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On the Convergence of Solutions of Certain Generalized Functional-Differential Equations

O zbieżności rozwiązań pewnych równań kotyngensowo-funkcjonałowych

О сходимости решений некоторых дифференциально-функциональных включений.

In this paper we show that the main results of J. Błaż [2] may be extended for generalized functional-differential equations. We shall prove three theorems which are the counterpart of theorems 1–3 in [2].

I. We accept the following notations and symbols:
$r < 0$ is a fixed real number, $R^+ = [0, \infty)$, R^n denotes a n-dimensional Euclidean space with the usual norm $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, where $x = (x_1, x_2, \dots, x_n)$, θ denotes the origin of R^n and $\{\theta\}$ denotes the subset of R^n, whose unique element is θ.
For $A, B \subset R^n$

$$\sigma(x, A) = \inf_{y \in A} |x - y|$$

$$d(A, B) = \max\{\sup_{x \in A} \sigma(x, B), \sup_{y \in B} \sigma(y, A)\}.$$

$\mathrm{Conv}\, R^n$ is the family of all convex compact and nonempty subsets of R^n. This family is metrized by the Hausdorff distance d. C is the space of all continuous functions φ: $[r, \infty) \to R^n$ with topology defined by an almost uniform convergence on $[r, \infty)$ (i.e. an uniform convergence on each compact subinterval of interval $[r, \infty)$). $[\varphi]_v$ denotes the function φ localized to the interval $[r, v]$, $\|\varphi\|_v = \max_{r \leqslant s \leqslant v} |\varphi(s)|$.

The set of all functions $[\varphi]_v$, where $\varphi \in C$ and $v \geqslant 0$, will be denoted by $\mathfrak{C}$. In this set set we introduce the metric as follows: by the distance two functions $[\varphi]_v$ and $[\psi]_w$ we mean the distance of graph of these functions (the graph being a subset of $R \times R^n$) in the Hausdorff sense (the so-called graph topology).

Let F be a multivalued mapping (in the abbreviation m.v.m.), $F\colon R^+\times\mathfrak{C}\to \operatorname{Conv} R^n$, let $\nu\colon R^+\to R^+$ and let $[\xi]_0\in\mathfrak{C}$.

We shall investigate the existence of solutions for two problems concerning the generalized functional-differential equations:

$$(1)\qquad \begin{cases}\varphi'(t)\in F(t,[\varphi]_{\nu(t)}), & 0\leqslant t,\\ \varphi(t)=\xi(t), & r\leqslant t\leqslant 0,\end{cases}$$

and

$$(M_i)\qquad \begin{cases}\varphi'_{M_i}(t)\in F(t,[\varphi_{M_i}]_{\nu(t)}), & 0\leqslant t\leqslant M_i,\\ \varphi_{M_i}(t)=\varphi_{M_i}(M_i), & M_i\leqslant t,\\ \varphi_{M_i}(t)=\xi(t), & r\leqslant t\leqslant 0,\end{cases}$$

where M_i is an arbitrary, but fixed, positive number. By a solution of (1) we mean any function $\varphi\in C$, which is absolutely continuous on each compact subinterval of the interval $R^+=[0,\infty)$, $\varphi'(t)\in F(t,[\varphi]_{\nu(t)})$ a.e. $t\geqslant 0$ (the abbreviation a.e.t is used for for almost every t in the Lebesque measure sense), $\varphi(t)=\xi(t)$, $r\leqslant t\leqslant 0$. Similarly, a solution of (M_i) is any function $\varphi_{M_i}\in C$ which is absolutely continuous on $[0, M_i]$, $\varphi'_{M_i}(t)\in F(t,[\varphi_{M_i}]_{\nu(t)})$ a.e.t, $0\leqslant t\leqslant M_i$, $\varphi_{M_i}(t)=\varphi_{M_i}(M_i)$ for $t\geqslant M_i$ and $\varphi_{M_i}(t)=\xi(t)$ for $r\leqslant t\leqslant 0$.

II. Assume the following:

1° The function ν is continuous and $\nu(t)\geqslant t$, $t\geqslant 0$.

2° The m.v.m. F satisfies conditions

a) $F(\cdot,[\varphi]_\nu)$ is Lebesque measurable for each $[\varphi]_v\in\mathfrak{C}$, *)

b) $F(t,\cdot)$ is continuous for each $t\geqslant 0$ and there exists a continuous function $L\colon R^+\to R^+$ such that

$$d\big(F(t,[\varphi]_{\nu(t)}),F(t,[\psi]_{\nu(t)})\big)\leqslant L(t)\,\|\varphi-\psi\|_{\nu(t)}\quad \text{for each } t\geqslant 0,$$

3° There exists a constant k, $k>1$, such that

$$d\big(F(t,[0]_{\nu()}),\{\theta\}\big)\leqslant kL(t),\ t\geqslant 0.$$

4° The following inequality holds

$$\sup_{0\leqslant t}\frac{e^{k\int_t^{\nu(t)}L(s)ds}}{k}=q<1\ \ **)$$

*) We say that a m.v.m. $G\colon R^+\to\operatorname{conv}(R^n)$ is Lebesque measurable iff the set $\{t\in R^+\colon G(t)\cap B\neq\emptyset\}$ is Lebesque measurable for each closed subset $B\subset R^n$.

**) Throughout this paper integrals are understood in the Lebesque sense.

III. Let p be constant such that

$$\frac{\|\xi\|_0+1}{1-q}\leqslant p.$$

Denote by C^* the family of all functions $\varphi\in C$ satisfying the condition

$$\|\varphi\| = \sup_{0\leqslant t}\frac{\|\varphi\|_t}{e^{k\int_0^t L(s)ds}}\leqslant p.$$

It is easy to verify that the set C^* with a norm $\|\cdot\|$ is a complete metric space.

We now state the following theorem:

Theorem 1. *If the hypotheses* $1^\circ-4^\circ$ *are fulfilled, then the problem* (1) *has at least one solution which belongs to* C^*.

Proof. Let us consider the m.v.m. Γ defined in C^* by formula

$$\Gamma\varphi = \left[\tilde{\varphi}\colon \tilde{\varphi}(t) = \begin{cases}\xi(0)+\int_0^t x(s)\,ds, & t\geqslant 0,\ \text{where } x \text{ is a Lebesque measurable selector of } F(\cdot,[\varphi]_{\nu(\cdot)})\\ \xi(t), & r\leqslant t\leqslant 0.\end{cases}\right]$$

It follows from Bridgland's Lemma [3, Lemma 2.8] (cf also [4]) that the m.v.m. $F(\cdot,[\varphi]_{\nu(\cdot)})\colon R^+\to \operatorname{Conv}R^n$ is measurable. Then in view of Kuratowski-Ryll-Nardzewski theorem [6] there exists a measurable selector x of $F(\cdot,[\varphi]_{\nu(\cdot)})$. Thus $\Gamma\varphi$ is nonempty for each $\varphi\in C^*$. Using the Bridgland's theorem [3, Theorem 3.1] we conclude that $\Gamma\varphi$ is closed in C^* for $\varphi\in C^*$. To show the inclusion $\Gamma\varphi\subset C^*$, $\varphi\in C^*$, first let us observe that

$$\begin{aligned} d\big(F(t,[\varphi]_{\nu(t)}),\{\theta\}\big) &\leqslant d\big(F(t,[\varphi]_{\nu(t)}),F(t,[0]_{\nu(t)})\big)+d\big(F(t,[0]_{\nu(t)}),\{\theta\}\big)\\ &\leqslant L(t)\|\varphi-0\|_{\nu(t)}+kL(t) = L(t)\|\varphi\|_{\nu(t)}+kL(t), \quad t\geqslant 0\end{aligned}$$

and let us choose arbitrary $\tilde{\varphi}\in\Gamma\varphi$. We have for $t\geqslant 0$

$$\tilde{\varphi}(t) = \xi(0)+\int_0^t x(s)\,ds.$$

Since $x(t)\in F(t,[\varphi]_{\nu(t)})$ a.e. $t\geqslant 0$, then following closely as in ([2], see the proof of Theorem 1]) we obtain for $t\geqslant 0$

(3)

$$|\tilde{\varphi}(t)|\leqslant|\xi(0)|+\int_0^t|x(s)|\,ds\leqslant|\xi(0)|+\int_0^t\big(L(s)\|\varphi\|_{\nu(t)}+kL(s)\big)\,ds\leqslant pe^{k\int_0^t L(s)ds}.$$

Obviously for $r \leqslant t \leqslant 0$

$$|\tilde{\varphi}(t)| = |\xi(t)| \leqslant \|\xi\|_0, \tag{4}$$

So in view of (3) and (4) $\|\tilde{\varphi}\| \leqslant p$. Consequently $\Gamma\varphi \subset C^*$. Now we shall prove that Γ is a contraction with constant q, i.e. that $D(\Gamma\varphi, \Gamma\psi) \leqslant q\|\varphi-\psi\|$ for each $\varphi, \psi \in C^*$, where D is the Hausdorff metric (in a family of all nonempty closed subsets of C^*) generated by the norm $\|\cdot\|$.

Let $\varphi, \psi \in C^*$, $\varphi \neq \psi$, and let $\tilde{\varphi} \in \Gamma\varphi$. Then, for $t \geqslant 0$,

$$\tilde{\varphi}(t) = \xi(0) + \int_0^t x(s)\,ds,$$

where x is measurable and $x(t) \in F(t, [\varphi]_{r(t)})$ a.e. $t \geqslant 0$. Since $d\big(F(t, [\varphi]_{r(t)}), F(t, [\psi]_{r(t)})\big) \leqslant L(t)\|\varphi-\psi\|_{r(t)}$, there is $y_t \in F(t, [\psi]_{r(t)})$ such that $|x(t) - y_t| \leqslant L(t)\|\varphi-\psi\|_{r(t)}$.

Let us put $K(t) = \{y_t \in R^n\colon |x(t)-y_t| \leqslant L(t)\|\varphi-\psi\|_{r(t)}\}$. $K(t)$ is a nonempty closed convex set and the m.v.m. $K\colon R^+ \to \operatorname{conv} R^n$ is Lebesque measurable. Then the m.v.m. $G\colon R^+ \to \operatorname{conv} R^n$ defined by $G(t) = \theta = F(t, [\psi]_{r(t)}) \cap K(t)$ is also Lebesque measurable (cf for example [4]). Let z be a measurable selector for G. Then we have $z(t) \in F(t, [\psi]_{r(t)})$ a.e. $t \geqslant 0$ and

$$|x(t) - z(t)| \leqslant L(t)\|\varphi-\psi\|_{r(t)} \quad \text{a.e. } t \geqslant 0.$$

Now define a function $\tilde{\psi}\colon [r, \infty) \to R^n$ by

$$\tilde{\psi}(t) = \begin{cases} \xi(0) + \int_0^t z(s)\,ds, & t \geqslant 0, \\ \xi(t), & r \leqslant t \leqslant 0. \end{cases}$$

Obviously $\tilde{\psi} \in \Gamma\psi$ and $|\tilde{\varphi}(t) - \tilde{\psi}(t)| = 0$ for $r \leqslant t \leqslant 0$. For $t \geqslant 0$

$$|\tilde{\varphi}(t) - \tilde{\psi}(t)| \leqslant \int_0^t |x(s) - z(s)|\,ds \leqslant \int_0^t L(s)\|\varphi-\psi\|_{r(s)}\,ds$$

and further identically as in [2, see the proof of Th. 1] we obtain the inequality

$$|\tilde{\varphi}(t) - \tilde{\psi}(t)| \leqslant q\|\varphi-\psi\| e^{k\int_0^t L(s)ds}.$$

Hence $\|\tilde{\varphi} - \tilde{\psi}\| \leqslant q\|\varphi-\psi\|$.

From this, and the analogues inequality obtained by interchanging the roles of φ and ψ, we get $D(\Gamma\varphi, \Gamma\psi) \leqslant q\|\varphi-\psi\|$.

So we see that the m.v.m. Γ fulfills all hypotheses of the contraction principle of Covitz and Nadler [5, Corollary 3] (Γ maps the complete metric space C^* into the family of all nonempty closed subset of C^* and is the contraction with constant $q < 1$) Therefore, there exists a function $\varphi \in C^*$ such that $\varphi \in \Gamma\varphi$ what means that

$$\begin{cases} \varphi'(t) \in F(t, [\varphi]_{r(t)}) & \text{a.e. } t \geqslant 0, \\ \varphi(t) = \xi(t), & r \leqslant t \leqslant 0. \end{cases}$$

This completes the proof of our Theorem.

IV⁰ Let

$$C_{M_i} = \{\varphi \in C\colon \varphi(t) = \varphi(M_i) \quad \text{for } t \geqslant M_i\},$$
$$C^*_{M_i} = \{\varphi \in C^*\colon \varphi(t) = \varphi(M_i) \quad \text{for } t \geqslant M\ \}.$$

Similarly as C^* in Theorem 1, the set $C^*_{M_i}$ with a norm $\|\cdot\|$ given by (2) is a complete metric space.

Define on $C^*_{M_i}$ the m.v.m. Γ by formula

$$\Gamma_{M_i}\varphi = \left[\tilde{\varphi}_{M_i}\colon \tilde{\varphi}_{M_i}(t) = \begin{cases} \xi(0) + \int\limits_0^t x_{M_i}(s)\,ds, & 0 \leqslant t \leqslant M_i, \\ \xi(0) + \int\limits_0^{M_i} x_{M_i}(s)\,ds, & M_i \leqslant t, \\ \xi(t), & r \leqslant t \leqslant 0. \end{cases} \quad \text{where } x_{M_i} \text{ is a measurable selector of } F(\cdot, [\varphi_{M_i}]_{r(\cdot)}), \right]$$

Considering this mapping in the same way as in previously section we get the following

Theorem 2. *If hypotheses* 1° – 4° *are fulfilled, the problem* (M_i) *has at least one solution which belongs to* $C^*_{M_i}$.

Remark 1. If F is a single-valued mapping, then the problems (1) and (M_i) have exactly one solution. This it follows immediately from proof of these theorems.

V⁰ In this section we prove a theorem which is a generalization of Theorem 3 in [2].

Theorem 3. *Let* $\{M_i\}_{i=1}^{\infty}$ *be an increasing sequence of real numbers such that* $\lim\limits_{i\to\infty} M_i = +\infty$.

a) *If* $\{\varphi_{M_i}\}_{i=1}^{\infty}$ *is a sequence of solutions of problems* (M_i) (*in* $C^*_{M_i}$ *respectively*), *then there exists a subsequence* $\{\varphi_{M_{i_j}}\}_{j=1}^{\infty}$ *which is uniformly convergent on each compact subinterval of* $[r, \infty)$ *to a function* φ *and* φ *is a solution* (*in* C^*) *of problem* (1).

b) *If* φ_0 *is a solution* (*in* C^*) *of problem* (1), *then there exists a sequence* $\{\varphi_{M_i}\}_{i=1}^{\infty}$ *of solutions* (*in* $C^*_{M_i}$ *respectively*) *of problems* (M_i), *which is uniform convergent on each compact subinterval of* $[r, \infty)$ *to the function* φ_0.

Proof. a) It is easy to verify that

$$|\varphi_{M_i}(t)| \leqslant pe^{k\int_0^t L(s)ds}, \quad r \leqslant t, \quad i = 1, 2, \ldots$$

and

$$|\varphi_{M_i}(t+h) - \varphi_{M_i}(t)| \leqslant \int_0^t \big(L(s)\,\|\varphi_{M_i}\|_{\nu(s)} + kL(s)\big)ds, \; 0 \leqslant t, \; 0 < h, \; i = 1, 2, \ldots$$

Since $\varphi_{M_i}(t) = \xi(t)$ for $r \leqslant t \leqslant 0$, consequently the functions φ_{M_i} are uniformly continuous on each compact subinterval of $[r, \infty)$. Thus, by well-known Arzela's theorem, there exists a subsequence $\{\varphi_{M_{ij}}\}_{j=1}^{\infty}$ which is almost uniformly convergent on $[r, \infty)$ to some function φ. Obviously $\varphi \in C^*$.

To prove that φ is the solution of (1), it suffices to show that φ satisfies the equation $\varphi'(t) \in F(t, [\varphi]_{\nu(t)})$ almost everywhere on each compact interval $[0, T] \subset R^+$.

Let us fix arbitrary $T > 0$ and let us define m.v. mappings G_j: $[0, T] \to$ $\to \operatorname{conv} R^n$ and G: $[0, T] \to \operatorname{conv} R^n$ by formulas

$$G_j(t) = F(t, [\varphi_{M_{ij}}]_{\nu(t)}), \quad 0 \leqslant t \leqslant T,$$

$$G(t) = F(t, [\varphi]_{\nu(t)}), \quad 0 \leqslant t \leqslant T.$$

Since $\varphi_{M_{ij}}$ converges uniform to φ in $[0, T^*]$, where $T^* = \max_{0 \leqslant t \leqslant T} \nu(t)$ we conclude that $\lim_{j\to\infty} d\big(G_j(t), G(t)\big) = 0$ on $[0, T]$.

From this it follows that

$$\sigma\big(\varphi'_{M_{ij}}(t), G(t)\big) \to 0 \quad \text{a.e. } t \in [0, T].$$

By virtue of Pliś's Lemma [6, Lemma 1] we get

$$\varphi'(t) \in G(t) = F(t, [\varphi]_{\nu(t)}) \quad \text{a.e. } t \in [0, T].$$

Therefore the proof of the part a) is completed.

Remark 2. In the case when F is a single-valued mapping the whole sequence $\{\varphi_{M_i}\}_{i=1}^{\infty}$ of solutions of (M_i) (which in view of Remark 1 are unique) converges to a solution of (1).

b) Now let φ_0 be a solution of (1). Let us define

$$f(t, [\chi]_\nu) = \begin{cases} \operatorname{Proj}\big(\varphi_0'(t)/F(t, [\chi]_\nu)\big) & \text{for } (t, [\chi]_\nu) \in R^+ \times \mathfrak{C} \text{ and if } \varphi_0'(t) \text{ exists}, \\ \operatorname{Proj}\big(\theta/F(t, [\chi]_\nu)\big) & \text{otherwise}, \end{cases}$$

where $\mathrm{Proj}(z/K)$ denotes the metric projection a point $z \in R^n$ onto a nonempty compact convex subset K of R^n, i.e.

$$\mathrm{Proj}(z/K) = \{y \in K\colon |y-z| = \inf_{k \in K} |k-z|\}.$$

Obviously f is the single-valued mapping.
According to the result in [1, Chapter VI, ν 3, Th. 3] f is continuous in $[\chi]_v$ for each fixed $t \in R^+$ and by Castaing's theorem [4, Th. 5.1] f is Lebesque measurable in $t \geqslant 0$ for each fixed $[\chi] \in \mathfrak{C}$. Moreover f satisfies the hypotheses 2° b) and 3°. Therefore the following problems

$$(1,f) \qquad \begin{cases} \varphi'(t) = f(t, [\varphi]_{\nu(t)}), & 0 \leqslant t, \\ \varphi(t) = \xi(t), & r \leqslant t \leqslant 0, \end{cases}$$

and

$$(M_i,f) \qquad \begin{cases} \varphi'_{M_i}(t) = f(t, [\varphi_{M_i}]_{\nu(t)}), & 0 \leqslant t \leqslant M_i, \\ \varphi_{M_i}(t) = \varphi_{Mi}(M_i), & M_i \leqslant t, \\ \varphi_{M_i}(t) = \xi(t), & r \leqslant t \leqslant 0, \end{cases}$$

have, in view of our Remark 1, exactly one solutions $\bar{\varphi}$ and $\bar{\varphi}_{M_i}$ and, by our Remark 2, $\bar{\varphi}_{M_i}$ converges to $\bar{\varphi}$. But the function φ_0 is the solution of $(1,f)$ too, because

$$f(t, [\varphi_0]_{\nu(t)}) = \mathrm{Proj}\big(\varphi_0'(t)/F(t, [\varphi_0]_{\nu(t)})\big) = \varphi_0'(t) \text{ a.e. } t \geqslant 0.$$

Thus must be $\bar{\varphi} = \varphi_0$.
Similarly the functions $\bar{\varphi}_{M_i}$ are solutions of problems (M_i) because

$$\bar{\varphi}'_{M_i}(t) = f(t, [\bar{\varphi}_{M_i}]_{\nu(t)}) \in F(t, [\bar{\varphi}_{M_i}]_{\nu(t)}) \text{ a.e. } t \in [0, M_i].$$

This proves the part b) and finally the proof of our Theorem 3 is completed.

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STRESZCZENIE

W pracy podano trzy twierdzenia. Pierwsze dwa — to twierdzenia o istnieniu rozwiązania w klasie C^* równań (1) i (M_i). Twierdzenie trzecie jest następujące:

a) Jeśli $\{\varphi_{M_i}\}_{i=1}^{\infty}$ jest ciągiem rozwiązań (w klasie C^*) równań (M_i), to istnieje podciąg $\{\varphi_{Mij}\}_{j=1}^{\infty}$ tego ciągu, który na każdym zwartym podprzedziale przedziału $[r, \infty)$ jest jednostajnie zbieżny do rozwiązania φ (w klasie C^*) równania (1).

b) Jeżeli φ jest rozwiązaniem (w klasie C^*) równania (1), to istnieje ciąg $\{\varphi_{M_i}\}_{i=1}^{\infty}$ rozwiązań (w klasie C^*) równań (M_i) zbieżny jednostajnie na każdym zwartym podporzedziale przedziału $[r, \infty)$ do funkcji φ.

РЕЗЮМЕ

В работе даны три теоремы. Две первые — это теоремы о существовании решения в классе C^* уравнений (1) и (M_i). Теорема третья следующая: Теорема 3: а) Если $\{\varphi_{M_i}\}_{i=1}^{\infty}$ последовательность решений (в классе C^*) уравнений (M_i), то из этой последовательности можно выделить подпоследовательность $\{\varphi_{Mij}\}г_{j=1}^{\infty}$, равномерно сходящиеся на каждом компактном интервале луча $[r, \infty)$ к решению φ (в классе C^*) уравнения (1).

в) Если φ решение (в классе C^*) уравнения (1), то существует последовательность $\{\varphi_{M_i}\}_{i=1}^{\infty}$ решений (в классе C^*) уравнений (M_i), равномерно сходящаяся на каждом компактном интервале луча $[r, \infty)$ к решению φ.

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O pewnym równaniu paratyngensowym z odchylonym argumentem.
15. W. Zygmunt: On Some Properties of a Certain Family of Solutions of Paratingent Equation.
O kilku własnościach pewnej rodziny rozwiązań równania paratyngensowego.

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