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Certain Classes of Meromorphic Functions

Pewne klasy funkcji meromorficznych

Abstract. The author is concerned with the class $\Sigma_p^*(\rho)$ of the functions f holomorphic in the punctured disc $0 < |z| < 1$ with the expansion $f(z) = z^{-p} + a_0 z^{-p+1} + \cdots$, starlike of order ρ, $0 \leq \rho < 1$.

Introduction. Let S denote the family of functions $f(z)$ which are regular and univalent in the unit disc E and which satisfy the conditions $f(0) = f'(0) - 1 = 0$. Let $S^*(\alpha)$ and C be the subclasses of S consisting of functions which are starlike of order α and close-to-convex in E, respectively. Let $P(\alpha)$ denote the class of all regular functions $h(z)$ in E which satisfy the conditions $h(0) = 1$, $\operatorname{Re} h(z) > \alpha$ $(0 \leq \alpha < 1)$, in particular $P(0) \equiv P$. Let Σ_p be the class of functions of the form

$$f(z) = z^{-p} + a_0 z^{-p+1} + \cdots + a_{n+p-1} z^n + \cdots \qquad (p = 1, 2, \ldots),$$

which are regular and p-valent in $E \setminus \{0\}$. Denote by $\Sigma_1 \equiv \Sigma$ the class of univalent meromorphic functions in $E \setminus \{0\}$. A function $f(z)$ of Σ_p is said to belong to $\Sigma_p^*(\rho)$, the class of p-valent meromorphic starlike functions of order ρ $(0 \leq \rho < 1)$, if and only if

$$\operatorname{Re}\{zf'(z)/f(z)\} < -p\rho, \qquad z \in E.$$

In particular case, the class Σ^* of univalent meromorphic starlike functions is identified by $\Sigma^* \equiv \Sigma_1^*(0)$. Then the functions $f(z)$ are called meromorphic Bazilević functions of type α if for each $f(z)$ there exisits a function $g(z) \in \Sigma^*$ satisfying

$$\operatorname{Re}\{-zf'(z)f(z)^{\alpha-1}g(z)^{-\alpha}\} > 0, \qquad z \in E,$$

where $\alpha > 0$ is any real number. Denote by $MB(\alpha, g)$ the class of meromorphic Bazilević functions of type α with respect to $g(z)$ [6]. For $\alpha = 1$ the class MC of meromorphic close-to-convex functions is identified by $MB(1, g) \equiv MC$

In [2], [4] and [5] the following theorems are proved :

Theorem A ([5] Theo.3.1, $\beta = 0$, $\gamma + 1 = c$). *Let α and c real constants such that $\alpha > 0$ and $c + 1 - p\alpha > 0$. If $f(z) \in \Sigma_p^*(\rho)$, then*

$$F(z) = \left\{(c + 1 - p\alpha) z^{-c-1} \int_0^z t^c f(t)^\alpha \, dt\right\}^{1/\alpha}$$

also belong to $\Sigma_p^*(\rho)$ for $F(z) \neq 0$ in $E \setminus \{0\}$.

Theorem B ([2], Theo.3). *Let $f(z)$ be close-to-convex with respect to $g(z)$,*

$$F(z) = c z^{-c-1} \int_0^z t^c f(t) \, dt \quad , \quad G(z) = c z^{-c-1} \int_0^z t^c g(t) \, dt \; , \quad c > 0 \; .$$

Then $F(z)$ is close-to-convex with respect to $G(z)$, for $G(z) \neq 0$ in $0 < |z| < 1$.

Theorem C ([2], Theo.4). *Let $F(z)$ belong to Σ^*,*

$$f(z) = \tfrac{1}{c}\left\{(c + 1)F(z) + zF'(z)\right\} \; , \quad c > 0 \; ,$$

then $f(z) \in \Sigma^*$ for $0 < |z| < \sqrt{\frac{c}{c+2}}$. The result is sharp.

Theorem D ([2], Theo.6). *Let $F(z)$ be close-to-convex with respect to $G(z)$, $c > 0$,*

$$f(z) = \tfrac{1}{c}\left\{(c + 1)F(z) + zF'(z)\right\} \quad , \quad g(z) = \tfrac{1}{c}\left\{(c + 1)G(z) + zG'(z)\right\} \; .$$

Then $f(z)$ is close-to-convex with respect to $g(z)$ for $0 < |z| < \dfrac{\sqrt{4 + 2c + c^2} - 2}{2 + c}$.

Theorem E ([4], Theo.1). *If is $f(z)$ in $S^*(\alpha)$ and $g(z)$ in $S^*(\gamma)$,*

$$F(z) = (c + 1) g(z)^{-c} \int_0^z t^{c-1} f(t) \, dt \qquad (c > 0) \; ,$$

then $F(z)$ is β-starlike for $|z| < \sigma$, where σ is the least positive root of the equation

$$1 - \beta - r\left[2(1 - \alpha) + 2c(1 - \gamma)\right] - r^2\left[2\alpha - 1 - \beta + 2c(1 - \gamma)\right] = 0 \; .$$

In this paper are generalized the above results of Goel and Sohi [2] and we obtain a result analogous to the Theorem E of Karunakaran and Ziegler [4] for functions meromorphic in the unit disc.

2. Main results. We require the following results to prove the theorems of this section.

Lemma A [5]. *A function $f(z)$ belongs to $\Sigma_p^*(\rho)$ $(0 \leq \rho < 1)$ if and only if there exists a function $w(z)$ regular and satisfying $w(0) = 0$, $|w(z)| < 1$ in E such that*

$$\frac{zf'(z)}{f(z)} = -p\frac{1 + (2\rho - 1)w(z)}{1 + w(z)} \; .$$

Lemma B ([1], p.25). *If $w(z)$ is regular in E and satisfies the conditions $w(0)=0$, $|w(z)|<1$ for $z \in E$, then*

$$|zw'(z)-w(z)| \le \frac{|z|^2-|w(z)|^2}{1-|z|^2}.$$

We shall now proceed to prove the following :

Theorem 1. *Let α and c be real constants such that $\alpha > 0$ and $c+1-p\alpha > 0$. If $F(z) \in \Sigma_p^*(\rho)$ and*

$$f(z) = \left\{(c+1-p\alpha)^{-1}\left(c+1+\alpha zF'(z)/F(z)\right)\right\}^{1/\alpha} F(z), \tag{1}$$

then $f(z) \in \Sigma_p^*(\rho)$ for $0 < |z| < \sqrt{\dfrac{c+1-p\alpha}{c+1+p\alpha(1-2\rho)}}$.

Proof. Since $F(z) \in \Sigma_p^*(\rho)$, by Lemma A there exists a function $w(z)$ regular in E with $w(0)=0$, $|w(z)|<1$ such that

$$-\frac{zF'(z)}{F(z)} = p\frac{1+(2\rho-1)w(z)}{1+w(z)}. \tag{2}$$

From (1) and (2) we have

$$-\frac{zf'(z)}{f(z)} = p\frac{1+(2\rho-1)w(z)}{1+w(z)} - \frac{b-1}{\alpha}\,\frac{zw'(z)}{(1+w(z))(1+bw(z))}$$

or

$$-\frac{zf'(z)}{f(z)} = \frac{1}{\alpha}\left\{\frac{A}{h(z)} + B + Dh(z) - (b-1)\frac{zw'(z)-w(z)}{(1+w(z))(1+bw(z))}\right\} \tag{3}$$

where $b = \dfrac{c+1+p\alpha(1-2\rho)}{c+1-p\alpha}$, $h(z) = \dfrac{1+bw(z)}{1+w(z)}$, $(b-1)A = b$, $(b-1)B = (b-1)c-2$ and $(b-1)D = 1-2p\alpha(1-\rho)$. Using Lemma B, we get from (3)

$$-\text{Re}\,\frac{zf'(z)}{f(z)} - p\rho \ge \frac{1}{\alpha}\left\{\text{Re}\left(\frac{A}{h(z)} + Dh(z) + B + \frac{r^2|h(z)-b|^2-|1-h(z)|^2}{(b-1)(1-r^2)|h(z)|}\right)\right\} - p\rho. \tag{4}$$

$h(z)$ is subordinate to the linear transformation $\dfrac{1+bz}{1+z}$ and from this it follows by elementary arguments that

$$|h(z)-a| \le d\,,\quad a = \frac{1-br^2}{1-r^2}\,,\quad d = \frac{(b-1)r}{1-r^2}\,,\quad \text{for } r^2 < \tfrac{1}{b}.$$

If we put $h(z) = Re^{i\theta}$ and denote the right hand side of (4) by $S(R,\theta)$, then

$$S(R,\theta) = \frac{1}{\alpha(b-1)}\left\{-T(R)\cos\theta + \frac{R^2+a^2-d^2}{R} + (b+1)\big(p\alpha(1-\rho)-1\big)\right\},$$

$$\cos\theta \geq \frac{R^2+a^2-d^2}{2aR}.$$

$$\frac{\partial S}{\partial\theta} = \frac{1}{\alpha(b-1)}T(R)\sin\theta$$

where $T(R) = 2a + \big(2p\alpha(1-\rho)-1\big)R - \frac{b}{R}$, $a-d \leq R \leq a+d$.

If $T(R) \leq 0$, then clearly $S(R,\theta) > 0$ inside the disc $|h(z)-a| \leq d$. To see this, note that if $T(R) \leq 0$, then

$$0 < R < \frac{\sqrt{a^2+b\big(2p\alpha(1-\rho)-1\big)}-a}{2p\alpha(1-\rho)-1} < \sqrt{\frac{b}{2p\alpha(1-\rho)-1}}.$$

The preceding inequalities in turn imply that

$$-T(R)\cos\theta + \frac{R^2+a^2-d^2}{2aR} \geq \frac{R^2+a^2-d^2}{2aR}\left\{\frac{b}{R} - \big(2p\alpha(1-\rho)-1\big)R\right\} > 0.$$

If $T(R) > 0$, then the minimum of $S(R,\theta)$ inside the disc $|h(z)-a| \leq d$ is attained at $\theta = 0$ and the minimum value is given by

$$S(R,0) = L(R) := \frac{1}{\alpha(b-1)}\left\{\frac{a^2-d^2+b}{R} - 2\big(p\alpha(1-\rho)-1\big)R - 2a + (b+1)\big(p\alpha(1-\rho)-1\big)\right\}.$$

$L(R)$ is a monotonic decreasing function of R and therefore its minimum is attained at $R = a+d$,

$$L(a+d) = \frac{p\alpha(1-\rho)(1-br^2) + \big(p\alpha(1-\rho)-1\big)(b-1)r}{\alpha(1+r)(1+br)}$$

and $L(a+d) > 0$ for $r^2 < \frac{1}{b}$. Thus

$$-\mathrm{Re}\,\frac{zf'(z)}{f(z)} > p\rho, \quad r^2 < \frac{1}{b}.$$

This completes the proof of Theorem 1.

Remark. The result of Theorem C turns out to be a particular case of the above theorem for $\alpha = p = 1$ and $\rho = 0$.

Theorem 2. *Let α and c be real constants, $\alpha > 0$, $c > 0$, $c+1-\alpha > 0$. If $g(z) \in \Sigma^*$ and $f(z) \in MB(\alpha, g)$,*

$$(5)\quad F(z) = \left(\frac{c+1-\alpha}{z^{c+1}}\int_0^z t^c f(t)^\alpha\,dt\right)^{1/\alpha}, \quad G(z) = \left(\frac{c+1-\alpha}{z^{c+1}}\int_0^z t^c g(t)^\alpha\,dt\right)^{1/\alpha}$$

then $F(z) \in MB(\alpha, G)$ *for* $G(z) \neq 0$ *in* $0 < |z| < 1$.

Proof. If we put $p = 1$ and $\rho = 0$ in Theorem A, we can see that $G(z) \in \Sigma^*$. Therefore it is sufficient to show that

$$-\mathrm{Re}\left\{zF'(z)F(z)^{\alpha-1}G(z)^{-\alpha}\right\} > 0 .$$

Let $w(z)$ be regular function defined in E by

$$-\frac{zF'(z)F(z)^{\alpha-1}}{G(z)^{\alpha}} = \frac{1-w(z)}{1+w(z)} \tag{6}$$

Clearly $w(0) = 0$ and $w(z) \neq -1$. From the definition $F(z)$ and $G(z)$ in (5), we have

$$\begin{aligned}(c+1-\alpha)f(z)^{\alpha} &= (c+1)F(z)^{\alpha} + \alpha zF'(z)F(z)^{\alpha-1} = \\ &= (c+1)F(z)^{\alpha} - \alpha\frac{1-w(z)}{1+w(z)}G(z)^{\alpha} .\end{aligned} \tag{7}$$

Differentiating (7) and using (6), we obtain

$$-\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}} = \frac{1-w(z)}{1+w(z)} - \frac{2zw'(z)}{(1+w(z))^2}\,\frac{1}{c+1+\alpha\frac{zG'(z)}{G(z)}}. \tag{8}$$

Now we claim that $|w(z)| < 1$ for otherwise by a lemma of Jack [3] there exists $z_0 \in E$ such that

$$z_0 w'(z_0) = m\,w(z_0)\,, \quad |w(z_0)| = 1 \quad \text{and } m \geq 1\,. \tag{9}$$

Thus from (8) it follows that

$$-\frac{z_0 f'(z_0)f(z_0)^{\alpha-1}}{g(z_0)^{\alpha}} = \frac{1-w(z_0)}{1+w(z_0)} - \frac{2mw(z_0)}{1+w(z_0))^2}\,\frac{1}{c+1+\alpha\frac{z_0G'(z_0)}{G(z_0)}} . \tag{10}$$

Since $G(z) \in \Sigma^*$, $-\dfrac{zG'(z)}{G(z)} \in P$ and hence

$$\left|\frac{zG'(z)}{G(z)} + a\right| \leq d\,, \quad |z| = r\,, \tag{11}$$

where $a = \dfrac{1+r^2}{1-r^2}$, $d = \dfrac{2r}{1-r^2}$. If we put

$$k(z) = \frac{1}{c+1+\alpha\frac{zG'(z)}{G(z)}}$$

then (11) gives

$$\left|k(z) - \frac{c+1-\alpha a}{(c+1-\alpha a)^2 - \alpha^2 d^2}\right| \leq \frac{\alpha d}{(c+1-\alpha a)^2 - \alpha^2 d^2} . \tag{12}$$

This implies that

$$\operatorname{Re} k(z) \geq \frac{1+r}{c+1-\alpha+(c+1+\alpha)r} > 0 .$$

Also, $\operatorname{Re} \dfrac{1-w(z_0)}{1+w(z_0)} = 0$ and $\operatorname{Re} \dfrac{w(z_0)}{(1+w(z_0))^2} = \dfrac{1}{2(1+\operatorname{Re} w(z_0))} > 0$, it follows from (10) and (12)

$$-\operatorname{Re} \frac{z_0 f'(z_0) f(z_0)^{\alpha-1}}{g(z_0)^\alpha} = -\frac{m}{1+\operatorname{Re} w(z_0)} \operatorname{Re} k(z_0) < 0 ,$$

which is a contradiction to our hypothesis that $f(z) \in MB(\alpha, g)$. Hence $|w(z)| < 1$ and the theorem follows from (6).

Remark. For $\alpha = 1$, this theorem reduces to the Theorem .

Theorem 3. *Let $\alpha > 0$, $c > 0$, $c+1-\alpha > 0$. If $G(z) \in \Sigma^*$ and $F(z) \in MB(\alpha, G)$,*

$$(c+1-\alpha) f(z)^\alpha = (c+1) F(z)^\alpha + \alpha z F'(z) F(z)^{\alpha-1} , \tag{13}$$

$$(c+1-\alpha) g(z)^\alpha = (c+1) G(z)^\alpha + \alpha z G'(z) G(z)^{\alpha-1} , \tag{14}$$

then $f(z) \in MB(\alpha, g)$ for $0 < |z| < r(\alpha, c) = \dfrac{\sqrt{c^2+2c+2\alpha+2} - \alpha - 1}{c+1+\alpha}$.

Proof. Since $G(z)$ is starlike, the Theorem 1 with $p = 1$, $\rho = 0$, gives $g(z) \in \Sigma^*$ for $0 < |z| < r_0 = \sqrt{\dfrac{c+1-\alpha}{c+1+\alpha}}$. $F(z)$ is a Bazilević function of type α with respect to $G(z)$, therefore we can write

$$-\frac{z F'(z) F(z)^{\alpha-1}}{G(z)^\alpha} = h(z) , \tag{15}$$

where $h(z) \in P$. Differentiating (15), with (13) and (14) we get, after a simple computation,

$$-\frac{z f'(z) f(z)^{\alpha-1}}{g(z)^\alpha} = h(z) + \frac{z h'(z)}{c+1+\alpha \dfrac{z G'(z)}{G(z)}} .$$

For $G(z) \in \Sigma^*$ we may write $-\dfrac{z G'(z)}{G(z)} = u(z) \in P$. It is well known that for a function of positive real part in E

$$|h'(z)| < \frac{2 \operatorname{Re} h(z)}{1-r^2} .$$

Then,

$$\operatorname{Re} A(z) = \operatorname{Re} \left\{ h(z) + \frac{z h'(z)}{c+1-\alpha u(z)} \right\} \geq \frac{\operatorname{Re} h(z)}{|c+1-\alpha u(z)|} \left\{ |c+1-\alpha u(z)| - \frac{2r}{1-r^2} \right\} ,$$

$|z| = r$.

Since $|u(z) - a| \le d$, $a = \dfrac{1+r^2}{1-r^2}$ and $d = \dfrac{2r}{1-r^2}$, we have further

$$
\begin{aligned}
|c+1-\alpha u(z)| - d &\ge |c+1-\alpha a| - \alpha|u(z)-a| - d \ge \\
&\ge |c+1-\alpha a| - (\alpha+1)d = \\
&= \frac{|c+1-\alpha-(c+1+\alpha)r^2| - 2(\alpha+1)r}{1-r^2} = \\
&= \frac{D(r)}{1-r^2}
\end{aligned}
\tag{16}
$$

where $D(r) = -(c+1+\alpha)r^2 - 2r(\alpha+1) + c + 1 - \alpha$ for $|z| < r_0$. Thus, $r = r(\alpha, c)$ being the positive root of the equation $D(r) = 0$, it is clear that $0 < r(\alpha, c) < r_0$. Therefore, it follows from (16) that $\operatorname{Re} A(z) > 0$ for $0 < |z| < r(\alpha, c)$. Thus, the proof is completed.

Remark. In particular for $\alpha = 1$, we obtain the result of Theorem D.

Theorem 4. *Let α and c be real constants $\alpha > 0$, $c > 0$, $c+1-\alpha > 0$. If $g(z) \in \Sigma^*(\gamma)$ and $f(z) \in \Sigma^*$,*

$$
F(z) = \Big\{(c+1-\alpha)g(z)^{c+1} \int_0^z t^c f(t)^\alpha \, dt\Big\}^{1/\alpha}, \tag{17}
$$

then $F(z) \in \Sigma^*$ for $0 < |z| < \dfrac{\alpha}{\alpha + 2(c+1)(1-\gamma)}$.

Proof. If $k(z) = \left(\dfrac{c+1-\alpha}{z^{c+1}} \displaystyle\int_0^z t^c f(t)\, dt\right)^{1/\alpha}$ then $F(z) = (z\, g(z))^{(c+1)/\alpha} \cdot k(z)$ and Theorem A implies $k(z)$ is in Σ^*. Differentating (17) we obtain

$$
-\frac{zF'(z)}{F(z)} = -\frac{c+1}{\alpha} - \frac{c+1}{\alpha}\frac{zg'(z)}{g(z)} - \frac{zk'(z)}{k(z)}.
$$

For $-\dfrac{zg'(z)}{g(z)} \in P(\gamma)$ and $-\dfrac{zk'(z)}{k(z)} \in P$ it is well known that

$$
-\operatorname{Re}\frac{zg'(z)}{g(z)} \ge \frac{1-(1-2\gamma)r}{1+r}, \quad \operatorname{Re}\frac{zk'(z)}{k(z)} \ge \frac{1-r}{1+r} \quad (|z| \le r)
$$

Therefore

$$
-\operatorname{Re}\frac{zF'(z)}{F(z)} \ge \frac{\alpha - (2(c+1)(1-\gamma)+\alpha)r}{\alpha(1+r)}
$$

and $-\operatorname{Re}\dfrac{zF'(z)}{F(z)} > 0$ for $0 < |z| < \dfrac{\alpha}{\alpha+2(c+1)(1-\gamma)}$.

Remark. Note that, this theorem is analogous to the Theorem E for functions meromorphic in the unit disc. And also note that, the limiting case $\gamma \to 1$ while $p = 1$ gives the result of Theorem A.

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STRESZCZENIE

Autor rozpatruje własności funkcji klasy $\Sigma_p^*(\rho)$ funkcji holomorficznych f w obszarze $0 < |z| < 1$, o rozwinięciu $f(z) = z^{-p} + a_0 z^{-p+1} + \cdots$, gwiaździstych rzędu ρ, $0 \le \rho < 1$.