ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA

VOL. XL, 18 SECTIO A 1986

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An Estimate of the Integral of Quasisymmetric Functions

Oszacowanie całki z funkcji quasisymetrycznych

Оценка интеграла квазисимметрических функций

Introduction. It is known that a quasiconformal mapping F of a Jordan domain G onto a Jordan domain G' can be extended to a homeomorphism of their closures. Hence it induces a homeomorphism of their boundaries C and C' respectively.

In view of the invariance of quasiconformal mappings under composition with conformal mappings the problem of characterizing the induced homeomorphism f can be reduced to the case, when $G = U = \{z : \operatorname{Im} z > 0\} = G'$. Then the boundary correspondence is determined by a monotone continuous function f, in this sense that the point $(x,0)$ corresponds to $(f(x),0)$.

According to Beurling and Ahlfors [1] an automorphism f of the real line can be extended to a $\tilde{K}$-quasiconformal automorphism $\tilde{F}$ of the upper half-plane that fixes the point at infinity if and only if there exists a constant $\rho = \rho(\tilde{K})$ such that

$$(1) \qquad \frac{1}{\rho} \leqslant \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leqslant \rho$$

holds for all $x \in R$ and $t > 0$. The function f satisfying the condition (1) is said to be a quasisymmetric function on R, the term being due to Kelingos [2].

Furthermore, if $K = K[F]$ is the quasiconformal dilatation of F then

(2) $$K \geq 1 + 0.2284 \log \rho(f)$$

for each quasiconformal extension F of f (see [1]), where $\rho(f)$ denotes the infimum of all ρ such that (1) holds. It was shown in [3] that there exists an extension $\tilde{F}$ for which

(3) $$\tilde{K} = K[\tilde{F}] \leq \min\{\rho^{3/2}, 2\rho - 1\} .$$

It should be noted that these bounds are still not the best (cf. [1], [5], [3], [4], [6], [8]). A good bound on $\tilde{K}$ is therefore of great importance for investigating quasiconformal mappings which are quasiconformal extensions of f to the upper half-plane.

A well-known and widely applied result due to Beurling and Ahlfors [1] states that the map $F[f,r]$ defined by

(4) $$2F[f,r](z) = \alpha(z) + \beta(z) + ir(\alpha(z) - \beta(z)) ,$$

where

(5) $$\alpha(z) = \int_0^1 f(x+ty)\,dt , \quad \beta(z) = \int_{-1}^0 f(x-ty)\,dt ,$$

$$z = x+iy , \quad r > 0 ,$$

is a quasiconformal extension of f to the upper half-plane U. In the case when f is ρ-quasisymmetric on an interval I, the quasiconformal extension F defined by (4) and (5) has a domain which is a right isosceles triangle with base I.

The extension technique applied by Beurling and Ahlfors assures that the function $F[f,r]$ is continuously differentiable everywhere on its domain, which follows by the property of ρ-quasisymmetric function and ([5], p. 84).

Since linear transformation do not effect the property of being ρ-quasisymmetric or the dilatation of a quasiconformal mapping, we can make certain simplifying assumptions when estimating the dilatation quotient of $F[f,r]$ at an arbitrary z. First, we may suppose that f is normalized, i.e. satisfies $f(0) = 0$ and $f(1) = 1$, and secondly, we may restrict ourselves to the point $z = i$.

The dilatation quotient K of $F[f,r]$ at i satisfies

$$(6) \qquad 2r(\varsigma+\eta)(K+K^{-1}) = (1+r^2)(\varsigma(1+\xi^2)+\varsigma^{-1}(1+\eta^2)) + 2(1-\varsigma\eta)(1-r^2) ,$$

where $\varsigma = \alpha_x/\beta_x$, $\xi = \alpha_y/\alpha_x$, $\eta = \beta_y/\beta_x$. Since h is normalized, one easily gets $\alpha_x(i) = 1$, $\beta_x(i) = -f(-1)$, $\alpha_y(i) = 1 - \int_0^1 f(t)\,dt$ and $\beta_y(i) = f(-1) - \int_{-1}^0 f(t)\,dt$.

The ρ-quasisymmetry of f immediately yields $\rho^{-1} \leqslant \varsigma \leqslant \rho$ By a lemma of Beurling and Ahlfors [1, p. 137],

$$(7)\qquad \mu \leqslant \int_0^1 f(t)\,dt \leqslant \lambda ,$$

where $\mu = (1+\rho)^{-1}$, $\lambda = \rho\mu$. It follows that ξ and η both lie in the interval $\langle \mu, \lambda \rangle$.

The bounds in (7), which play the most important role in the main problem of getting the best estimation of the dilatation quotient K of $F[f,r]$, are not the best possible. Equality on, say, the right hand side of (7) holds for the non-quasisymmetric majorant P for normalized ρ-quasisymmetric function introduced by R. Salem [7]. Let $P(0) = 0$, $P(1) = 1$ and

$$(8)\qquad P\left(\frac{2k+1}{2^n}\right) = (1-\lambda)P\left(\frac{k}{2^{n-1}}\right) + \lambda P\left(\frac{k+1}{2^{n-1}}\right) ,$$

$k = 0,1,\dots,2^{n-1}-1$; $n = 1,2,\dots$, and extend the definition of P to the numbers $x \in (0,1)$ with non-terminating binary representation by continuity. If

$N_\rho = \{ f : f(0) = 0 ,\ f(1) = 1 ,$ and f is ρ-qs on $R \}$, then

$$(9)\qquad f(x) \leqslant P(x)$$

for all $x \in \langle 0,1 \rangle$ with a finite binary representation, and by continuity on the whole interval. By (8) we see that

$$(10)\qquad \int_0^1 P(x)\,dx = \lambda ,$$

$$(11)\qquad \int_{k/2^n}^{(k+1)/2^n} P(x)\,dx = \frac{1}{2^n}\left[(1-\lambda)P\left(\frac{k}{2^n}\right) + \lambda P\left(\frac{k+1}{2^n}\right)\right] =$$

$$= \frac{1}{2^n} P\left(\frac{2k+1}{2^{n+1}}\right)$$

where $k = 0,1,\ldots,2^n-1$, $n = 1,2,\ldots$.

Making use of very elementary technique we obtain a better estimation of (7), which in the first step convers the result obtained by Lehtinen [4], whose method is founded on very interesting observation that the singular function P has locally convexity points.

Main result. Suppose now that $f \in N_\rho$ then for every $0 \leqslant x < y \leqslant 1$ and $\lambda = \rho(1+\rho)^{-1}$, we have

$$(12)\qquad \lambda f(x) + (1-\lambda)f(y) \leqslant f\left(\frac{x+y}{2}\right) \leqslant (1-\lambda)f(x) + \lambda f(y) .$$

By this

$$(13)\qquad f\left(\tfrac{1}{2}-h\right) + f\left(\tfrac{1}{2}+h\right) \leqslant 1 + \frac{2\lambda-1}{\lambda} f\left(\tfrac{1}{2}+h\right)$$

for $h \geqslant 0$ and $0 < \frac{1}{2} - h \leqslant \frac{1}{2} \leqslant \frac{1}{2} + h \leqslant 1$.

An integration of f over $\langle \frac{3}{8}, \frac{5}{8} \rangle$ yields

$$(14)\qquad \int_{\frac{1}{2}-\frac{1}{8}}^{\frac{1}{2}+\frac{1}{8}} f(t)\,dt \leqslant \int_0^{\frac{1}{8}} \left[1 + \frac{2\lambda-1}{\lambda} f\left(\tfrac{1}{2}+t\right)\right] dt \leqslant$$

$$\leqslant \frac{1}{8} + \frac{2\lambda-1}{\lambda}\int_0^{\frac{1}{8}} F(\tfrac{1}{2}+t)\,dt = \frac{1}{8} + \frac{2\lambda-1}{\lambda}\,\frac{1}{8}\quad(1+\lambda^2-\lambda^3)$$

by which

$$(15)\qquad \int_{\frac{3}{8}}^{\frac{5}{8}} F(t)\,dt - \int_{\frac{3}{8}}^{\frac{5}{8}} f(t)\,dt \geqslant \frac{1}{8}\lambda(\lambda-1)^2(2\lambda-1) = A\ .$$

Let now $\alpha_0 = \frac{3}{4}$, $\alpha_1 = \frac{3}{8}, \ldots, \alpha_n = \frac{3}{4\cdot 2^n}, \ldots$,

and let

$$S_n = \int_{\alpha_{n+1}}^{\alpha_n} f(t)\,dt\ , \quad n = 0,1,\ldots ,$$

then we have

$$S_{n+1} = \int_{\alpha_{n+2}}^{\alpha_{n+1}} f(t)\,dt \leqslant \int_{\alpha_{n+2}}^{\alpha_{n+1}} \left[(1-\lambda)f(0) + \lambda f(2t)\right] dt =$$

$$= \lambda\int_{\alpha_{n+2}}^{\alpha_{n+1}} f(2t)\,dt = \frac{\lambda}{2}\int_{\alpha_{n+1}}^{\alpha_n} f(x)\,dx = \frac{\lambda}{2} S_n\ .$$

From this it follows that

$$\int_0^{\frac{5}{8}} f(t)\,dt = \int_{\frac{3}{8}}^{\frac{5}{8}} f(t)\,dt + S_1 + \sum_{n=2}^{\infty} S_n \leqslant$$

$$\leqslant \int_{\frac{3}{8}}^{\frac{5}{8}} f(t)\,dt + \frac{\lambda}{2}\sum_{n=1}^{\infty} S_n + \frac{\lambda}{2} S_o =$$

$$= \int_{\frac{3}{8}}^{\frac{5}{8}} f(t)\,dt + \frac{\lambda}{2}\int_{0}^{\frac{3}{8}} f(t)\,dt + \frac{\lambda}{2}\left[\int_{\frac{3}{8}}^{\frac{5}{8}} f(t)\,dt + \right.$$

$$\left. + \int_{\frac{3}{8}}^{\frac{3}{4}} f(t)\,dt \right],$$

then

(17) $$\int_{0}^{\frac{5}{8}} f(t)\,dt \leqslant \frac{2}{2-\lambda}\int_{\frac{3}{8}}^{\frac{5}{8}} f(t)\,dt + \frac{\lambda}{2-\lambda}\int_{\frac{5}{8}}^{\frac{3}{4}} f(t)\,dt .$$

On the other hand let $\beta_o = \frac{1}{4}$, $\beta_1 = \frac{5}{8}$, ... , $\beta_{n+1} = \frac{\beta_n + 1}{2}$ and let

$$R_{n+1} = \int_{\beta_{n+1}}^{\beta_{n+2}} f(t)dt \leqslant \int_{\beta_{n+1}}^{\beta_{n+2}} \left[(1-\lambda)f(2t-1) + \lambda f(1)\right]dt =$$

$$= (1-\lambda)\int_{\beta_{n+1}}^{\beta_{n+2}} f(2t-1)dt + \lambda(\beta_{n+2} - \beta_{n+1}) =$$

$$= \frac{1-\lambda}{2}\int_{\beta_n}^{\beta_{n+1}} f(x)dx + \lambda(\beta_{n+2} - \beta_{n+1}) =$$

$$= \frac{1-\lambda}{2} R_n + \lambda(\beta_{n+2} - \beta_{n+1}) \quad , \text{ for } n=0,1,2,\ldots \quad .$$

By this

$$(18) \quad \int_{\frac{5}{8}}^{1} f(t)dt = \sum_{n=1}^{\infty} R_n \leqslant \frac{1-\lambda}{2} \sum_{n=0}^{\infty} R_n + \lambda \sum_{n=1}^{\infty} (\beta_{n+1} - \beta_n) =$$

$$= \frac{1-\lambda}{2} \int_{\frac{1}{4}}^{1} f(t)dt + \frac{3}{8}\lambda \leqslant \frac{1-\lambda}{2} \int_{\frac{5}{8}}^{1} f(t)dt + \frac{1-\lambda}{2} \int_{\frac{1}{4}}^{\frac{5}{8}} f(t)dt +$$

$$+ \frac{3}{8}\lambda \ .$$

Hence

$$(19) \quad \int_{\frac{5}{8}}^{1} f(t)dt \leqslant \frac{1-\lambda}{1+\lambda} \int_{\frac{1}{4}}^{\frac{5}{8}} f(t)dt + \frac{6}{8} \frac{\lambda}{1+\lambda} \leqslant$$

$$\leqslant \frac{1-\lambda}{1+\lambda} \int_{\frac{1}{4}}^{\frac{3}{8}} f(t)dt + \frac{1-\lambda}{1+\lambda} \int_{\frac{3}{8}}^{\frac{5}{8}} f(t)dt + \frac{3}{4} \frac{\lambda}{1+\lambda} \ .$$

Finally, the application of the above inequalities gives rise to the next estimation

$$(20) \quad \int_0^1 f(t)dt = \int_0^{\frac{5}{8}} f(t)dt + \int_{\frac{5}{8}}^{1} f(t)dt \leqslant$$

$$\leqslant \frac{2}{2-\lambda}\int_{\frac{3}{8}}^{\frac{5}{8}} f(t)dt + \frac{\lambda}{2-\lambda}\int_{\frac{5}{8}}^{\frac{3}{4}} P(t)dt + \frac{1-\lambda}{1+\lambda}\int_{\frac{1}{4}}^{\frac{3}{8}} P(t)dt +$$

$$+ \frac{1-\lambda}{1+\lambda}\int_{\frac{3}{8}}^{\frac{5}{8}} f(t)dt + \frac{3}{4}\,\frac{\lambda}{1+\lambda} \leqslant$$

$$\leqslant \frac{2}{2-\lambda}\left\{\int_{\frac{3}{8}}^{\frac{5}{8}} P(t)dt - A\right\} + \frac{\lambda}{2-\lambda}\int_{\frac{5}{8}}^{\frac{3}{4}} P(t)dt + \frac{1-\lambda}{1+\lambda}\int_{\frac{1}{4}}^{\frac{3}{8}} P(t)dt +$$

$$+ \frac{1-\lambda}{1+\lambda}\left\{\int_{\frac{3}{8}}^{\frac{5}{8}} P(t)dt - A\right\} + \frac{3}{4}\,\frac{\lambda}{1+\lambda} =$$

$$= \frac{1-\lambda}{1+\lambda}\int_{\frac{1}{4}}^{\frac{3}{8}} P(t)dt + \frac{4-\lambda+\lambda^2}{(2-\lambda)(1+\lambda)}\int_{\frac{3}{8}}^{\frac{5}{8}} P(t)dt + \frac{\lambda}{2-\lambda}\int_{\frac{5}{8}}^{\frac{3}{4}} P(t)dt +$$

$$+ \frac{3}{4}\,\frac{\lambda}{1+\lambda} - \frac{4-\lambda+\lambda^2}{(2-\lambda)(1+\lambda)}\,A \quad .$$

By (11)

$$\int_{\frac{1}{4}}^{\frac{3}{8}} P(t)dt = \frac{\lambda^2}{8}\,(1+\lambda-\lambda^2)\ , \qquad \int_{\frac{3}{8}}^{\frac{5}{8}} P(t)dt = \frac{\lambda}{8}\,(1+3\lambda-2\lambda^2)$$

and

$$\int_{\frac{5}{8}}^{\frac{3}{4}} P(t)dt = \frac{\lambda}{8}\,(1+2\lambda-3\lambda^2+\lambda^3) \quad .$$

Then, by (20)

$$\int_0^1 f(t)dt \leqslant \frac{\lambda}{(2-\lambda)(1+\lambda)}(2+\lambda-\lambda^2) - \frac{4-\lambda+\lambda^2}{(2-\lambda)(1-\lambda)} A =$$

$$= \frac{\varrho}{1+\varrho}\left[1 - \frac{1}{8}\frac{\varrho-1}{(\varrho+1)^3}\left(2 - \frac{3}{2(\varrho+\varrho^{-1})+5}\right)\right] = D(\varrho) .$$

Denoting by $B(\varrho) = \frac{1}{8}\left(2 - \frac{3}{2(\varrho+\varrho^{-1})+5}\right)$ we see that

$\frac{5}{24} \leqslant B(\varrho) \leqslant \frac{6}{24}$ by which

$$(21) \qquad \int_0^1 f(t)dt \leqslant \lambda\left[1 - \frac{5}{24}\frac{\varrho-1}{(\varrho+1)^3}\right] .$$

A replacement of $f(x)$ by $1 - f(1-x)$ produces the left-hand side estimation of this integral. Let a, b, $0 \leqslant a < b \leqslant 1$ be real numbers. Putting $L(t) = (b-a)t + a$ we see that the function

$(f \circ L - f(a))(f(b) - f(a))^{-1} \in N_\varrho$

thus

$$\int_0^1 \frac{f \circ L(t) - f(a)}{f(b) - f(a)} dt \leqslant D(\varrho)$$

and consequently

$$\frac{1}{b-a}\int_a^b f(t)dt = \int_0^1 f \circ L(t)dt \leqslant (f(b) - f(a))D(\varrho) + f(a) .$$

This inequality leads to

$$(22) \qquad \int_a^b f(t)dt \leqslant (b-a)(D(\varrho)P(b) + (1 - D(\varrho))P(a)) .$$

Making use of (11) and (22) we obtain for $a = \frac{1}{4}$, $b = \frac{3}{8}$ what follows

$$\int_{\frac{1}{4}}^{\frac{3}{8}} P(t)dt - \int_{\frac{1}{4}}^{\frac{3}{8}} f(t)dt \geqslant \frac{1}{8}(\lambda - D(\varrho))(P(\tfrac{3}{8}) - P(\tfrac{1}{4})) =$$

$$= \frac{\lambda}{8} B(\varrho) \frac{\varrho - 1}{(\varrho+1)^3} (P(\tfrac{3}{8}) - P(\tfrac{1}{4}))$$

and similarly

$$\int_{\frac{5}{8}}^{\frac{3}{4}} P(t)dt - \int_{\frac{5}{8}}^{\frac{3}{4}} f(t)dt \geqslant \frac{\lambda}{8} B(\varrho) \frac{\varrho - 1}{(\varrho+1)^3} (P(\tfrac{3}{4}) - P(\tfrac{5}{8})) \quad .$$

Substituting these inequalities to (19) and (17) respectively and by (20) we obtain an improvement of (21) in the form

$$\int_0^1 f(t)dt \leqslant \lambda \left[1 - B(\varrho) \frac{\varrho - 1}{(\varrho+1)^3} (1 + C(\varrho))\right] \quad ,$$

where $C(\varrho) = \frac{3}{8} \frac{\varrho^2}{(\varrho+1)^2(\varrho+2)(2\varrho+1)}$. We summarize this as

<u>Theorem. If</u> f <u>is a</u> ϱ-quasisymmetric function of the class N_ϱ <u>then</u>

$$\int_0^1 f(t)dt \leqslant \frac{\varrho}{\varrho+1}\left[1 - \frac{1}{8}\left(2 - \frac{3}{2(\varrho + \varrho^{-1} + 5)}\right) \cdot \frac{\varrho - 1}{(\varrho+1)^3}\left(1 + \frac{3}{8}\frac{\varrho^2}{(\varrho+1)^2(\varrho+2)(2\varrho+1)}\right)\right] .$$

This estimation enables us to get a better estimation of the dilatation quotient K of K[f,r] which will be published later.

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STRESZCZENIE

Niech $H_0(\varrho)$ oznacza klasę funkcji ϱ-quasisymetrycznych unormowanych, tzn. $f(0) = 0$, $f(1) = 1$ dla każdej funkcji $f \in H_0(\varrho)$. Funkcjonałem odgrywającym podstawową rolę w oszacowaniu rzędu quasikonforemnego rozszerzenia Beurlinga-Ahlforsa funkcji ϱ-quasisymetrycznej jest całka $\int_0^1 f(t)\,dt$ gdy $f \in H_0(\varrho)$. Ostatnie znane jego oszacowanie podał M. Lehtinen w pracy [3]. W przedstawionej przez nas pracy uzyskujemy wzmocnienie tego oszacowania.

РЕЗЮМЕ

Через $H(\varrho)$ мы обозначаем множества всех возрастающих гомеоморфизмов h прямой $\mathbb{R}$ таких, что $\varrho^{-1} \le \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le \varrho$ для всех $x \in \mathbb{R}$, $t > 0$ а через $H_0(\varrho)$ его подмножество состоящееся из функций h нормированных условиями $h(0) = 0$, $h(1) = 1$. Чтобы получить оценку на K для K-квазиконформного расширения Берлинга-Альфорса гомеоморфизма $h \in H(\varrho)$ нужно эценивать интеграл $\int_0^1 \theta(t)\,dt$ для $\theta \in H_0(\varrho)$. Последни известную оценку такого интеграла получил М. Лехтинен в работе [3]. В данной работе мы получили более точную оценку этого интеграла.