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Some Remarks on Bi-univalent Functions

Kilka uwag o funkcjach bi-jednolistnych

Несколько замечаний о би-однолистных функциях

1. Introduction. Let S be the class of functions:

$$g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \tag{1.1}$$

which are regular and univalent in the open disc:

$$D = \{z \in \mathbf{C} : |z| < 1\}.$$

The inverse function $\mathfrak{g}(w)$ of $g \in S$ has the power series expansion:

$$\mathfrak{g}(w) = w + B_2 w^2 + B_3 w^3 + \cdots, \quad |w| < \frac{1}{4} \tag{1.2}$$

It can happen that the inverse function $\mathfrak{g}(w)$ admits analytic continuation on D which is univalent in D. Therefore we may consider the class σ of functions:

$$f(z) = z + a_2 z^2 + \cdots = z + a_2 f(z^2) + \cdots, \quad |z| < 1. \tag{1.3}$$

such that both $f(z)$ and its inverse $\mathfrak{f}(w)$ belong to the class S. Such functions are called bi-univalent. There are many open problems concerning the class σ, among them the classical problem of finding:

$$a_2^* = \sup_{f \in \sigma} |a_2(f)|. \tag{1.4}$$

According to a conjecture due to D.A Brannan ([1] p. 561) $a_2^* = \sqrt{2}$.
M. Lewin [7] using Grunsky's inequalities and the properties of Jabotinsky's l-sequences has proved that $a_2^* < 1.51$. D.Styer and D.Wright [9] by considering a special bi-univalent function g have showed that $a_2^* > 4/3 + 0.02$. In this paper we apply well-known estimates of the functional $|a_3(g) - \lambda a_2^2(g)|$ for g ranging over S and some subclasses in order to obtain estimates of a_2^* in σ and its subclasses. The estimate for a_2^* in σ obtained in [7] is slightly better then (2.2) but our proof has the advantage of great simplicity. The estimates of a_2^* for the subclasses seem to be new and permit us to eliminate some subclasses from the competition so far as the precise estimate of a_2^* in σ is concerned.

D.A Brannan ([1] p. 559) has also stated the following problem:

Let V_n denote the class of polynomials:

$$P_n(z) = z + a_2 z^2 + \cdots + a_n z^n \tag{1.5}$$

bi-univalent in D. Determine $\max\limits_{V_n} |a_2|$ and $\max\limits_{V_n} |a_n|$. The above stated problem was solved by H.V.Smith [8] for polynomials of degree three with real coefficients. We obtain estimates of $|a_2|$ in V_n for $n = 4, 5, 6, 7$. The author wishes to express his sincere gratitude to Prof. J.G.Krzyż for the valuable advice and encouragement.

2. Estimates of $|a_2|$. To this end we need following lemmas, some of them being well-known classical results.

Lemma 1 [3]. *If*

$$g(z) = z + b_2 z^2 + b_3 z^3 + \cdots, \quad |z| < 1\,, \tag{2.1}$$

belongs to the class S then the following inequality holds:

$$|b_3 - tb_2^2| \le F(t) = 1 + 2\exp\left[\frac{2t}{(t-1)}\right], \quad 0 \le t \le 1\,.$$

Lemma 2 [5]. *For each starlike function g of the form* (2.1) *we have:*

$$|b_3 - tb_2^2| \le G(t) = \max(1, |4t - 3|), \quad 0 \le t \le 1\,.$$

Lemma 3 [5]. *For each convex function g of the form* (2.1) *we have:*

$$|b_3 - tb_2^2| \le H(t) = max\left(\frac{1}{3}, 1 - t\right), \quad 0 \le t \le 1\,.$$

In the above lemmas the estimates are sharp for all $t \in [0, 1]$. The class of starlike functions will be denoted be ST and the class of convex functions by CV (cf.[4]).

Theorem 1. *Let $f(z) = z + a_2 z^2 + a_3 f(z^3) + \cdots$ be a bi-univalent function. Then:*

$$|a_2| < 1.5894\,. \tag{2.2}$$

Proof. The inverse function of $f(z)$ is given by:

$$\check{f}(w) = w + A_2 w^2 + A_3 w^3 + \cdots$$

where

$$A_2 = -a_2, \qquad A_3 = 2a_2^2 - a_3 . \tag{2.3}$$

The function f and $\tilde{f}$ both belong to the class S and therefore, by Lemma 1, we have:

$$|a_3 - xa_2^2| \le F(x), \qquad 0 \le x \le 1 . \tag{2.4}$$

$$|A_3 - yA_2^2| \le F(y), \qquad 0 \le y \le 1 . \tag{2.5}$$

Adding the inequalities (2.4) and (2.5) and taking into account (2.3) we obtain:

$$(2 - x - y)|a_2|^2 \le F(x) + F(y), \qquad 0 \le x, y \le 1 .$$

Then (2.2) follows by putting $x = y = 0.46$. We used computer to establish x and y in order to obtain possibly sharp estimate of $|a_2|$.

Theorem 2. *Let the functions $f(z) = z + a_2 z^2 + \cdots$ and $\tilde{f}(w)$ be starlike. Then $|a_2| \le \sqrt{2}$.*

Proof. Using Lemma 2 we obtain analogously:

$$(2 - x - y)|a_2|^2 \le G(x) + G(y), \qquad 0 \le x, y \le 1 .$$

and putting $x = y = 0.5$ we obtain our result.

Now, we are going to present estimates of $|a_2|$ for some subclasses of bi-univalent functions.

Theorem 3. *Let $f(z) = z + a_2 z^2 + \cdots$ be such that $f \in ST$, and its inverse $\tilde{f} \in S$. Then $|a_2| < 1.507$.*

Proof. By Lemmas 1 and 2 we have:

$$(2 - x - y)|a_2|^2 \le F(x) + G(y), \qquad 0 \le x, y \le 1 .$$

We obtain our inequality by putting $x = 0.49$, $y = 0.5$.

Theorem 4. *Let $f(z) = z + a_2 z^2 + \cdots$ be such that $f \in CV$, and its inverse $\tilde{f} \in S$. Then $|a_2| < 1.224$.*

Proof. By Lemmas 1 and 3 we have:

$$(2 - x - y)|a_2|^2 \le F(x) + H(y), \qquad 0 \le x, y \le 1 .$$

Taking $x = 0.57$, $y = 0$, we get the stated result.

Remark 1. If the functions f and $\tilde{f}$ are convex then the analogous problem is trivial. The function $\dfrac{z}{(1 - z)}$ is extremal in the class CV with respect to the modulus of coefficients and it simultaneously belongs to the class σ. But it seems to be interesting that a sharp estimate also follows from Lemma 3.
We have $(2 - x - y)|a_2|^2 \le H(x) + H(y), \quad 0 \le x, y \le 1$.
Putting $x = \dfrac{1}{3}$, $y = \dfrac{2}{3}$ we obtain $|a_2| \le 1$.

We shall now give an estimate of $|a_2|$ in the case when one of the functions f, $\tilde{f}$ is bounded.

Let $S(M)$ denote the set of all functions f belonging to the class S, such that $|f(z)| < M$ for all $z \in D$. Then we have the following

Lemma 4 [6]. *Let $h(z) = z + c_2z^2 + c_3z^3 + \cdots$ belongs to $S(M)$, $b = \frac{1}{M}$, $t \in [-2,2]$. Then:*

$$\operatorname{Re}\,(a_3 - a_2^2 + 2ta_2) \le J(t,b)$$

where

$$J(t,b) = \begin{cases} 1 - b^2 - t^2 \log b, & 0 \le |t| \le 2b \\ 1 - b^2 - t^2 \log |t|/2 + 1.5t^2 - 4|t|b, & 2b \le |t| \le 2 \end{cases}$$

$$J(t,0) = \lim_{b \to 0_+} J(t,b) = 1 - t^2 \log |t|/2 + 1.5t^2.$$

Hence we can obtain:

Theorem 5. *Let $f(z) = z + a_2z^2 + \cdots$ belongs to S, and let $\check{f} \in S(M)$ with $M = 4$. Then $|a_2| < 1.32$.*

Proof. Making use of (2.3) and Lemma 4 for the functions f, $\check{f}$ with $t = x$ and $t = -y$ and b equal 0 and $\frac{1}{M}$ respectively we have:

$$\operatorname{Re}\,(a_3 - a_2^2 + 2xa_2) \le J(x,0)$$

$$\operatorname{Re}\,(a_2^2 - a_3 + 2ya_2) \le J(-y,b).$$

Adding these inequalities we have:

$$2(x+y)\operatorname{Re}\ a_2 \le J(x,0) + J(-y,b). \tag{2.5}$$

By putting $\frac{1}{b} = 0.25$, $x = 0.6$, $y = 1.2$ we obtain our assertion.

Corollary. *The estimate of* $|a_2|$ given in Theorem 1 can be obtained also by making use of (2.5) with $b = 0$, $x = y = 0.88$.

Remark 2. It follows from Theorems 4 and 5, that the extremal function for $|a_2|$ in σ can not be a convex function, or a function from $S(M)$ where $M < 4$.

By the considerations entirely analogous to those in Theorem 5 and Lemma 5 we can estimate the modulus of the second coefficient for bi-univalent polynomials of degree 4, 5, 6, 7.

Lemma 5 [2]. *Let $W(z)$ be a univalent polynomial of degree n. Then there exists $M = M_n$ such that $W \in S(M)$. For $n = 4,5,6,7$ we can take $M = 3.61$,* 5.64, 7.73, 10.49 respectively.

Theorem 6. *Let $P_n(z) = z + a_2z^2 + \cdots + a_nz^n \in V_n$* and $A_n^* = \sup_{V_n} |a_2(P_n)|$. *Then we have $A_4^* \le 1.29$, $A_5^* \le 1.4$, $A_6^* \le 1.46$, $A_7^* \le 1.5$.*

Proof. We obtain our estimates by putting in (2.5):

$$\frac{1}{b} = 3.61 \quad x = 0.56 \quad y = 1.24 \quad \text{for} \quad n = 4$$
$$\frac{1}{b} = 5.64 \quad x = 0.68 \quad y = 1.08 \quad \text{for} \quad n = 5$$
$$\frac{1}{b} = 7.73 \quad x = 0.72 \quad y = 1.04 \quad \text{for} \quad n = 6$$
$$\frac{1}{b} = 10.49 \quad x = 0.76 \quad y = 1 \quad \text{for} \quad n = 7$$

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STRESZCZENIE

Niech S będzie klasa funkcji $f(z) = z + a_2 z^2 + \cdots$ regularnych i jednolistnych w kole $|z| < 1$. Mówimy, że funkcja f z klasy S jest funkcją bi-jednolistną, jeśli funkcja odwrotna $\bar{f}$ także należy do S. W pracy tej podaję oszacowania $|a_2|$ w pewnych podklasach funkcji bi-jednolistnych.

РЕЗЮМЕ

Пусть S будет классом функций $f(z) = z + a_2 z^2 + \cdots$ регулярных и однолистных в круге $|z| < 1$. Функцию f с класса S назовем би-однолистной, если обратная функция $\bar{f}$ тоже принадлежит к классу S. Данная работа дает оценку $|a_2|$ в некоторых подклассах функций би-однолистных.