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On a Certain Paratingent Equation with a Deviated Argument

O pewnym równaniu paratyngensowym z odchylonym argumentem

О некотором паратингентном уравнении с отклоненным аргументом

In this paper we shall prove a theorem on the existence of solutions of a paratingent equation of the form

$$(Ptx)(t) \subset F([x]_{\nu(t)}) \text{ for } t \geqslant 0$$

with the initial condition

$$x(t) = \xi(t) \text{ for } t \leqslant 0 .$$

Preliminaries

Let $\operatorname{Comp} E$ denote the set of all compact and nonempty subsets of a metric space E. If additionally E is a linear space, $\operatorname{Conv} E$ denotes the set of all elements of $\operatorname{Comp} E$ which are convex. Having two metric spaces E and E', a mapping Γ: $E \to \operatorname{Comp} E'$ is called upper semi-continuous (usc) when for each point $a \in E$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that the inclusion $\Gamma(x) \subset K(\Gamma(a), \varepsilon)$ holds for all $x \in K(a, \delta)$. $K(a, \delta) = \{x \in E: \varrho(x, a) < \delta\}$, ϱ – a metric in E. $K(\Gamma(a), \delta) = \{y \in E': \text{there exists } z \in \Gamma(a) \text{ such that } \varrho'(y, z) < \varepsilon\}$, ϱ' – a metric in E'. The following fact has been established in [6] (Proposition 4.1).

Lemma 1. *Let E and E' be two metric spaces. A mapping Γ: $E \to \operatorname{Comp} E'$ is usc if and only if for all sequences $\{x_i\} \in E$, $\{y_i\} \in E'$ such that $x_i \to x_0$ and $y_i \in \Gamma(x_i)$, $i = 1, 2, \ldots$, there exists a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ which is convergent to y_0 and $y_0 \in \Gamma(x_0)$.*

Let R be the real line, R^n be the n-dimensional Euclidean space with norm $|x| = \max(|x_1|, \ldots, |x_n|)$ where $R^n \ni x = (x_1, \ldots, x_n)$ Let C be the space of all continuous functions φ: $R \to R^n$ with the topology defined by an almost uniform convergence (i.e. a uniform convergence on each compact interval of R). It is well known that the almost uniform convergence in C is equivalent to the convergence by metric d defined as follows

$$d(\varphi, \psi) := \sum_{i=1}^{\infty} \frac{1}{2^i} \min(1, \sup_{-i \leqslant s \leqslant i} |\varphi(s) - \varphi(s)|),$$

for $\varphi, \psi \in C$.

Then C is a metric locally convex linear topological space. Let $\beta < 0$ be a fixed real number and let $I = \langle 0, \infty) \subset R$. Given a function $\varphi \in C$ the symbol $[\varphi]_t$ will denote the function $\varphi|_{\langle \beta, t\rangle}$ (i.e. φ localized within the interval $\langle \beta, t\rangle$) where $t \in I$, and the symbol $\|\varphi\|_t$ will denote the maximum of $|\varphi(s)|$in $\langle \beta, t\rangle$, i.e. $\|\varphi\|_t := \max_{\beta \leqslant s \leqslant t} |\varphi(s)|$.

Let $\mathfrak{C}$ denote the metric space the elements of which are functions $[\varphi]_t$, $[\psi]_u$ etc and the distance $\varrho([\varphi]_t, [\psi]_u)$ between the two functions $[\varphi]_t$ and $[\psi]_u$ in $\mathfrak{C}$ being understood as a distance of graphs of these functions (the graph being subsets of $R \times R^n$) in the Hausdorff sense.
A more detailed study on the properties of the space $\mathfrak{C}$ can be found in [7]. The following lemma will be most useful for us:

Lemma 2. *Let $\varphi_i, \varphi \in C$, $i = 1, 2, \ldots$. If $\varphi_i \to \varphi$, then to every $\varepsilon > 0$ there exist $\delta > 0$ and $N > 0$ such that the inequality $\varrho([\varphi_i]_{t_1}, [\varphi]_{t_2}) < \varepsilon$ holds for all $t_1, t_2 \in (T - \delta, T + \delta) \cap I$and $i \geqslant N$.*

Proof. Let us fix $T \in I$ and choose anarbitrary $\varepsilon > 0$. Since the function φ is continuous, there exists $\delta > 0$ such that

$$|\varphi(\tau) - \varphi(\sigma)| < \varepsilon/2 \quad \text{for } \tau, \sigma \in (T - \delta, T + \delta) \cap I.$$

Hence it follows immediately that

$$\varrho([\varphi]_{t_1}, [\varphi]_{t_2}) < \varepsilon/2 \quad \text{for } t_1, t_2 \in (T - \delta, T + \delta) \cap I.$$

Since $\varphi_i \to \varphi$, then the sequence $\{\varphi_i\}$ is uniformly convergent to a function φ on the interval $\langle \beta, T + \delta\rangle$, in particular. Thus there exists $N > 0$ such that $|\varphi_i(s) - \varphi(s)| < \varepsilon/2$ for $s \in \langle \beta, T + \delta\rangle$ and $i \geqslant N$. Then we obviously have

$$\varrho([\varphi_i]_t, [\varphi]_t) < \varepsilon/2 \quad \text{for } t \in \langle 0, T + \delta\rangle \text{ and } i \geqslant N.$$

Finally for $t_1, t_2 \in (T - \delta, T + \delta) \cap I$ and $i \geqslant N$ we have

$$\varrho([\varphi_i]_{t_1}, [\varphi]_{t_2}) \leqslant \varrho([\varphi_i]_{t_1}, [\varphi]_{t_1}) + \varrho([\varphi]_{t_1}, [\varphi]_{t_2}) \leqslant \varepsilon$$

which completes the proof of our lemma.

Having a function $\varphi \in C$ and with $t \in I$ the set of all limit points

$$x = \frac{\varphi(t_i) - \varphi(s_i)}{t_i - s_i}$$

where $s_i, t_i \in I$, $s_i \to t$, $t_i \to t$ and $s_i \neq t_i$ $(i = 1, 2, \ldots)$, is called the paratingent of φ at the point t and denoted by $(Pt\varphi)(t)$. It is easy to see that $Pt\varphi\colon I \to R^n$ maps the interval I into the family of the nonempty closed subsets of R^n (cf. [3], [10]). By the paratingent equation we understood a relation

$$(*) \qquad (Ptx)(t) \subset F([x]_{\nu(t)}), \quad t \in I$$

where a mapping $F\colon \mathfrak{C} \to \operatorname{Comp} R^n$ is usc and ν is nonnegative, real-valued, continuous function defined on I. Every function $\varphi \in C$ satysfying (*) will be called the solution of these equation.

The main theorem

Let $\nu(t) \geqslant 0$, $M(t) \geqslant 0$ and $N(t) \geqslant 0$ be real-valued continuous functions defined on the interval I, let $0 < a \leqslant 1$ be a fixed number and let

$$(1) \qquad \Lambda(t) = \int_0^t L(u)\,du, \quad \text{where} \quad L(t) = M(t) + N(t).$$

Let us assume that

$$(2) \qquad \Lambda(\nu(t)) \leqslant a^{-1}\big(\Lambda(t + e^{-1})\big).$$

Let $\xi \in C$ and $A \geqslant \max(1, \|\xi\|_0)$ be a fixed number. Furthemore, let us assume that a mapping $F\colon \mathfrak{C} \to \operatorname{Conv} R^n$ is usc satisfying the condition

$$(3) \qquad F([x]_t) \subset K\big(\theta, M(t) + N(t)(\|x\|_t)^a\big) \quad \text{for} \quad t \in I, \ \theta \text{ — an origin}$$

of R^n. Then there exists a function $\varphi \in C$ such that

$$(4) \qquad (Pt\varphi)(t) \subset F([\varphi]_{\nu(t)}) \quad \text{for } t \geqslant 0$$

and

$$(5) \qquad \varphi(t) = \xi(t) \quad \text{for } t \leqslant 0.$$

This solution φ of our paratingent equation satisfies the inequality

$$(6) \qquad |\varphi(t)| \leqslant A \exp[e\Lambda(t)] \quad \text{for } t \geqslant 0.$$

Before proving this theorem we shall give some lemmas.

Lemma 3 (cf lemma 3 in [7]). *If $\varphi, \psi \in C$ and*

$$(Pt\psi)(t) \subset K\big(\theta, M(t) + N(t)(\|\varphi\|_{\nu(t)})^a\big) \quad \textit{for } t \geqslant 0$$

then for all $t \geqslant 0$ *and* $h > 0$

$$|\psi(t+h)-\psi(t)| \leqslant \int_t^{t+h} \big(M(u)+N(u)(\|\varphi\|_{\nu(u)})^a\big)\,du. \tag{7}$$

Proof. It is completely analogous to the proof of lemma 3 in [7].

Lemma 4. *Let* $\varphi, \psi, \varphi_i, \psi_i \in C$ $(i = 1, 2, \ldots)$. *If* $\varphi_i \to \varphi$, $\psi_i \to \psi$ *and*

(a) $(Pt\psi_i)(t) \subset F([\varphi_i]_{\nu(t)})$ *for* $t \geqslant 0$ *and* $i = 1, 2, \ldots$

(b) $\psi_i(t) = \xi(t)$ *for* $t \leqslant 0$ *and* $i = 1, 2, \ldots,$

then $(Pt\psi)(t) \subset F([\varphi]_{\nu(t)})$ *for* $t \geqslant 0$

$$\psi(t) = \xi(t) \quad \text{for } t \leqslant 0.$$

Proof. The second condition is obvious. To prove that the first condition is satisfied let us fix $t_0 \in I$ and choose an arbitrary $\varepsilon > 0$. Let $T = \nu(t_0) \geqslant 0$. From the continuity of function $\nu(t)$, lemma 2 and the upper semi-continuity of the mapping F it follows that there exists a neighbourhood $\theta(t_0)$ of the point t_0 and number $N > 0$ such that

$$(Pt\psi_i)(t) \subset F([\varphi_i]_{\nu(t)}) \subset K\big(F([\varphi]_T), \varepsilon\big) \quad \text{for } t \in \theta(t_0) \cap I,\ i \geqslant N.$$

Since the sequence of functions $\{\psi_i\}$ is uniformly convergent to ψ on the same set $\theta(t_0)$, in view of lemma 8 in [7] (cf also Theorem 2.6 in [10] and [4]) we obtain

$$(Pt\psi)(t) \subset K(F([\varphi]_T), \varepsilon) \quad \text{for } t \in \theta(t_0) \cap I.$$

In particular we have

$$(Pt\psi)(t_0) \subset K(F([\varphi]_T), \varepsilon)$$

and, owing to the optionality of ε, we conclude

$$(Pt\psi)(t_0) \subset F([\varphi]_{\nu(t_0)}).$$

Thus the first condition is satisfied and in this way lemma 4 is proved.

Lemma 5. *Let* $\varphi, \psi \in C$ *and* $G(t) = F([\varphi]_{\nu(t)})$ *for* $t \geqslant 0$. *Then following statements are equivalent:*

(c_1) $(Pt\psi)(t) \subset G(t)$ *for* $t \geqslant 0$

(c_2)

$$\bigwedge_{t \in I} \bigwedge_{\varepsilon > 0} \bigvee_{\delta > 0} \bigwedge_{\substack{\tau, \sigma \in I \\ \tau \neq \sigma}} \left\{ (|\tau - t| < \delta \text{ and } |\sigma - t| < \delta) \Rightarrow \frac{\varphi(\tau) - \varphi(\sigma)}{\tau - \sigma} \in K(G(t), \varepsilon) \right\}.$$

Proof. It is easy to see that the mapping G is usc on I and the implication $(c_2)\Rightarrow(c_1)$ is obvious. To prove that the implication $(c_1)\Rightarrow(c_2)$ holds, let us suppose that the condition (c_2) is not satisfied. Thus

$$\bigvee_{t_0\in I}\bigvee_{\varepsilon_0>0}\bigwedge_{\delta>0}\bigvee_{\substack{\tau,\sigma\in I\\ \tau\neq\sigma}}|\zeta-t|<\delta \quad\text{and}\quad |\sigma-t|<\delta \quad\text{and}$$
$$\frac{\varphi(\tau)-\varphi(\sigma)}{\tau-\sigma}\notin\overline{K(G(t_0),\varepsilon_0)}.$$

Puting $\delta=1/i$, $i=1,2,\ldots$, we can choose sequences $\{\tau_i\}\subset I$, $\{\sigma_i\}\subset I$ such that $\tau_i\to t_0$, $\sigma_i\to t_0$, $\sigma_i\neq\tau_i$ and

$$\frac{\varphi(\tau_i)-\varphi(\sigma_i)}{\tau_i-\sigma_i}\notin\overline{K(G(t_0),\varepsilon_0)},\quad i=1,2,\ldots.$$

On the other hand, from the upper semicontinuity of the mapping G and in view of Lemma 9 in [7] (cf also Lemma 6 in [3] and Lemma 2.5 in [10]) it follows that the difference quotients $[\varphi(\tau_i)-\varphi(\sigma_i)]/(\tau_i-\sigma_i)$ are uniformly bounded. Then there exist subsequences $\{\tau_{i_j}\}\subset\{\tau_i\}$ and $\{\sigma_{i_j}\}\subset\{\sigma_i\}$ such that

$$\lim_{j\to\infty}\frac{\varphi(\tau_{i_j})-\varphi(\sigma_{i_j})}{\tau_{i_j}-\sigma_{i_j}}\in(Pt\varphi)(t_0)\not\subset G(t_0).$$

But this contradicts the condition (c_1). Thus there must be $(c_1)\Rightarrow(c_2)$.

Lemma 6. *Let $\varphi\in C$ and $G(t)=F([\varphi]_{\nu(t)})$ for $t\geqslant 0$. There exists a function $\psi\in C$ such that*

$$(Pt\psi)(t)\subset G(t)\quad \textit{for } t\geqslant 0$$

and

$$\psi(t)=\xi(t)\quad \textit{for } t\leqslant 0.$$

Proof. Since the mapping G is usc on I, there exists a measurable selection g of G (cf [8], Theorem in § 2) such that $g(t)\in G(t)$ for $t\geqslant 0$. Defining

$$\psi(t)=\begin{cases}\xi(0)+\int_0^t g(s)\,ds & \text{for } t\geqslant 0\\ \xi(t) & \text{for } t\leqslant 0\end{cases}$$

we conclude that $\psi(t)$ is an absolutely continuous function for $t\geqslant 0$ and then obviously the relation $\psi'(t)\in G(t)$ holds a.e. (= almost everywhere) in I. We shall show that $(Pt\psi)(t)\subset G(t)$ for all $t\in I$. Let us fix arbitrary $t_0\in I$. From the upper semicontinuity of G it follows that to any given

$\varepsilon > 0$ there is $\delta > 0$ such that the condition $|t_0 - t| \leqslant \delta$, $t \in I$, implies $G(t) \subset K(G(t_0), \varepsilon)$. Hence $\psi'(t) \in \overline{H}(G(t_0), \varepsilon)$ a.e. in $Q(t_0) = \{t \in I : |t_0 - t| \leqslant \delta\}$ and by the Ważewski's lemma (Lemma in [9])

$$\frac{\psi(\tau)-\psi(\sigma)}{\tau-\sigma} \in \overline{K(G(t_0), \varepsilon)} \text{ for all } \tau, \sigma \in Q(t_0),\ \tau \neq \sigma.$$

Therefore in view of our Lemma 5 we obtain $(Pt\psi)(t_0) \subset G(t_0)$. Since t_0 is arbitrary, we have finally

$$(Pt\psi)(t) \subset G(t) \quad \text{for } t \geqslant 0$$

and

$$\psi(t) = \xi(t) \quad \text{for } t \leqslant 0.$$

Proof of the theorem. Let Φ denote a family of all functions φ belonging to C and satisfying the following three conditions

$$\text{(i)} \qquad |\varphi(t)| \leqslant A\exp[eA(t)] \quad \text{for } t \geqslant 0$$

$$\text{(7)} \qquad \text{(ii)} \qquad |\varphi(t+h)-\varphi(t)| \leqslant A\int_t^{t+h} eL(u)\exp[eA(u)]\,du \quad \text{for } t \geqslant 0,\ h > 0$$

$$\text{(iii)} \qquad \varphi(t) = \xi(t) \quad \text{for } t \leqslant 0.$$

We see at once that this family is a nonempty, compact and convex subset of the space C. Given a function $\varphi \in \Phi$, by $\mathscr{F}\varphi$ we denote the set of all functions $\psi \in C$ such that $(Pt\psi)(t) \subset F([\varphi]_{r(t)})$ for $t \geqslant 0$ and $\varphi(t) = \xi(t)$ for $t \leqslant 0$. Let us consider the correspondence $:\varphi \to \mathscr{F}\varphi$ First let us note that the inequality $\|\varphi\|_t \leqslant A\exp[eA(t)]$ for $t \geqslant 0$ is equivalent to the inequality $|\varphi(t)| \leqslant A\exp[eA(t)$ for $t \geqslant 0$, i.e. if $t \in I$ and $\|\varphi\|_t \leqslant A\exp[eA(t)]$ then $|\varphi(t)| \leqslant A\exp[eA(t)]$ and, vice versa, if $|\varphi(s) \leqslant A\exp[eA(s)]$ for $0 \leqslant s \leqslant t$ then

$$\|\varphi\|_t \leqslant A\exp[eA(t)].$$

For every $\varphi \in \Phi$ the set $\mathscr{F}\varphi$ is nonemtpy according to the Lemma 6, it is convex which is easily concluded from Lemma 5 and closed in view of Lemma 4. Similarly, if $\psi \in \mathscr{F}\varphi$, then by Lemma 3 and conditions (1), (3), (7i) and (2) we have

$$|\psi(t+h)-\psi(t)| \leqslant A\int_t^{t+h} eL(u)\exp[eA(u)]\,du \quad \text{for } t \geqslant 0,\ h > 0$$

and

$$|\psi(t)| \leqslant A\exp[eA(t)] \quad \text{for } t \geqslant 0$$

and obviously

$$\psi(t) = \xi(t) \quad \text{for } t \leqslant 0.$$

This means that $\psi \in \Phi$. Thus $\mathcal{F}\varphi \subset \Phi$.

Moreover, all functions ψ belonging to $\mathcal{F}\varphi$ are uniformly bounded and equicontinuous on each compact interval of R. Therefore in view of closedness of $\mathcal{F}\varphi$ we may conclude that $\mathcal{F}\varphi$ is compact, too. Now, we see that the correspondence $\mathcal{F}$ maps the set Φ into the family of the nonempty compact and convex subsets of Φ. We shall prove that $\mathcal{F}$ is usc on Φ. Indeed, let $\varphi_i, \varphi, \psi_i \in \Phi$, $\varphi_i \to \varphi$ and $\psi_i \in \mathcal{F}\varphi_i$, $i = 1, 2, \ldots$. In view of compactness of Φ there exists a subsequence $\{\psi_{i_j}\} \subset \{\psi_i\}$ which converges to ψ. Thence from the lemma 4 it follows immediately that $(Pt\psi)(t) \subset F([\varphi]_{\nu(t)})$ for $t \geqslant 0$ and $\psi(t) = \xi(t)$ for $t \leqslant 0$. Thus $\psi \in \mathcal{F}\varphi$ and in view of lemma 1 a correspondence $\mathcal{F}$ is usc.

Now, we see that $\mathcal{F}$ fulfils all the hypotheses of the well known theorem by Kakutani – K. Fan on a fixed point for multivalued mappings (cf [1]) Therefore, there exists a function $\varphi_0 \in \Phi$ such that $\varphi_0 \in \mathcal{F}\varphi_0$ what means that

$$\big(Pt\varphi_0\big)(t) \subset F\big([\varphi_0]_{\nu(t)}\big) \quad \text{for } t \geqslant 0,$$

$$\varphi_0(t) = \xi(t) \quad \text{for } t \leqslant 0$$

and

$$|\varphi_0(t)| \leqslant A\exp[e\Lambda(t)] \quad \text{for } t \geqslant 0.$$

Our theorem is thus proved.

Remarks

1. Conditions (2) and (3) given in the assumption of our thorem come from A. Bielecki's paper [2] on the existence of solutions of ordinary differential equation with a deviated argument. These conditions were subsequently used by T. Dłotko [5], with some modifications, showing the existence of solutions of an ordinary differential equation with an advanced argument $\varphi'(t) = f(\{\varphi\}_{t,k(t)})$ where $\{\varphi\}_{t,k(t)}$ denotes the function φ localized within interval $\langle t, k(t)\rangle$, $k(t) \geqslant t$.

2. If $\nu(t) \equiv t$, then we obtain the paratingent equation with a retarded argument which has been precisely examined by B. Krzyżowa [7]. In this case, every function $\varphi \in C$ satisfying $(Pt\varphi)(t) \subset F([\varphi]_t)$ for $t \geqslant 0$ must also fulfill the inequality $|\varphi(t)| \leqslant A\exp[e\Lambda(t)]$ for $t \geqslant 0$. But if $\nu(t) > t$ then we know nothing about the evaluation of the growth of the function φ which is the solution of the paratingent equation (4).

REFERENCES

[1] Berge Cl., *Topological Spaces*, Oliver Boyd. Edinburgh and London 1963.

[2] Bielecki A., *Certaines conditions suffissantes pour l'existence d'une solution de l'équation* $\varphi'(t) = f(t, \varphi(t), \varphi(y(t)))$, Folia Soc. Sci. Lublinensis 2 (1962), 70-73.

[3] Bielecki A., *Sur certaines conditions nécessaires et suffissantes pour l'unicité des solutions des systèmes d'équations différentielles ordinaires et des équations au paratingent*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A, 2 (1948), 49-106.

[4] Bielecki A., *Extension de la méthode du rétracte de T. Ważewski aux équations au paratingent*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A, 9 (1955), 37-61.

[5] Dłotko T., *O istnieniu rozwiązań pewnego równania różniczkowego z wyprzedzającym argumentem*, Zesz. Naukowe WSP w Katowicach, Sekcja Matematyki, 4 (1964), 79-83.

[6] Hukuhara M., *Sur l'application semi-continue dont la valeur est un compact convexe*, Funkcial. Ekvac., 10 (1967), 43-66.

[7] Krzyżowa B., *Équations au paratingent à argument retardé*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A, 17 (1965), 7-18.

[8] Kuratowski K., Ryll-Nardzewski, Cz., *A General Theorem on Selectors*, Bull. Acad. Polon. Sci. Sér. Mat., 13 (1965). 397-403.

[9] Ważewski T., *Sur une condition équivalente à l'équation au contingent*, Bull. Acad. Polon. Sci. Sér. Math., 9 (1961), 865-867.

[10] Zaremba S. K., *O równaniach paratyngensowych*, Dodatek do Rocznika Polskiego Towarzystwa Matematycznego, 9 (1935), 1-22.

STRESZCZENIE

W pracy rozważa się problem istnienia rozwiązania równania paratyngensowego z odchylonym argumentem postaci

$$(*) \qquad (Ptx)(t) \subset F([x]_{\nu(t)}), \quad t \geqslant 0$$

z warunkiem początkowym

$$x(t) = \xi(t), \quad t < 0.$$

Korzystając z twierdzenia Kakutani-Fana o punkcie stałym dowodzi się przy stosowanych założeniach o funkcjach ν, ξ i odwzorowaniu F, istnienia funkcji φ określonej na całej osi R, mającej z góry zadane wartości na przedziale $(-\infty, 0\rangle$ oraz takiej, że jej paratyngens $(Pt\varphi)(t)$ w momencie t zawiera się w zbiorze $F([x]_{\nu(t)})$. Zbiór $F([\varphi]_{\nu(t)})$ zmienia się w zależności od całego przebiegu funkcji φ na zmiennym przedziale $\langle\beta, \nu(t)\rangle$, gdzie $\beta < 0$, $\nu(t) \geqslant 0$. Rozwiązanie φ spełnia warunek

$$|\varphi(t)| \leqslant A\exp[eA(t)], \quad t \geqslant 0.$$

W przypadku, gdy $\nu(t) > t$, to równanie (*) obejmuje równania i nierówności z wyprzedzającym argumentem.

РЕЗЮМЕ

В работе рассматривается проблема существования решения паратингентного уравнения с отклоняющим аргументом вида

$$(*) \qquad (Ptx)(t) \subset F([x]_{\nu(t)}), \quad t \geqslant 0$$

с начальным условием

$$x(t) = \xi(t), \quad t < 0.$$

При помощи принципа Какутани-Фана о неподвижной точке доказывается при соответственных предположениях о функциях ν, ξ и отображении F существование функции φ, определенной на всей оси R, совпадающей на отрезке $(-\infty, 0)$ с заданной начальной функцией ξ, паратингент $(Pt\varphi)(t)$ которой в момент t включается во множество $F([\varphi]_{\nu(t)})$. Множество $F([\varphi]_{\nu(t)})$ зависит от всего течения функции ψ на переменном интервале $<\beta, \nu(t)>$ где $\beta<0$, $\nu(t) > 0$. Решение ψ удовлетворяет условию

$$|\varphi(t)| < A\exp[e\Lambda(t)], \quad t > 0.$$

В случае, когда $\nu(t) > t$ уравнение (*) охватывает диференциальные уравнения и неравенства с опереджающим аргументом.