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A Univalence Criterion and the Structure of Some Subclasses of Univalent Functions

Pewne kryterium jednolistności i związane z nim podklasy funkcji jednolistnych

Один признак однолистности и его связь с некоторыми классами однолистных функций

Let H denote the class of holomorphic functions in E, where $E_r = \{z : |z| < r\}$, $E_1 = E$, $\bar{E}$ - the closure of E and $S_0 \subset H$ be the subclass of univalent functions in E. Denote by $\Omega_0 \subset H$ the class of functions ω such that $|\omega(z)| < 1$ for $z \in E$ and by $\Omega \subset \Omega_0$ the class of functions ω satisfying the assumptions of Schwarz's lemma.

Next, for arbitrary fixed numbers A and B, $|A| \leqslant 1$, $|B| \leqslant 1$ we denote by $P(A,B)$ the family of functions

(1.1) $$p(z) = b_0 + b_1 z + \dots$$

holomorphic in E and such that

(1.2) $$p(z) = \frac{1 + A\omega(z)}{1 - B\omega(z)}, \quad \omega \in \Omega_0, \quad z \in E .$$

If additionaly we suppose that

(1.3) $$1 - |A + B| + \operatorname{Re}\{\bar{A} \cdot B\} > 0 ,$$

then the class $P(A,B)$ of functions p is a subclass of the class P of functions with positive real part, i.e. of functions $p(z)$ for which $\operatorname{Re} p(z) > 0$ in E. The class of those functions we denote by $\tilde{P}(A,B) \subset P$.

For $b_0 = 1$, $\tilde{P}(A,B)$ is a subclass of Caratheodory's functions. In the case $-1 \leq A \leq 1$ and $-A < B \leq 1$ the class $P(A,B)$ was introduced by W. Janowski [8].

If A and B are complex numbers and $\omega \in \Omega$, then this class $P(A,B)$ was studied, among others, by Z.J. Jakubowski [4]-[7], J. Stankiewicz and J. Waniurski [13] and MacGregor [10].

Following Ch. Pommerenke [11] and J. Becker [2] we introduce a normalized chain $f(z,t) = e^t z + \dots$ of subordinate functions over the interval $I = \langle 0, \infty)$, analytic in E and such that for almost every $t \in I$ the Lowner-Kufariew equation [2]

$$\frac{\partial f(z,t)}{\partial t} = zp(z,t)\frac{\partial f(z,t)}{\partial z}, \quad z \in E \tag{1.4}$$

is satisfied, where

$$p(z,t) = \frac{1 + A\omega(z,t)}{1 - B\omega(z,t)} \in \tilde{P}(A,B), \quad \omega \in \Omega, \quad z \in E, \quad t \in I. \tag{1.5}$$

It is well-known that the function $f(z,t)$ satisfying the condition (1.4) determine a family of univalent functions in E for each $t \in I$. The class of these functions we denote by $\tilde{S}_0(A,B)$.

L.V. Ahlfors [1] and J. Becker [2] gave the following sufficient univalence condition:

<u>Theorem A-B. Let</u> $f \in H$ <u>and</u> $f'(z) \neq 0$. <u>If there exists a constant</u> $c \in \bar{E} \setminus \{1\}$ <u>such that</u>

(1.6)
$$\left|(1-|z|^2)\frac{zf''(z)}{f'(z)} - c|z|^2\right| \leqslant 1$$

holds for $z \in E$, then $f \in S_0$.

In the case $c = 0$ this theorem was given earlier by Duren, Sharpio and Shilds [3]. A generalization of the theorem A-B was obtained by Z. Lewandowski [9].

Theorem L. Let $f \in H$, $f'(z) \neq 0$. If there exists a function $\omega \in \Omega$ such that

(1.7)
$$\left|\omega(z)|z|^2 - (1-|z|^2)\left(\frac{z\omega'(z)}{1-\omega(z)} + \frac{zf''(z)}{f'(z)}\right)\right| \leqslant 1$$

holds for $z \in E$, then $f \in S_0$.

The purpose of this note is to characterize the structure of functions which satisfy the further given sufficient univalence condition and that have a K - quasiconformal (K-q.c.) extension for some subclass of the class $\tilde{S}_0(A,B)$.

2. We now state the following

Theorem 1. Let $f \in H$, $f'(z) \neq 0$, $z \in E$. If there exists a function $\omega \in \Omega_0$, $\omega \neq \frac{1}{B}$, $|B| \leqslant 1$ such that

(2.1)
$$\left|\frac{(A+B)\omega(z)}{2+(A-B)\omega(z)}|z|^2 - (1-|z|^2)\left[\frac{(A+B)z\omega'(z)}{(1-B\omega(z))(2+(A-B)\omega(z))} + \frac{zf''(z)}{f'(z)}\right] - \frac{(\bar{A}-\bar{B})(A+B)}{4-|A-B|^2}\right| \leqslant \frac{2|A+B|}{4-|A-B|^2}, \quad z \in E,$$

where

$$(2.2)\qquad 1-|A+B|+\operatorname{Re}\{\bar{A}\cdot B\}>0\ ,\quad |A|\leq 1\ ,\quad |B|\leq 1\ ,$$

then f is a univalent function in E, imbedded in the class $\tilde{S}_0(A,B)$ of univalent chains $f(z,t)$ over the interval I, $f(z,0)=f(z)$.

Proof. Let $\tau(z,t)$ be an analytic function in E for each $t\in I$. Moreover, let $\tau(0,t)=1$, $\tau(z,t)\neq 0$, $\operatorname{Re}\{\tau(z,t)\}\geq 0$, $z\in E$, $t\in I$. Let

$$(2.3)\qquad f(z,t)=f(ze^{-t})+(e^{t}-e^{-t})zf'(ze^{-t})\tau(ze^{-t})\ ,\quad t\in I\ .$$

For each $t\in I$ $f(z,t)$ is an analytic function for $|z|<e^{t}$, $f'(z,0)\neq 0$.

It is enough to prove that $f(z,t)$ ($|z|<e^{t}$, $t\in I$) satisfies the Lowner equation (1.4) for almost every $t\in I$ if chain (2.3) satisfies inequality (2.1). Hence $f(z,t)$ is a chain of subordinate functions over interval I.

From relations (1.5) and (1.4) we have

$$(2.4)\qquad \omega(z,t)=\frac{f^{\bullet}-zf'}{Azf'+Bf^{\bullet}}\ ,$$

where $f^{\bullet}=\frac{\partial f}{\partial t}$, $f'=\frac{\partial f}{\partial z}$.

Hence by (2.3) we obtain

$$(2.5)\qquad \omega(z,t)=\frac{-2\omega(z,t)}{A+B+(A-B)\omega(z,t)}\ ,$$

where

$$(2.6)\qquad \omega(z,t)=\Big(\frac{1}{\tau(ze^{-t})}-1\Big)e^{-2t}+(1-e^{-2t})\Big(\frac{ze^{-t}\tau'(ze^{-t})}{\tau(ze^{-t})}+\frac{ze^{-t}f''(ze^{-t})}{f'(ze^{-t})}\Big)$$

From (2.5) and since $\omega \in \Omega_0$, putting $ze^{-t} = \zeta$ for $|z| = 1$, we have

$$(2.7) \quad \left| \left(\frac{1}{\tau(\zeta)} - 1\right)|\zeta|^2 + (1-|\zeta|^2)\left(\frac{\zeta\tau'(\zeta)}{\tau(\zeta)} + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right) - \frac{(\overline{A}-\overline{B})(A+B)}{4-|A-B|^2} \right| \leqslant \frac{2|A+B|}{4-|A-B|^2} .$$

Now we put in (2.7) $\frac{1}{\tau} - 1 = -\frac{(A+B)\omega}{2+(A-B)\omega}$, $\omega \in \Omega_0$. Hence there exists a function $\omega \in \Omega_0$ such that the chain (2.3) satisfies the inequality (2.1).

In virtue of theorem 4 [11], $f(z,t)$ is a Löwner chain i.e. the chain of subordinate functions over interval I .

Moreover, for each $t \in I$ $f(z,t)$ is a univalent function in E . In particular $f(z,0) = f(z)$ is a univalent function in E imbedded in the class $\widetilde{S}_0(A,B)$.

The family of the functions f satisfying (2.1) we denote by $S_0(A,B) \subset \widetilde{S}_0(A,B)$. Let us note that by suitable choice of A and B (2.1) gives a continuous transition from the Becker's univalence condition 2 to the Lewandowski's condition 9.

3. Let $f \in H$ satisfy in E the equation

$$(3.1) \quad \frac{(A+B)z\omega'(z)}{(1-B\omega(z))(2+(A-B)\omega(z))} + \frac{zf''(z)}{f'(z)} = \frac{(A+B)\cdot\varphi(z)}{2-(A-B)\varphi(z)} ,$$

$|A| \leqslant 1$, $|B| \leqslant 1$ and $1-|A+B| + \operatorname{Re}\{\overline{A}\cdot B\} > 0$ for any fixed function $\omega \in \Omega_0$ and for a function φ satisfying the assumptions of Schwarz's lemma. The class of these functions f we denote by $\hat{S}_0(A,B)$.

It is easy to see that the function f given by (3.1) satisfies the inequality (2.1), hence $\hat{S}_0(A,B) \subset S_0(A,B)$.

From (3.1) we obtain at once

$$(3.2)\qquad f'(z) = \frac{1-B\omega(z)}{2+(A-B)\omega(z)}\cdot\frac{2+(A-B)\omega(0)}{1-B\omega(0)}\cdot\frac{g(z)}{z}$$

where

$$(3.3)\qquad g(z) = z\exp\int_0^z \frac{(A+B)\varphi(t)}{t\,[2-(A-B)\varphi(t)]}\,dt\ ,\qquad z\in E\ .$$

By (3.3) we have

$$(3.4)\qquad \left|\frac{zg'(z)}{g(z)}-1\right| \leqslant \frac{|A+B|}{2-|A-B|}\ ,\qquad z\in E\ ,$$

and so g belongs to the class $S^*(A,B)\subset S^*$ - the class of functions starlike with respect to the origin.

The relation (3.2) can be rewritten in the form

$$(3.5)\qquad \frac{zf'(z)}{g(z)} = \frac{1-B\omega(z)}{2+(A-B)\omega(z)}\cdot\frac{2+(A+B)\omega(0)}{1-B\omega(0)}\ .$$

In particular, the class $\hat{S}_0(A,B)$ contains known subclasses of the class of univalent functions.

1°. Let $\varphi(z) = -\omega(z)$ and $f(0) = 0$, then $f = g$, where g is given by formula (3.3).

2°. If we put $\omega(0) = 0$ into (3.5), then we obtain some subclass of the class of close-to-convex functions contained in $\hat{S}_0(A,B)\subset S_0(A,B)$.

3°. Putting into (3.5) $\omega(z)\equiv c\in \overline{E}\setminus\{1\}$, where c is a constant, we obtain $zf' = g$, hence f is a convex function.

4°. If we put $\varphi(z) = 0$, $\omega(0) = 0$ into (3.5), then f is a univalent function of bounded rotation (i.e. $\operatorname{Re} f'(z) > 0$, $z\in E$).

Moreover, we remark that by a suitable choice of A and B satisfying the condition (2.2) we can obtain different subclasses of univalent functions contained in $S_0(A,B)$, characterized by the functions p $P(A,B) \subset P$ (see W. Jankowski [8]).

4. Let C denote the complex plane $\overline{C} = C \cup \{\infty\}$ and let $S \subset S_0$ denote the class of functions f such that $f(0) = 0$, $f'(0) = 1$. By S^* we denote the class of mappings $F : C \to \overline{C}$ -q.c. such that $F|_E = f \in S$. (The symbol $F|_E$ denotes the restriction of the function F to the set E).

It is well-known (see [12] p.149) that the class S^K is a compact family with respect to the topology of uniform convergence on compact sets.

Let $S^K(A,B) \subset S_0(A,B)$, ($A,B$ - satisfy the relation (2.2)) denote the class of functions satisfying the condition

$$(4.1)\quad \left| \frac{(A+B)\omega(z)}{2+(A-B)\omega(z)}|z|^2 - (1-|z|^2)\left[\frac{(A+B)z\omega'(z)}{(1-B\omega(z))[2+(A-B)\omega(z)]} + \frac{zf''(z)}{f'(z)}\right] - \frac{(\bar{A}-\bar{B})(A+B)\varrho^2}{4-|A-B|^2\varrho^2}\right| \leqslant \frac{2|A+B|\varrho}{4-|A-B|^2\varrho^2}, \quad z \in \overline{E},$$

where $\omega \in \Omega$, $|\omega(z)| \leqslant \varrho < 1$.

Ahlfors (see [12] p. 169) gave for the subclass of the class $S^K(A,B)$ generated by inequality (4.1) with the function $\omega(z) \equiv c \in \overline{E}\setminus\{1\}$, c=const., $A=1$, $B=1$, the K-q.c. extension $F \in S^K(A,B)$, such that $F(\infty) = \infty$.

We now prove

Theorem 2. Let $f \in S^K(A,B)$. Then the function F given by the formula

$$(4.2)\quad F(z) = \begin{cases} f(z) & \text{for } |z| < 1, \\ f\left(\frac{1}{\bar{z}}\right) + \frac{|z|^2-1}{\bar{z}} \cdot \frac{2+(A-B)\omega\left(\frac{1}{\bar{z}}\right)}{2\left(1-B\omega\left(\frac{1}{\bar{z}}\right)\right)} \cdot f'\left(\frac{1}{\bar{z}}\right) & \text{for } |z| \geqslant 1 \end{cases}$$

belongs to $S^K(A,B)$, $F(\infty) = \infty$ and F is a q.c. extension of order $K = \frac{2+\varrho(|A+B|-|A-B|)}{2-\varrho(|A+B|+|A-B|)}$ of the function f.

For the proof it is enough to see that there exists a function $\tilde{\varphi} \in \Omega$, $|\tilde{\varphi}(z)| \leq \varrho < 1$, $z \in E$, such that the function $G(z) = \frac{1}{r}f(rz)$, $z \in \bar{E}$, $0 < r < 1$, satisfies the condition (4.1).

For the function $F(z)$ given by formula (4.2) the modulus of the complex dilatation $\kappa = F_{\bar{z}}/F_z$ is not greater than $\varrho|A+B|/(2-\varrho|A-B|)$. Hence following Ahlfors [1], we obtain the assertion of our theorem.

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STRESZCZENIE

Jeśli f jest funkcją lokalnie jednolistną w kole jednostkowym E, taką, że $f(o) = f'(o) - 1 = 0$ i ω jest funkcją holomorficzną w E taką, że $|\omega(z)| \le |z|$ w E, to warunek (2.1) ze stałymi A, B spełniającymi warunki (2.2) zapewnia jednolistność f.

РЕЗЮМЕ

Пусть f локально однолистна в единичном круге E функция такая, что $f(o)= f'(o)-1 = o$ и ω голоморфна в E исполняет $|\omega(z)| \le |z|$ в E. Тогда исполняющая условия (2.1) ,(2.2) является однолистной в E.