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Quasisymmetric Functions and Quasihomographies of the Unit Circle

Funkcje quasisymetryczne i quasihomografie okręgu jednostkowego

Abstract. The relationship between traditional quasisymmetric functions of the unit circle and a new representation for the boundary values of arbitrary quasiconformal automorphisms of the unit disc, called quasihomographies, is studied in this paper.

Introduction. A characterization of the boundary values of a K-quasiconformal (K-qc) automorphism F of the unit disc $\Delta = \{z : |z| < 1\}$, with fixed point zero, was given by J. G. Krzyż (see [K1] and [K2]). Using the conformal configuration connected with harmonic measure, he has obtained a class of ρ-quasisymmetric (ρ-qs) functions of $T = \partial\Delta$, representing boundary automorphisms $f = F|_T$, such that

$$\text{(K)} \qquad \frac{1}{\rho} \le \frac{|f(\eta_1)|}{|f(\eta_2)|} \le \rho$$

holds for each pair of disjoint adjacent open subarcs η_1, η_2 of T, with equal length and $\rho = \lambda(K)$ (cf. [LV]).

The class of all sense-preserving automorphisms of T satisfying ρ-condition with a constant ρ, $\rho \ge 1$, is said to be the class of ρ-qs functions of T and is denoted by $Q_T(\rho)$. It is invariant under composition only with the group of rotations of T. Then, by $Q_T^\circ(\rho)$, we denote the subclass of $Q_T(\rho)$ consisting of all f normalized $f(1) = 1$. This characterization does not comprise the boundary values of arbitrary K-qc automorphisms of Δ (see Example of [Z3]).

A new representation of the boundary values for arbitrary K-qc automorphisms of a generalized disc of the extended plane $\overline{\mathbf{C}}$, was obtained by the author in [Z1] and [Z3]. This new representation has some advantages not shared by quasisymmetric functions.

Suppose that Φ_K is the distortion function in the quasiconformal version of the Schwarz Lemma (see [PH]). It was extensively studied by G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen [AVV1], [AVV2] and [VV]. A number of results on Φ_K function was obtained by the author (see [Z3]).

By the generalized circle of the extended complex plane $\overline{\mathbb{C}}$ we mean the stereographic projection of a circle on the sphere $B = \{(x, y, u) : x^2 + y^2 + u^2 - u = 0\}$, onto the plane $\overline{\mathbb{C}}$. Suppose that Γ is such a circle on $\overline{\mathbb{C}}$ that z_1, z_2, z_3, z_4 is an ordered quadruple of distinct points of Γ. The expression

$$[z_1, z_2, z_3, z_4] = \left\{ \frac{z_3 - z_2}{z_3 - z_1} : \frac{z_4 - z_2}{z_4 - z_1} \right\}^{1/2} \tag{0.1}$$

is invariant under homographies and its values range over $(0, 1)$, for each ordered quadruple of distinct points of Γ.

By $A_\Gamma(K)$ we denote the class of all sense-preserving automorphisms f of Γ, such that

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]) \leq [f(z_1), f(z_2), f(z_3), f(z_4)] \leq \Phi_K([z_1, z_2, z_3, z_4]) \tag{0.2}$$

holds for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$, with a constant $K \geq 1$.

A function $f \in A_\Gamma(K)$ is said to be K-quasihomography (K-qh) of Γ. This class of functions represents the boundary values of K-qc automorphisms (of complementary domains D and D^* of Γ) (see [Z3, part 2]). It is invariant under self-homographies of Γ and has a number of properties very close to those of K-qc mappings (see [Z1], [Z3] and [Z4]).

The aim of this paper is to explain the relationship between $Q_T(\rho)$ and $A_T(K)$, without any quasiconformal extension.

1. Quasihomographies as quasisymmetric functions. We begin with proving

Theorem 1. *For each $K \geq 1$ and $f \in A_T(K)$, there exists a constant $\rho = \rho(f, K)$, such that $f \in Q_T(\rho)$ and*

$$\rho \leq \lambda(K) \cot^2(\varphi_f/4) , \tag{1.1}$$

where

$$\varphi_f = \min_{z\in T} \min\left\{ \arg \frac{f(-z)}{f(z)},\ 2\pi - \arg \frac{f(-z)}{f(z)} \right\} \tag{1.2}$$

Proof. Suppose that z_1, z_2, z_3, z_4 is an arbitrary quadruple of distinct points of T. Then

$$\frac{[z_1, z_2, z_3, z_4]^2}{[z_2, z_3, z_4, z_1]^2} = \frac{|z_3 - z_2|\, |z_4 - z_1|}{|z_4 - z_3|\, |z_2 - z_1|} . \tag{1.3}$$

For an arbitrary $f \in A_T(K)$ let $w_i = f(z_i)$ and $\zeta_i = h(w_i)$, $i = 1, 2, 3, 4$, where h is a homography mapping T onto itself, such that $\zeta_2 = -\zeta_1$. Consider a quadruple of ordered points $z_1, z_2, z_3, z_4 \in T$, such that $z_4 = -z_2$ and $|z_2 - z_1| = |z_3 - z_2|$. Thus

$[z_1, z_2, z_3, z_4] = [z_2, z_3, z_4, z_1]$. By the definition of K-qh, $h \circ f \in A_T(K)$ and, because of (1.3), we have

$$(1.4) \qquad 1/\lambda(K) \leq \frac{|\zeta_3 - \zeta_2|\,|\zeta_4 - \zeta_1|}{|\zeta_4 - \zeta_3|\,|\zeta_2 - \zeta_1|} \leq \lambda(K)\,,$$

with $\lambda(K) = \Phi_K^2(1/\sqrt{2})/\Phi_{1/K}^2(1/\sqrt{2})$.

Let

$$(1.5) \qquad \alpha = \arg\frac{\zeta_2 - \zeta_4}{\zeta_1 - \zeta_4} \quad \text{and} \quad \beta = \arg\frac{\zeta_3 - \zeta_4}{\zeta_2 - \zeta_4}\,, \quad 0 < \alpha\,, \beta < \pi/2\,.$$

Then

$$(1.6) \qquad \frac{|\zeta_3 - \zeta_2|\,|\zeta_4 - \zeta_1|}{|\zeta_4 - \zeta_3|\,|\zeta_2 - \zeta_1|} = \frac{\tan\beta}{\tan\alpha}$$

and

$$(1.7) \qquad 1/\lambda(K) \leq \frac{\tan\beta}{\tan\alpha} \leq \lambda(K)\,.$$

Now, by the concavity of arctan x for $x \geq 0$, and the well-known Jensen's inequality, we have

$$(1.8) \qquad \beta \leq \arctan(\lambda(K)\tan\alpha) \leq \lambda(K)\arctan(\tan\alpha) = \lambda(K)\alpha$$

and similarly

$$\alpha \leq \lambda(K)\beta\,.$$

Thus

$$(1.9) \qquad 1/\lambda(K) \leq \frac{\beta}{\alpha} \leq \lambda(K)\,.$$

Let arc $(z_1, z_2) = \{z \in T : \arg z_1 < \arg z < \arg z_2\}$ and $|\text{arc }(z_1, z_2)| = |\arg z_2 - \arg z_1|$ stands for its measure. Then, for an arbitrary subarc η of T, we see that

$$(1.10) \qquad \frac{1-|a|}{1+|a|}\,|\eta| \leq |h^{-1}(\eta)| = \int_\eta |(h^{-1})'(z)|dz| \leq \frac{1+|a|}{1-|a|}\,|\eta|\,,$$

where $a = |h(0)|$. Since $f = h^{-1} \circ (h \circ f)$, we have

$$(1.11) \qquad \begin{aligned} \frac{|\text{arc }(w_2, w_3)|}{|\text{arc }(w_1, w_2)|} &= \frac{|h^{-1}(\text{arc }(\zeta_2, \zeta_3))|}{|h^{-1}(\text{arc }(\zeta_1, \zeta_2))|} \leq \left(\frac{1+|a|}{1-|a|}\right)^2 \frac{|\text{arc }(\zeta_2, \zeta_3)|}{|\text{arc }(\zeta_1, \zeta_2)|} \\ &\leq \lambda(K)\cot^2(\varphi_f/4)\,, \end{aligned}$$

with φ_f given by (1.2) (cf. [V], p.13). This completes our proof.

The constant $\lambda(K)\cot^2(\varphi_f/4)$ may depend only on K, when we confine ourselves to the normalized K-qh of T. Let

$$(1.12) \qquad A_T^\circ(K) = \{f \in A_T(K) : f(z) = z,\ z^3 = 1\}\,.$$

then we have

Lemma. *For each $K \geq 1$, $f \in A_T^\circ(K)$, and $z \in T$*

$$(1.13)\qquad |f(z) - z| \leq |\arg f(z) - \arg z| \leq \frac{4}{\sqrt{3}} \Lambda(K)\ ,$$

where

$$(1.14)\qquad \Lambda(K) = \begin{cases} 1 - \left(\frac{K+1}{3K-1}\right)^2 & \text{for } 1 \leq K \leq 3/2\,, \\ 1 - (2K-1)^{-2} & \text{for } 3/2 < K \leq 4\,, \\ 1 - 4^{1-K} & \text{for } K > 4 \end{cases}$$

is such that

$$(1.15)\qquad \max_{0\leq t\leq 1} [\Phi_K^2(\sqrt{t}) - t] \leq \Lambda(K) \quad \text{cf. [Z3, Theorem 3].}$$

Proof. Without any loss of generality, suppose that $z \in T$ is such that $0 < \arg z < 2\pi/3$ and $\alpha = \arg z - \pi/3, |\alpha| \leq \pi/3$. If $z_l = e^{2\pi i l/3}$, $l = 1, 2, 3$, then

$$(1.16)\qquad [z_1, z, z_2, z_3]^2 = (1 - \sqrt{3}\tan\frac{\alpha}{2})/2\ .$$

For an arbitrary $f \in A_T^\circ(K)$ and $\beta = \arg f(z) - \pi/3$, we have

$$(1.17)\qquad \Phi_{1/K}^{-2}\left(\frac{(1-\sqrt{3}\tan\frac{\alpha}{2})^{1/2}}{\sqrt{2}}\right) \leq \frac{1-\sqrt{3}\tan\frac{\beta}{2}}{2} \leq \Phi_K^2\left(\frac{(1-\sqrt{3}\tan\frac{\alpha}{2})^{1/2}}{\sqrt{2}}\right).$$

On the other hand

$$(1.18)\qquad |f(z) - z| = 2\sin\frac{|\beta-\alpha|}{2} \leq |\beta - \alpha| \leq 2\tan\frac{|\beta-\alpha|}{2} \leq 2|\tan\frac{\beta}{2} - \tan\frac{\alpha}{2}|\ .$$

Then, by (1.15) and (1.17), we have

$$(1.19)\qquad \begin{aligned} |f(z) - z| &\leq \frac{4}{\sqrt{3}} \max_{0\leq t\leq 1} \max\{|\Phi_K^2(\sqrt{t}) - t|\ ,\ |\Phi_{1/K}^2(\sqrt{t}) - t|\} \\ &= \frac{4}{\sqrt{3}} \max_{0\leq t\leq 1} |\Phi_K^2(\sqrt{t}) - t| \leq \frac{4}{\sqrt{3}} \Lambda(K)\ , \end{aligned}$$

which completes the proof.

Now we prove

Theorem 2. *For each $K \geq 1$ there exists a constant $\rho \geq 1$ such that $A_T^\circ(K) \subset Q_T^\circ(\rho)$, where*

$$(1.20)\qquad \rho \leq \begin{cases} \lambda(K)\left(\frac{1+\tan\frac{2}{\sqrt{3}}\Lambda(K)}{1-\tan\frac{2}{\sqrt{3}}\Lambda(K)}\right)^2 & \text{for } 1 \leq K \leq K_0\ , \\ \frac{1}{8}\lambda(K)16^{K-1}(5+\sqrt{2})^{2K} & \text{for } K > K_0\ . \end{cases}$$

with Λ *given by* (1.14) *and* $1.425 < K_0 < 1.426$, *that satisfies*

$$1+\tan\frac{2}{\sqrt{3}}\,\Lambda(K) = \frac{5+\sqrt{2}}{\sqrt{3}}\,\left(1-\tan\frac{2}{\sqrt{3}}\,\Lambda(K)\right)\,. \tag{1.21}$$

Proof. With the notations of Lemma, for $z = e^{i\pi/3}$, we have

$$\arg f(z) = \beta + \frac{\pi}{3} \geq \frac{\pi}{3} - 2\arctan\left((2\Phi_K^2(1/\sqrt{2})-1)/\sqrt{3}\right) \tag{1.22}$$

and then

$$\begin{aligned}\varphi_t &> \min\left\{\frac{2\pi}{3}, \frac{2\pi}{3} - 4\arctan\left((2\Phi_K^2(1/\sqrt{2})-1)/\sqrt{3}\right)\right\}\\ &= \frac{2\pi}{3} - 4\arctan\left((2\Phi_K^2(1/\sqrt{2})-1)/\sqrt{3}\right)\,.\end{aligned} \tag{1.23}$$

The inequality (1.23) remains the same for $z = e^{\pi i}$ and $z = e^{5\pi i/3}$. If

$$r = (1-\sqrt{3}\tan\frac{\alpha}{2})/2\,, \quad \text{then } \tan\frac{\alpha}{2} = (1-2r)/\sqrt{3}\,. \tag{1.24}$$

Setting $\beta = \pi/3 - \alpha$, we get

$$1-\sqrt{3}\tan\frac{\beta}{2} = \frac{2-4r}{2-r}\,. \tag{1.25}$$

Since $4^{1-K}t^K \leq \Phi_{1/K}(t) \leq t^K$ and $\Phi_{1/K}((1-t)(1+t)) = (1-\Phi_K(t))/(1+\Phi_K(t))$, for $0 \leq t \leq 1$ and $K \geq 1$ (cf. [AVV1]), then

$$\begin{aligned}\cot^2(\varphi_f/4) &< \left(\frac{\sqrt{3}+(2\Phi_K^2(1/\sqrt{2})-1)/\sqrt{3}}{1-\sqrt{3}(2\Phi_K^2(1/\sqrt{2})-1)/\sqrt{3}}\right)^2\\ &= \frac{1}{3}\left((1+\Phi_K^2(1/\sqrt{2}))/(1-\Phi_K^2(1/\sqrt{2}))\right)^2\\ &< \frac{1}{3}\,\Phi_{1/K}^2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{1}{3}\,\Phi_{1/K}^{-2}\left(\frac{1}{5+2\sqrt{2}}\right)\\ &\leq \frac{1}{3}\left(4^{1-K}\left(\frac{1}{5+2\sqrt{2}}\right)^K\right)^{-2} = \frac{1}{3}\,16^{K-1}(5+2\sqrt{2})^{2K}\,.\end{aligned} \tag{1.26}$$

Using the above estimate to (1.1) we obtain the case $K > K_0$ in (1.20). This result is not sharp, since for $K = 1$, the upper bound is $(5+2\sqrt{2})^2/3$. To get such result, we have to use our Lemma. By (1.13) we have

$$\varphi_f > \min_{z\in T}\left(\pi - 2|\arg f(z) - \arg z|\right) \geq \pi - 2\frac{4}{\sqrt{3}}\,\Lambda(K) = \pi - \frac{8}{\sqrt{3}}\,\Lambda(K)\,.$$

thus

$$\cot^2(\varphi_f/4) \leq \left(\frac{1+\tan\frac{2}{\sqrt{3}}\,\Lambda(K)}{1-\tan\frac{2}{\sqrt{3}}\,\Lambda(K)}\right)^2 \tag{1.27}$$

Then, the case $1 \le K \le K_0$ in (1.20) follows by applying (1.27) to (1.1) where K_0 satisfies the equation

$$\frac{1+\tan\frac{2}{\sqrt{3}}\Lambda(K)}{1-\tan\frac{2}{\sqrt{3}}\Lambda(K)} = \min_{1\le K<\infty} 4^{K-1}\frac{(5+\sqrt{2})^K}{\sqrt{3}} = \frac{5+\sqrt{2}}{\sqrt{3}} .$$

This makes our proof complete.

2. Quasisymmetric functions as quasihomographies. An opposite inclusion is presented by

Theorem 3. *For each $\rho \ge 1$ there exists $K \ge 1$ such that $Q_T(\rho) \subset A_T(K)$ and*

$$K \le \begin{cases} \chi(\nu(2C_\rho^2)) & \text{for } 1 \le \rho \le \rho_0 , \\ \chi(\nu(M_\rho - 1)) & \text{for } \rho > \rho_0 , \end{cases} \tag{2.1}$$

with $\rho_0 = (50\pi+1)/(50\pi-1)$, where:

$$C_\rho = \frac{64^{\nu(\rho)-1}}{\left(1-\frac{\pi}{3}\,\frac{\rho-1}{\rho+1}\right)^{\nu(\rho)}}\,\frac{\sqrt{\rho+1}+\sqrt{2\pi(\rho-1)}}{\sqrt{\rho+1}-4.1\sqrt{2\pi(\rho-1)}} ; \tag{2.2}$$

$$M_\rho = \frac{1}{2}\,\pi^2 4^{7\nu(\rho)-4} ; \tag{2.3}$$

$$\nu(r) = \begin{cases} \dfrac{e^{2\sqrt{r-1}}}{1-2^{-m}e^{1/m}} , \ m = \mathrm{Ent}\left\{\frac{1}{\sqrt{r-1}}\right\} , & \text{for } 1 \le r \le \frac{5}{4} , \\ 3.41\log_2(1+r) , & \text{for } \frac{5}{4} < r \le 6 , \\ (\ln 2)\left(1-\left(\log_2\left(\frac{2}{r}\log_2(1+r)\right)\right)^{-1}\right)(1+r) , & \text{for } r > 6 \end{cases} \tag{2.4}$$

with $\nu(r) \simeq (\ln 2)(1+r)$, when $r \to \infty$;

$$\chi(r) = \begin{cases} r\left(\dfrac{r-1}{\log_4(31/33)}+1\right)^{-1} & \text{for } 1 \le r \le r_0 , \\ 2r & \text{for } r > r_0 \end{cases} \tag{2.5}$$

with $r_0 = 1+\log_{16}(33/31)$. The functions ν and χ were introduced by the author in [Z1] and [Z2], respectively, in a connection with the distortion function Φ_K.

Proof. Suppose that $f \in Q_T(\rho)$, $1 \le \rho < \infty$, is arbitrarily chosen. Without any loss of generality, we may assume, that $f(1) = 1$ and $f(-1) = -1$ (cf. [K2]). Let $h(z) = i(1-z)/(1+z)$, $h(T) = \overline{\mathbf{R}}$. For an arbitrary symmetric triple $a-t, a, a+t \in \mathbf{R}$, with $t > 0$, we have

$$[\infty, a-t, a, a+t]^2 = \frac{1}{2} . \tag{2.6}$$

For each quadruple, as on the left hand side of (2.6), there exists a positively ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in T$ and positive numbers $\alpha, \beta, \gamma, \delta$, such that $z_1 = h^{-1}(\infty) = -1$, $z_2 = h^{-1}(a-t) = e^{2i\alpha}$, $z_3 = h^{-1}(a) = e^{2i(\alpha+\beta)}$ and $z_4 = h^{-1}(a+t) = e^{2i(\alpha+\beta+\gamma)}$. Moreover, $\alpha+\beta+\gamma = \pi - \delta$. Thus, by invariance of (0.1) under homographies

$$[z_1, z_2, z_3, z_4]^2 = \frac{\sin\beta}{\sin(\alpha+\beta)}\,\frac{\sin\delta}{\sin(\alpha+\delta)} = \frac{1}{2}\,. \tag{2.7}$$

Without loss of generality we assume that $\alpha \le \gamma$. Then $\alpha \le \frac{\pi}{2}$. Let $f(z_2) = e^{i\alpha'}$, $f(z_3) = e^{2i(\alpha'+\beta')}$ and $f(z_4) = e^{2i(\alpha'+\beta'+\gamma')}$, where α', β', γ' are positive and there exists a positive δ', such that $\alpha'+\beta'+\gamma' = \pi - \delta'$. If $g : \mathbf{R} \to \mathbf{R}$ is an increasing homeomorphism such that $f(e^{ix}) = e^{ig(x)}$, and normalized ($g(0) = 0$, $g(\pi) = \pi$) then, by a result of J. G. K r z y ż [K2], $g \in Q_{\mathbf{R}}(\rho)$, and the inequality

$$|g(x) - x| \le \pi\frac{\rho-1}{\rho+1} \tag{2.8}$$

holds for $0 \le x \le \pi$.

We intend to show that for an arbitrary $1 \le \rho \le \rho_0$, where $2\pi(\rho_0-1)/(\rho_0+1) = 1/25$, there exists a constant C_ρ, $1 \le C_\rho < \infty$, such that the inequality

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{1}{C_\rho}\,\frac{\sin\beta}{\sin(\alpha+\beta)} \tag{2.9}$$

is satisfied by all admissible independent α, β and α', β' defined here. Assuming that $\varepsilon^2 = 2\pi(\rho-1)/(\rho+1)$, $0 \le \varepsilon \le 1/5 < \pi/4$, we will consider a few special cases:

$$\beta < \alpha + \beta \le \varepsilon\,. \tag{I}$$

Then, by (2.8) [Z3, Theorem 13], [Z2, Theorem 4], and the Wang–Hübner inequality (cf. [AVV1]), there exists K', $1 \le K' \le \nu(\rho)$, where ν is given by (2.4), such that

$$\begin{aligned}\frac{\sin\beta'}{\sin(\alpha'+\beta')} &\ge \frac{\beta'}{\alpha'+\beta'}\,\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2} \ge \Phi^2_{1/K'}\left(\sqrt{\frac{\beta}{\alpha+\beta}}\right)\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\\ &\ge 16^{1-K'}\left(\frac{\beta}{\alpha+\beta}\right)^{K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\\ &\ge 16^{1-K'}\left(\frac{\sin\beta}{\sin(\alpha+\beta)}\right)^{K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\,.\end{aligned} \tag{2.10}$$

Since $\alpha \le \gamma$, we have $|z_2 - z_4| \ge \frac{1}{2}|z_1 - z_4|$ and by (2.7) we see that

$$\sin(\alpha+\beta) \le 4\sin\beta\,. \tag{2.11}$$

Therefore

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge 64^{1-K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\,\frac{\sin\beta}{\sin(\alpha+\beta)}\,; \tag{2.12}$$

(II) $$\beta \le \varepsilon < \alpha + \beta < \pi - \varepsilon .$$

Then using (2.8) and (2.11) we have

$$\sin\beta \ge \frac{1}{4}\sin(\alpha+\beta) \ge \frac{1}{4}\sin\varepsilon \,, \tag{2.13}$$

hence

$$\frac{\sin\beta'}{\sin\beta} \ge \frac{\sin(\beta-\varepsilon^2)}{\sin\beta} = \cos\varepsilon^2 - \cot\beta\sin\varepsilon^2 \ge \cos\varepsilon^2 - 4\frac{\sin\varepsilon^2}{\sin\varepsilon} \tag{2.14}$$

and

$$\frac{\sin(\alpha'+\beta')}{\sin(\alpha+\beta)} \le \frac{\sin(\varepsilon+\varepsilon')}{\sin\varepsilon} . \tag{2.15}$$

Therefore

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \left(\cos\varepsilon^2 - 4\frac{\sin\varepsilon^2}{\sin\varepsilon}\right)\frac{\sin\varepsilon}{\sin(\varepsilon+\varepsilon^2)}\,\frac{\sin\beta}{\sin(\alpha+\beta)} ; \tag{2.16}$$

(III) $$\varepsilon \le \beta < \alpha + \beta \le \pi - \varepsilon .$$

It follows from (2.8) that

$$\frac{\sin\beta'}{\sin\beta} \ge \frac{\sin(\varepsilon-\varepsilon^2)}{\sin\varepsilon} \tag{2.17}$$

and further, by (2.15),

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{\sin(\varepsilon-\varepsilon^2)}{\sin(\varepsilon+\varepsilon^2)}\,\frac{\sin\beta}{\sin(\alpha+\beta)} . \tag{2.18}$$

(IV) $$\varepsilon \le \beta \le \pi - \varepsilon \le \alpha + \beta .$$

In this case we can see that $\gamma+\delta \le \varepsilon$ and, because $\alpha \le \gamma$, it follows that $\alpha < \gamma+\delta \le \varepsilon$, hence $\alpha + \gamma + \delta < 2\varepsilon$. Using the same arguments as in (I), we conclude that

$$\begin{aligned}\frac{\sin(\gamma'+\delta')}{\sin(\gamma'+\delta'+\alpha')} &\le \frac{\gamma'+\delta'}{\gamma'+\delta'+\alpha'}\,\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)} \le \Phi^2_{K'}\left(\sqrt{\frac{\gamma+\delta}{\gamma+\delta+\alpha}}\right)\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}\\ &\le 16^{1-1/K'}\left(\frac{\gamma+\delta}{\gamma+\delta+\alpha}\right)^{1/K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}\\ &\le 16^{1-1/K'}\left(\frac{\sin(\gamma+\delta)}{\sin(\gamma+\delta+\alpha)}\right)^{1/K'}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{1/K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)} .\end{aligned} \tag{2.19}$$

Since $\alpha \le \gamma$, we have $|z_1 - z_3| \ge \frac{1}{2}|z_2 - z_3|$. Thus

$$\sin(\alpha+\beta) \ge \frac{1}{2}\sin\beta\,. \tag{2.20}$$

Using (2.19) and (2.20), we have

$$\begin{aligned}\frac{\sin\beta'}{\sin(\alpha'+\beta')} &= \frac{\sin(\alpha'+\gamma'+\delta')}{\sin(\gamma'+\delta')}\\ &\ge 32^{-1+1/K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{1/K'}\frac{\sin(2\varepsilon+\varepsilon^2)}{2\varepsilon+\varepsilon^2}\,\frac{\sin\beta}{\sin(\alpha+\beta)}\,.\end{aligned} \tag{2.21}$$

$$\pi - \varepsilon \le \beta < \alpha + \beta\,. \tag{V}$$

Then, $\gamma + \delta < \alpha + \mu + \delta \le \varepsilon$, and following (IV), we arrive at

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge 32^{-1+1/K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{1/K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\,\frac{\sin\beta}{\sin(\alpha+\beta)}\,. \tag{2.22}$$

Hence, by (2.12), (2.16), (2.18), (2.21) and (2.22), we see that

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{1}{C_{K'}}\,\frac{\sin\beta}{\sin(\alpha+\beta)}\,, \tag{2.23}$$

with

$$\frac{1}{C_{K'}} = \min\left\{\frac{1}{C^1_{K'}}, \frac{1}{C^2_{K'}}, \frac{1}{C^3_{K'}}, \frac{1}{C^4_{K'}}, \frac{1}{C^5_{K'}}\right\} = \min\left\{\frac{1}{C^1_{K'}}, \frac{1}{C^2_{K'}}, \frac{1}{C^3_{K'}}, \frac{1}{C^4_{K'}}\right\}\,. \tag{2.24}$$

where $C^l_{K'}$ are constants described by (2.12), (2.16), (2.18), (2.21) and (2.22), respectively, when $l = 1, 2, 3, 4, 5$.

In a similar way

$$\frac{\sin\delta'}{\sin(\alpha'+\delta')} \ge \frac{1}{C_{K'}}\,\frac{\sin\delta}{\sin(\alpha+\beta)}\,. \tag{2.25}$$

Therefore, by (2.23), (2.25) and (2.7), we have the inequality

$$\begin{aligned}[f(z_1), f(z_2), f(z_3), f(z_4)]^2 &= \frac{\sin\beta'}{\sin(\alpha'+\beta')}\,\frac{\sin\delta'}{\sin(\alpha'+\delta')} \ge \frac{1}{C^2_{K'}}\,[z_1, z_2, z_3, z_4]\\ &= \frac{1}{2C^2_{K'}}\,,\end{aligned} \tag{2.26}$$

which holds for our quadruple $z_1, z_2, z_3, z_4 \in T$. Similar arguments give the inequality

$$\begin{aligned}[f(z_2), f(z_3), f(z_4), f(z_1)]^2 &= \frac{\sin\gamma'}{\sin(\beta'+\gamma')}\,\frac{\sin\alpha'}{\sin(\beta'+\alpha')}\\ &\ge \frac{1}{C^2_{K'}}\,[z_2, z_3, z_4, z_1] = \frac{1}{C^2_{K'}}\left(1-\frac{1}{2}\right) = \frac{1}{2C^2_{K'}}\,,\end{aligned} \tag{2.26'}$$

with the same points as in (2.26).

Let $F = h \circ f \circ h^{-1}$, that is a sense-preserving homeomorphism of $\mathbf{R}$ onto itself. Moreover, by (2.26) and (2.26'),

(2.27)
$$\frac{1}{\dfrac{F(a+t)-F(a)}{F(a)-F(a-t)}+1} = \frac{F(a)-F(a-t)}{F(a+t)-F(a-t)} = [\infty, F(a-t), F(a), F(a+t)]^2$$
$$= [h\circ f(z_1), h\circ f(z_2), h\circ f(z_3), h\circ f(z_4)]^2$$
$$= [f(z_1), f(z_2), f(z_3), f(z_4)]^2 \geq \frac{1}{2C_{K'}^2}$$

and similarly

(2.27')
$$\frac{1}{\dfrac{F(a)-F(a-t)}{F(a+t)-F(a)}+1} = \frac{F(a+t)-F(a)}{F(a+t)-F(a-t)} = [F(a-t), F(a), F(a+t), \infty]^2$$
$$= [f(z_2), f(z_3), f(z_4), f(z_1)]^2 \geq \frac{1}{2C_{K'}^2} .$$

Using (2.27) and (2.27'), we see that

$$\frac{1}{2C_{K'}^2-1} \leq \frac{F(a+t)-F(a)}{F(a)-F(a-t)} \leq 2C_{K'}^2 - 1 , \tag{2.28}$$

hence $F \in Q_R(2C_{K'}^2 - 1)$. Now, by [Z3, Theorem 13], we see that there exists K, $1 \leq K < \infty$, such that $F \in A_{\mathbf{R}}(K)$ and $K \leq \chi(\nu(2C_{K'}^2 - 1))$. Thus, for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in T$,

$$\begin{aligned}[f(z_1), f(z_2), f(z_3), f(z_4)]^2 &= [F\circ h(z_1), F\circ h(z_2), F\circ h(z_3), F\circ h(z_4)]^2 \\ &\leq \Phi_K^2([h(z_1), h(z_2), h(z_3), h(z_4)]) = \Phi_K^2([z_1, z_2, z_3, z_4])\end{aligned} \tag{2.29}$$

and then $f \in A_T(K)$, for $1 \leq \rho \leq \rho_0$, $\rho_0 = (50\pi + 1)/(50\pi - 1)$.

Now let $\rho > \rho_0$. If $0 < \beta' < \pi/2$, then

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \geq \frac{2}{\pi}\,\frac{\beta'}{\alpha'+\beta'} . \tag{2.30}$$

If $\pi/2 \leq \beta' < \alpha' + \beta' < \pi$, then

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \geq 1 . \tag{2.31}$$

By applying (2.8) [Z3, Theorem 13], [Z2, Theorem 4] and the Wang–Hübner inequality (cf. [AVV1]), we see that there exists K', $1 \leq K' < \nu(\rho)$, such that

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \geq \frac{2}{\pi}\,\frac{\beta'}{\alpha'+\beta'} \geq \frac{2}{\pi}\,\Phi_{1/K'}^2\left(\sqrt{\frac{\beta}{\alpha+\beta}}\right) \geq \frac{2}{\pi}\,16^{1-K'}\left(\frac{\beta}{\alpha+\beta}\right)^{K'} . \tag{2.32}$$

In what follows we consider four possibilities (I') – (IV').

(I') $$0 < \beta < \alpha + \beta \le \frac{\pi}{2} .$$

Then

(2.33) $$\frac{\beta}{\alpha + \beta} \ge \frac{2}{\pi} \frac{\sin \beta}{\sin(\alpha + \beta)} ;$$

(II') $$0 < \beta < \frac{\pi}{4} \quad \text{and} \quad \frac{\pi}{2} \le \alpha + \beta < \pi .$$

Then, in view of $\alpha \le \frac{\pi}{2}$

(2.34) $$\frac{\beta}{\beta + \alpha} \ge \frac{4}{3\sqrt{2}\,\pi} \frac{\sin \beta}{\sin(\alpha + \beta)} ;$$

(III') $$\frac{\pi}{4} \le \beta \le \frac{\pi}{2} < \alpha + \beta < \pi .$$

Then, by (2.20),

(2.35) $$\frac{\beta}{\alpha + \beta} \ge \frac{1}{8} \frac{\sin \beta}{\sin(\alpha + \beta)} ;$$

(IV') $$\frac{\pi}{2} \le \beta < \alpha + \beta < \pi .$$

Then again, by (2.20),

(2.36) $$\frac{\beta}{\alpha + \beta} \ge \frac{1}{2} \ge \frac{1}{4} \frac{\sin \beta}{\sin(\alpha + \beta)} .$$

Since

(2.37) $$\frac{1}{8} = \min\left\{ \frac{2}{\pi} , \frac{4}{3\sqrt{2}\,\pi} , \frac{1}{8} , \frac{1}{4} \right\}$$

then, by (2.32),

(2.38) $$\frac{\sin \beta'}{\sin(\alpha' + \beta')} \ge \frac{2}{\pi}\, 16^{1-K'} \left(\frac{1}{8} \frac{\sin \beta}{\sin(\alpha + \beta)} \right)^{K'} .$$

In a similar way we can show that

(2.38') $$\frac{\sin \delta'}{\sin(\delta' + \alpha')} \ge \frac{2}{\pi}\, 16^{1-K'} \left(\frac{1}{8} \frac{\sin \delta}{\sin(\delta + \alpha)} \right)^{K'} .$$

Following our considerations presented by (2.26), (2.26'), (2.27), (2.27'), then by (2.28) with (2.38) and (2.28'), we see that $F \in A_{\overline{\mathbb{R}}}(K)$, and $1 \le K \le \chi(\nu(M_{K'} - 1)$, where $M_{K'} = \frac{1}{2}\,\pi^2 4^{7K'-4}$.

Let us call our attention back to (2.24). For $0 \le \varepsilon \le \varepsilon_0 = 1/5$ we have

$$\max\Big\{\frac{2\varepsilon + \varepsilon^2}{\sin(2\varepsilon + \varepsilon^2)}\,,\ \frac{\sin(\varepsilon + \varepsilon^2)}{\sin\varepsilon\cos\varepsilon^2 - 4\sin\varepsilon^2}\Big\} \le \frac{1+\varepsilon}{1 - 4\varepsilon/\cos\varepsilon} \le \frac{1+\varepsilon}{1 - 4.1\varepsilon}\,.$$

Hence

$$C_{K'} = \max\Big\{64^{K'-1}\Big(\frac{\varepsilon}{\sin\varepsilon}\Big)^{K'}\frac{\varepsilon+\varepsilon^2}{\sin(\varepsilon+\varepsilon^2)}\,,\ \frac{\sin(\varepsilon+\varepsilon^2)}{\sin\varepsilon}\,\frac{\sin\varepsilon}{\sin\varepsilon\cos\varepsilon^2 - 4\sin\varepsilon^2}\,,$$

$$\frac{\sin(\varepsilon+\varepsilon^2)}{\sin(\varepsilon-\varepsilon^2)}\,,\ 32^{1-1/K'}\Big(\frac{\varepsilon}{\sin\varepsilon}\Big)^{1/K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}\Big\}$$

$$\le \max\Big\{64^{K'-1}\Big(\frac{\varepsilon}{\sin\varepsilon}\Big)^{K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}\Big\}\,,\ \frac{\sin(\varepsilon+\varepsilon^2)}{\sin\varepsilon\cos\varepsilon^2 - 4\sin\varepsilon^2}\Big\} \tag{2.39}$$

$$\le 64^{K'-1}\Big(\frac{\varepsilon}{\sin\varepsilon}\Big)^{K'}\frac{1+\varepsilon}{1-4\varepsilon/cos\varepsilon} \le 64^{K'-1}\Big(\frac{\varepsilon}{\sin\varepsilon}\Big)^{K'}\frac{1+\varepsilon}{1-4.1\varepsilon}$$

$$\le 64^{K'-1}\frac{1}{(1-\frac{1}{6}\,\varepsilon^2)^{K'}}\,\frac{1+\varepsilon}{1-4.1\varepsilon} = \frac{64^{K'-1}}{(\frac{\pi}{3}\ \frac{\rho-1}{\rho+1})^{K'}}\ \frac{\sqrt{\rho+1}+\sqrt{2\pi(\rho-1)}}{\sqrt{\rho+1}-4.1\sqrt{2\pi(\rho-1)}}\,.$$

Since $\chi_{K'}$ and $\nu_{K'}$ are increasing functions so, in view of (2.2) and (2.3) we obtain the estimate (2.1). This completes the proof.

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STRESZCZENIE

Celem pracy jest wykazanie związków pomiędzy funkcjami quasisymetrycznymi okręgu jednostkowego a nową klasą automorfizmów reprezentujących wartości brzegowe dowolnych automorfizmów quasikonforemnych koła jednostkowego, zwanych quasihomografiami.

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