\delta}(\underset{\gamma}{\eta})\underset{\delta}{\overset{0}{l^\beta}}(X) - \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(\underset{\gamma}{\eta})\overset{0}{k^\beta}(X)$$

$$(\overset{0}{\nabla}_X \underset{\alpha}{\omega})(\underset{\gamma}{\eta}) = -\underset{\alpha}{\varepsilon}\underset{\alpha}{A^\beta}(\underset{\gamma}{\eta})\underset{\beta}{\overset{0}{l}}(X) + \underset{\alpha}{\omega}(\underset{\gamma}{\overset{0}{L}}X) - \underset{\alpha}{\omega}(\underset{\gamma}{\eta})\overset{0}{k}(X)$$

$$\underset{\alpha}{A^\beta}(\underset{\gamma}{\eta})\underset{\beta}{\overset{0}{L}}X + \underset{\alpha}{\varepsilon}\underset{\alpha}{\omega}(\underset{\gamma}{\eta})\overset{0}{K}X - \underset{\alpha}{\psi}(\underset{\gamma}{\overset{0}{L}}X) = 0 .$$

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STRESZCZENIE

W pracy tej badane są 3-struktury na pewnych podrozmaitościach wymiaru 3 i $4n-4$ rozmaitości Riemanna M^{4n} z zadaną na niej 3-strukturą $\{\underset{\alpha}{\tilde{F}}\}$. Obie rozmaitości są podrozmaitościami pewnej hiperpowierzchni zanurzonej w M^{4n} z indukowaną 3-strukturą.

Następnie rozważane są koneksje indukowane na tych podrozmaitościach oraz warunki całkowalności dystrybucji wyznaczających te podrozmaitości.