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The Number of Distinct Zeros of the Product of a Polynomial and its Successive Derivatives

O ilości różnych zer iloczynu wielomianu i jego kolejnych pochodnych

Об оценке числа различных нулей произведения многочлена и его последовательных производных

0. Let $p(z)\left(\equiv p^{(0)}(z)\right)$ be a polynomial of degree n and let $p^{(j)}(z)$ denote the j-th derivative of $p(z)$. How many distinct zeros does the product $P(z) = \prod_{j=0}^{n-1} p^{(j)}(z)$ have? This is the essence of a question asked by T. Popoviciu. We wish to investigate this problem in the present paper. In § 1 we consider polynomials whose zeros are all real. In § 2 we allow the zeros to be complex.

The notation "$f(z) \approx g(z)$" will stand for "$f(z)$ is a constant multiple of $g(z)$".

1.0 Let $x_1, x_2, \ldots, x_m$ be the distinct zeros of a polynomial $p(x)$ of degree n with only real zeros. We suppose $x_1 < x_2 < \ldots < x_m$. If n_j is the multiplicity of the zero at x_j then

$$p(x) \approx \prod_{j=1}^{m} (x - x_j)^{n_j}, \quad \sum_{j=1}^{m} n_j = n.$$

According to Rolle's theorem there is at least one zero of $p^{(1)}(x)$ in each of the $m-1$ intervals (x_j, x_{j+1}), $j = 1, 2, \ldots, m-1$. But taking multiplicity into account $p^{(1)}(x)$ has a total of $n-m$ zeros at the points x_j, $j = 1, 2, \ldots, m$. This means that only $m-1$ of its zeros remain to be accounted for. Hence $p^{(1)}(x)$ has one and only one zero (necessarily simple) in each of the intervals (x_j, x_{j+1}), $j = 1, 2, \ldots, m-1$. We find this observation very useful in our study of the above question for polynomials with only real zeros.

1.1 If the zeros of $p(z)$ are coincident then $P(z)$ has only one distinct zero.

1.2 Let $p(z)$ be a polynomial with noncoincident zeros. If $[a, b]$ is the smallest interval containing all the zeros of $p(z)$ then both a and b are zeros of $p(z)$. Let k be the multiplicity of the zero at a and l the multiplicity of the zero at b. Since the product $(\pm 1)\prod_{j=0}^{n-1} p^{(j)}(a+b-z)$ has the same number of distinct zeros as the product $P(z) = \prod_{j=0}^{n} p^{(j)}(z)$ we may assume $k \leqslant l$. Besides, if

$$f(z) \equiv p\left(\frac{(a+b)-(a-b)z}{2}\right) \text{ then } P(z) = \prod_{j=1}^{n-1} p^{(j)}(z) \text{ and } F(z) = \prod_{j=0}^{n-1} f^{(j)}(z)$$

have the same number of distinct zeros, and hence there is no loss of generality in assuming $a = -1$, $b = 1$.

1.2.1 In the case when $k+l = n$, i.e. $p(z)$ has two distinct zeros, we distinguish the following subcases.

i) $p(z) \approx (z+1)(z-1)^{n-1}$,
ii) $p(z) \approx (z+1)^2(z-1)^{n-2}$,
iii) $p(z) \approx (z+1)^k(z-1)^l$ where $3 \leqslant k < l$,
iv) $p(z) \approx (z+1)^k(z-1)^l$ where $k = l = n/2$.

1.2.1. (i). If $p(z) \approx (z+1)(z-1)^{n-1}$ then for $j = 1, 2, \ldots, n-2$ the j-th derivative $p^{(j)}(z)$ has a zero of multiplicity $n-1-j$ at 1 and a simple zero at $-1+2j/n$. Hence along with the zero $(n-2)/n$ of $p^{(n-1)}(z)$ the product $\prod_{j=0}^{n-1} p^{(j)}(z)$ has precisely $n+1$ distinct zeros.

1.2.1. (ii) If $p(z) \approx (z+1)^2(z-1)^2$ then elementary direct calculation shows that $P(z)$ has 5 $(= n+1)$ distinct zeros.

Now let $p(z) \approx (z+1)^2(z-1)^{n-2}$ where $n \geqslant 5$. Then

$$p^{(1)}(z) \approx \{nz+(n-4)\}(z+1)(z-1)^{n-3},$$

$$p^{(2)}(z) \approx \{n(n-1)z^2+2(n-1)(n-4)z+n^2-9n+16\}(z-1)^{n-4},$$

$$p^{(3)}(z) \approx \{n(n-1)z^2+2(n-1)(n-6)z+(n-4)(n-9)\}(z-1)^{n-5},$$

whereas

$$p^{(4)}(z) \approx nz+(n-6) \text{ or } \approx \{(n(n-1)z^2+2(n-1)(n-8)z+(n^2-17n+64)\}(z-1)^{n-6}$$

according as $n = 5$ or $n \geqslant 6$.

The product $p(z)p^{(1)}(z)$ has 3 distinct zeros, namely -1, $-(n-4)/n$, $+1$. The second derivative $p^{(2)}(z)$ has a simple zero $c_{2,1}$ in the open interval

$(-1, -(n-4)/n)$ and a simple zero $c_{2,2}$ in the open interval $(-(n-4)/n, 1)$. Thus the product $p(z)p^{(1)}(z)p^{(2)}(z)$ has 5 distinct zero. The third derivative $p^{(3)}(z)$ has a simple zero $c_{3,1}$ in the open interval $(c_{2,1}, c_{2,2})$, a simple zero $c_{3,2}$ in the open interval $(c_{2,2}, 1)$ and a zero of multiplicity $n-5$ at 1 if $n \geqslant 6$. By direct substitution we see that $p^{(3)}(-(n-4)/n) \neq 0$, i. e. neither $c_{3,1}$ nor $c_{3,2}$ can be equal to $-(n-4)/n$. Hence neither of the two numbers $c_{3,1}, c_{3,2}$ is a zero of $p(z)p^{(1)}(z)p^{(2)}(z)$. The product $p(z)p^{(1)}(z)p^{(2)}(z)p^{(3)}(z)$ has therefore 7 distinct zeros. If $n = 5$ then $p^{(4)}(z)$ has only one (simple) zero at $-(n-6)/n$ which is obviously not a zero of $p^{(3)}(z)$. It can be verified directly that it is also not a zero of $p^{(1)}(z)$ or of $p^{(2)}(z)$. Hence the product $P(z) = \prod_{j=0}^{4} p^{(j)}(z)$ has 8 $(= n+3)$ distinct zeros. If $n \geqslant 6$ then $p^{(4)}(z)$ has two simple zeros $c_{4,1}, c_{4,2}$ in the open interval $(-1, 1)$. In fact, $c_{4,1}$ lies in the open interval $(c_{3,1}, c_{3,2})$ whereas $c_{4,2}$ lies in $(c_{3,2}, 1)$. Both these zeros are different from $-(n-4)/n$ since $p^{(4)}(-(n-4)/n) \neq 0$. They are also different from the two zeros of $p^{(2)}(z)$. In fact, $p^{(2)}(\beta) = 0$, $p^{(4)}(\beta) = 0$ imply that $\beta = 1$ or $\beta = -(n-6)/(n-1)$. But since

$$p^{(2)}\big(-(n-6)/(n-1)\big) \neq 0,$$

$p^{(2)}(z), p^{(4)}(z)$ have no common zero except possibly 1. It fololws that the product $p(z)p^{(1)}(z)\ldots p^{(4)}(z)$ has 9 distinct zeros. In particular, if $n = 6$ then $P(z)$ has at least $n+3$ distinct zeros. If $n \geqslant 7$ then for $5 \leqslant j \leqslant n-2$ the largest zero $c_{j,2}$ of $p^{(j)}(z)$ in the open interval $(-1, 1)$ is simple and $c_{j-1,2} < c_{j,2}$. Hence the product $p(z)p^{(1)}(z)p^{(2)}(z)\ldots p^{(n-2)}(z)$ has at least $9+(n-2-5+1) = n+3$ distinct zeros.

1.2.1. (iii). If $p(z) \approx (z+1)^k(z-1)^l$ where $3 \leqslant k < l$ then

$$p^{(1)}(z) \approx \{(k+l)z+(l-k)\}(z+1)^{k-1}(z-1)^{l-1},$$

$$p^{(2)}(z) \approx \{(k+l)(k+l-1)z^2+2(k+l-1)(l-k)z+(l-k)^2-k-l\}(z+1)^{k-2}(z-1)^{l-2},$$

$$p^{(3)}(z) \approx \{k(k-1)(k-2)(z-1)^3+3kl(k-1)(z-1)^2(z+1)+3kl(l-1)(z-1)(z+1)^2+l(l-1)(l-2)(z+1)^3\}(z+1)^{k-3}(z-1)^{l-3},$$

whereas

$$p^{(4)}(z) \approx \{24l(z-1)^3+36l(l-1)(z-1)^2(z+1)+12l(l-1)(l-2)(z-1)(z+1)^2+l(l-1)(l-2)(l-3)(z+1)^3\}(z-1)^{l-4}$$

or

$$\approx \{k(k-1)(k-2)(k-3)(z-1)^4+4kl(k-1)(k-2)(z-1)^3(z+1) -6kl(k+l-kl-1)(z-1)^2(z+1)^2+4kl(l-1)(l-2)(z-1)(z+1)^3+ +l(l-1)(l-2)(l-3)(z+1)^4\}(z+1)^{k-4}(z-1)^{l-4}$$

according as $k=3$ or $k>3$. The product $p(z)p^{(1)}(z)$ has 3 distinct zeros namely $-1, -(l-k)/(l+k), 1$. The second derivative $p^{(2)}(z)$ has a simple zero $\gamma_{2,1}$ in the open interval $(-1, -(l-k)/(l+k))$ and another simple zero $\gamma_{2,2}$ in the open interval $(-(l-k)/(l+k), 1)$. The product $p(z)p^{(1)}(z)p^{(2)}(z)$ has therefore 5 distinct zeros. The third derivative $p^{(3)}(z)$ has a simple zero in each of the open intervals $(-1, c_{2,1}), (c_{2,1}, c_{2,2}), (c_{2,2}, 1)$. None of these zeros can be equal to $-(l-k)/(l+k)$ since $p^{(3)}(-(l-k)/(l+k)) \neq 0$ if $k \neq l$ which is in fact the case. Thus the product $p(z)\times p^{(1)}(z)p^{(2)}(z)p^{(3)}(z)$ has 8 distinct zeros. The fourth derivative $p^{(4)}(z)$ has 3 or 4 simple zeros in the open interval $(-1, 1)$ according as $k=3$ or $k\geqslant 4$. It is clear that none of these zeros can be a zero of $p^{(3)}(z)$. Besides, it is easily checked that $p^{(3)}(-(l-k)/(l+k)) \neq 0$ and hence $p^{(4)}(z), p^{(1)}(z)$ have no zero in common except possibly $-1, 1$. Now we wish to show that the zeros $\gamma_{2,1}, \gamma_{2,2}$ of $p^{(2)}(z)$ cannot both be zeros of $p^{(4)}(z)$. Suppose if possible that both $\gamma_{2,1}, \gamma_{2,2}$ are zeros of $p^{(4)}(z)$. Then

$$p^{(4)}(z) = \{(k+l)(k+l-1)z^2+2(k+l-1)(l-k)z+(l-k)^2-k-l\}q_1(z)$$

where $q_1(z)$ is a polynomial. But

$$p^{(4)}(z) = \frac{d^2}{dz^2}[\{(k+l)(k+l-1)z^2+2(k+l-1)(l-k)z+(l-k)^2-k-l\}\times$$
$$\times(z+1)^{k-2}(z-1)^{l-2}] = [2(k+l)(k+l-1)(z^2-1)+2\{2(k+l)(k+l-1)z$$
$$+2(k+l-1)(l-k)\}\{(k+l-4)z+(l-k)\}]\times$$
$$\times(z+1)^{k-3}(z-1)^{l-3}+\{(k+l)(k+l-1)z^2+2(k+l-1)(l-k)z+$$
$$+(l-k)^2-k-l\}\frac{d^2}{dz^2}\{(z+1)^{k-2}(z-1)^{l-2}\}$$

Hence

$$[2(k+l)(k+l-1)(z^2-1)+2\{2(k+l)(k+l-1)z+2(k+l-1)(l-k)\}$$
$$\times\{(k+l-4)z+(l-k)\}](z+1)^{k-3}(z-1)^{l-3} = \{(k+l)(k+l-1)z^2+2(k+$$
$$+l-1)(l-k)z+(l-k)^2-k-l\}q_2(z)$$

where $q_2(z)$ is again a polynomial. This is possible only if $(k+l)(k+l-1)(2k+2l-7)z^2+4(k+l-1)(l-k)(k+l-2)z+(k+l-1)\{2(l-k)^2-k-l\}$ is a constant multiple of $(k+l)(k+l-1)z^2+2(k+l-1)(l-k)z+\{(l-k)^2-k-l\}$. But such is not the case as one can easily see. Thus the product $p(z)p^{(1)}(z)\dots p^{(4)}(z)$ has at least 10 distinct zeros if $k=3$ and at least 11 distinct zeros if $k\geqslant 4$. In particular if $k=3$, $l=4$ then $P(z)$ has at least 10 $(=n+3)$ distinct zeros. If $k=3$ and $l\geqslant 5$ then for $5\leqslant j\leqslant n-3$ the largest zero $\gamma_{j,3}$ of $p^{(j)}(z)$ in the open interval $(-1, 1)$ is simple and $p^{(i)}(\gamma_{j,3})\neq 0$ for $i<j$. Hence the product $P(z)$ has at least $10+(n-3)-5+1=n+3$ distinct zeros. If $k=4$ then for $5\leqslant j\leqslant n-4$

the largest zero $\gamma_{j,4}$ of $p^{(j)}(z)$ in $(-1, 1)$ is simple and $p^{(i)}(\gamma_{j,4}) \neq 0$ for $i < j$. Hence again the product $P(z)$ has at least $11+(n-4)-5+1 = n+3$ distinct zeros. If $k \geqslant 5$ then for $5 \leqslant j \leqslant k$ the smallest zero $\gamma_{j,1}$ and the largest zero $\gamma_{j,j}$ of $p^{(j)}(z)$ in $(-1,1)$ are simple and $p^{(i)}(\gamma_{j,1}) \neq 0$, $p^{(i)}(\gamma_{j,j}) \neq 0$ for $i < j$. Besides, for $k < j \leqslant l$ the largest zero $\gamma_{j,k}$ of $p^{(j)}(z)$ in $(-1, 1)$ is simple and $p^{(i)}(\gamma_{j,k}) \neq 0$ for $i < j$. Hence the product $P(z)$ has at least $11+2(k-5+1)+l-k = n+3$ distinct zeros.

1.2.1. (iv). Let $p(z) \approx (z^2-1)^{n/2}$.

If $n = 6$ then $p^{(1)}(z)$ has a double zero at each of the points $-1, 1$ and a simple zero at the origin. The second derivative has a simple zero at each of the points $-1, -1/\sqrt{5}, 1/\sqrt{5}, 1$. The third derivative vanishes at the points $-\sqrt{(3/5)}$, 0, $\sqrt{(3/5)}$, the fourth at $-1/\sqrt{5}$, $1/\sqrt{5}$, whereas the fifth derivative vanishes at the origin. Hence $P(z)$ has precisely $7\ (= n+1)$ distinct zeros.

Now let $n = 2k$ where $k \geqslant 4$. The polynomial $p^{(1)}(z)$ has zeros of multiplicity $k-1$ at the points $-1, 1$ and a simple zero at the origin. The second derivative $p^{(2)}(z)$ has zeros of multiplicity $k-2$ at $-1, 1$ and simple zeros at $\gamma_{2,1} = -1/\sqrt{(2k-1)}$, $\gamma_{2,2} = 1/\sqrt{(2k-1)}$. The third derivative $p^{(3)}(z)$ has zeros of multiplicity $k-3$ at $-1, 1$ and simple zeros at $\gamma_{3,1} = -\sqrt{\{3/(2k-1)\}}$, $\gamma_{3,2} = 0$, $\gamma_{3,3} = \sqrt{\{3/(2k-1)\}}$. Now we note that $p^{(4)}(z)$ which is a constant multiple of $(z^2-1)^{k-4}\{(2k-1)(2k-3)z^4-6(2k-3)z^2+3\}$ has four simple zeros $\gamma_{4,1}, \gamma_{4,2}, \gamma_{4,3}, \gamma_{4,4}$ in the open interval $(-1,1)$ and none of these zeros is a zero of $p^{(j)}(z), j < 4$. Hence the product $p(z)p^{(1)}(z)\ldots p^{(4)}(z)$ has 11 distinct zeros. This implies that if $k = 4$ (i. e $n = 8$) then $P(z)$ has at least $n+3$ distinct zeros. If $k \geqslant 5$ then for $5 \leqslant j \leqslant k$ the smallest zero $\gamma_{j,1}$ and the largest zero $\gamma_{j,j}$ of $p^{(j)}(z)$ in $(-1, 1)$ are simple and $p^{(i)}(\gamma_{j,1}) \neq 0$, $p^{(i)}(\gamma_{j,j}) \neq 0$, $i<j$. Hence the product $p(z)p^{(1)}(z)\ldots p^{(k)}(z)$ and a fortiori $P(z)$ has at least $11+2(k-5+1) = n+3$ distinct zeros.

1.2.2. Let $k+l = n-1$.

First we consider the subcase $k = 1, l = n-2$, i. e. $-1, 1$ are supposed to be zeros of $p(z)$ of multiplicity 1, $n-2$ respectively. Let c be the zero of $p(z)$ which lies in $(-1, 1)$. Then for $n \geqslant 5$:

$$p(z) \approx (z+1)(z-c)(z-1)^{n-2},$$

$$p^{(1)}(z) \approx [nz^2+\{-(n-1)c+(n-3)\}z-(n-3)c-1](z-1)^{n-3},$$

$$p^{(2)}(z) \approx [n(n-1)z^2+\{-(n-1)(n-2)c+(n-1)(n-6)\}z-(n-2)\times(n-5)c-2(n-3)](z-1)^{n-4},$$

$$p^{(3)}(z) \approx [n(n-1)z^2+\{-(n-1)(n-3)c+(n-1)(n-9)\}z-\{(n-3)\times\times(n-7)c+3(n-5)\}](z-1)^{n-5}.$$

The polynomial $p(z)$ has 3 distinct zeros. The derivative $p^{(1)}(z)$ has a simple zero $c_{1,1}$ in the interval $(-1, c)$ and another simple zero $c_{1,2}$ in the interval $(c, 1)$. The second derivative $p^{(2)}(z)$ has a simple zero $c_{2,1}$ in the interval $(c_{1,1}, c_{1,2})$ and another simple zero $c_{2,2}$ in the interval $(c_{1,2}, 1)$. However, if $c = -(n-3)/(n-1)$ then $c_{2,1} = c$ and $p^{(2)}(z)$ contributes only one new zero to the product $p(z)p^{(1)}(z)p^{(2)}(z)$. Thus the product $p(z)p^{(1)}(z)p^{(2)}(z)$ has 6 or 7 distinct zeros according as $c = -(n-3)/(n-1)$ or $c \neq -(n-3)/(n-1)$. Now if $c = -(n-3)/(n-1)$ then $p^{(3)}(z)$ has a simple zero $c_{3,1}$ in the interval $(c, c_{2,2})$ and a simple zero $c_{3,2}$ in the interval $(c_{2,2}, 1)$. It is clear that $c_{3,2}$ is necessarily a new zero. Also $c_{3,1}$ is a new zero. For it is clearly not a zero of $p(z)$ or of $p^{(2)}(z)$. Besides, if it were a zero of $p^{(1)}(z)$ then $p^{(1)}(c_{3,1}) = 0$, $p^{(3)}(c_{3,1}) = 0$ would together lead to the conclusion that $c_{3,1} = -(2n^2-13n+17)/\{(n-1)(2n-3)\}$. Thus we would have

$$-(2n^2-13n+17)/\{(n-1)(2n-3)\} = c_{3,1} = c_{1,2} = -\{(n-1)(n-3)-$$
$$-\sqrt{(5n-9)(n-1)}\}/\{n(n-1)\}$$

or

$$(2n-3)\sqrt{(5n-9)(n-1)} = 2n^2+n-9,$$

which is false for $n \geqslant 5$. Hence the product $p(z)p^{(1)}(z)p^{(2)}(z)p^{(3)}(z)$ has 8 distinct zeros. In case $c \neq -(n-3)/(n-1)$ it is enough for our purpose to observe that the largest zero of $p^{(3)}(z)$ in $(-1, 1)$ is not a zero of $p(z)p^{(1)}(z)p^{(2)}(z)$ and therefore $p(z)p^{(1)}(z)p^{(2)}(z)p^{(3)}(z)$ has at least 8 distinct zeros. For $4 \leqslant j \leqslant n-2$ the largest zero of $p^{(j)}(z)$ in the open interval $(-1, 1)$ is simple and is not a zero of the product $p(z)p^{(1)}(z) \ldots$ $\ldots p^{(j-1)}(z)$. Consequently the product $p(z)p^{(1)}(z) \ldots p^{(n-2)}(z)$ and a fortiori $P(z)$ has at least $8+(n-2-4+1) = n+3$ distinct zeros.

If $n = 4$ then direct calculation shows that $P(z)$ has $7 (= n+3)$ distinct zeros if $c = -1/3$ or $1/3$; otherwise it has 8 distinct zeros.

If $n = 3$ then $P(z)$ has 5 or 6 distinct zeros according as $c = 0$ or $c \neq 0$.

Now we consider polynomials of the form $(z+1)^k(z-c)(z-1)^{n-k-1}$ where $2 \leqslant k \leqslant n-k-1$, $-1 < c < 1$. We have

$$p^{(1)}(z) \approx [nz^2+\{(n-2k-1)-(n-1)c\}z-1-$$
$$-(n-2k-1)c](z+1)^{k-1}(z-1)^{n-k-2},$$

$$p^{(2)}(z) \approx [n(n-1)z^3+(n-1)\{2(n-2k-1)-(n-2)c\}z^2+\{(n-2k-1)^2-$$
$$-3(n-1)-2(n-2)(n-2k-1)c\}z-2(n-2k-1)-\{(n-2k-1)^2-$$
$$-n+1\}c](z+1)^{k-2}(z-1)^{n-k-3}.$$

The first derivative $p^{(1)}(z)$ has a simple zero $c_{1,1}$ in the open interval $(-1, c)$ and a simple zero $c_{1,2}$ in $(c, 1)$. The second derivative $p^{(2)}(z)$ has a simple zero $c_{2,1}$ in $(-1, c_{1,1})$, a simple zero $c_{2,2}$ in $(c_{1,1}, c_{1,2})$ and a simple zero $c_{2,3}$ in $(c_{1,2}, 1)$. However, $c_{2,2} = c$ if $c = -(n-2k-1)/(n-1)$. Hence the product $p(z)p^{(1)}(z)p^{(2)}(z)$ has 7 distinct zeros or 8 distinct zeros according as $c = -(n-2k-1)/(n-1)$ or $c \neq -(n-2k-1)/(n-1)$. Now if $k = 2$, $c \neq -(n-2k-1)/(n-1)$ then for $3 \leqslant j \leqslant n-3$ the largest zero of $p^{(j)}(z)$ in the open interval $(-1, 1)$ is simple and is not a zero of $p^{(i)}(z)$ for $i < j$. Hence the product $p(z)p^{(1)}(z) \ldots p^{(n-3)}(z)$ has at least $8+n-3-3+1 = n+3$ distinct zeros. If $k \geqslant 3$, $c \neq -(n-2k-1)/(n-1)$ then for $3 \leqslant j \leqslant k$ the smallest zero of $p^{(j)}(z)$ is simple and is not a zero of $p^{(i)}(z)$ for $i<j$; the same is true of the largest zero of $p^{(j)}(z)$, $3 \leqslant j \leqslant n-k-1$ in $(-1, 1)$. Hence the product $\prod_{j=0}^{n-k-1} p^{(j)}(z)$ has at least $8+k-3+1+n-k-1+3+1 = n+3$ distinct zeros. We remark that the third derivative $p^{(3)}(z)$ has a simple zero in each of the intervals $(c_{2,1}, c)$, $(c, c_{2,3})$ but we have ignored these zeros to allow the possibility that they may be zeros of $p(z)p^{(1)}(z)$. We verify that if $c = -(n-2k-1)/(n-1)$ then they cannot both be zeros of $p(z)p^{(1)}(z)$. It is clear that neither of the two is a zero of $p(z)$. If both are zeros of $p^{(1)}(z)$ then we must have

$$n(z+1)^{k-1}(z-1)^{n-k-2}+2\{nz+(n-2k-1)\}\{(n-3)z+(n-2k-1)\}(z+1)^{k-2}(z-1)^{n-k-3} = \{nz^2+2(n-2k-1)z-1+(n-2k-1)^2/(n-1)\}A(z).$$

where $A(z)$ is a polynomial. This is possible only if

$$n(2n-5)z^2+2(2n-3)(n-2k-1)z-n+2(n-2k-1)^2 \approx nz^2+2(n-2k-1)z-1+(n-2k+1)^2/(n-1).$$

But this is obviously false unless $n = 5$, $k = 2$. Excluding this latter case we may now argue as above to conclude that the product $P(z)$ has at least $n+3$ distinct zeros. In the case just excluded $P(z)$ has $7 (= n+2)$ distinct zeros.

1.2.3. Now let $k+l \leqslant n-2$.

In this case $l = \max(k, l) \leqslant n-3$. For $k \leqslant j \leqslant n-2$ the smallest zero $a^{(j)}$ of $p^{(j)}(z)$ is simple and

$$-1 < a^{(k)} < a^{(k+1)} < \ldots < a^{(n-2)}.$$

Besides, for $l \leqslant j \leqslant n-2$ the largest zero $b^{(j)}$ of $p^{(j)}(z)$ is simple and

$$1 > b^{(l)} > b^{(l+1)} > \ldots > b^{(n-2)} > a^{(n-2)}.$$

Thus the product $p(z)p^{(1)}(z) \ldots p^{(n-2)}(z)$ has at least $2n-k-l$ distinct zeros, namely, $-1, a^{(k)}, a^{(k+1)}, \ldots, a^{(n-2)}, b^{(n-2)}, b^{(n-3)}, \ldots, b^{(l)}, 1$. Including the zero $(a^{(n-2)}+b^{(n-2)})/2$ of $p^{(n-1)}(z)$ the product $P(z)$ has at least $n+3$ distinct zeros.

The following theorem summarizes our discussion of polynomials with only real zeros.

Theorem 1. *If $p(z)$ is a polynomial of degree n with real zeros then the product $P(z) = p(z)p^{(1)}(z) \ldots p^{(n-1)}(z)$ has*

i) *1 distinct zero if $p(z) \approx (z-a)^n$,*

ii) *$n+1$ distinct zeros if $p(z) \approx (z-a)(z-b)^{n-1}$ or $p(z) \approx (z-a)^2(z-b)^2$ or $p(z) \approx (z-a)^3(z-b)^3$,*

iii) *$n+2$ distinct zeros if $p(z+b)$ is a constant multiple of $z(z^2-a^2)$ or of $z(z^2-a^2)^2$ for some b,*

iv) *at least $n+3$ distinct zeros in any other case.*

In the above theorem we only need to assume that the zeros of $p(z)$ are collinear.

2.0 Now we wish to consider polynomials whose zeros are not collinear. Let us denote the convex hull of the zeros of $p(z)$ by H_p. According to Gauss-Lucas theorem

$$H_p \supseteq H_{p^{(1)}} \supseteq \ldots \supseteq H_{p^{(n-1)}}.$$

If the zeros of $p^{(k-1)}(z)$ are not collinear, $p^{(k)}(\xi) = 0$ for some $\xi \in \partial H_{p^{(k-1)}}$ and some k $(1 \leq k \leq n-1)$ if and only if ξ is a multiple zero of $p^{(k-1)}(z)$ $\left(p^{(0)}(z) \equiv p(z)\right)$.

We note that if $H(z_1, \ldots, z_m)$ is the convex hull of the points $z_1, \ldots, z_m \in C$ and $v_1, \ldots, v_k$ are the vertices of $H(z_1, \ldots, z_m)$ then $\{v_1, \ldots, v_k\} \subseteq \{z_1, \ldots, z_m\}$.

The centroid of the zeros of a polynomial is invariant under differentiation. This trivial fact will be often used without being mentioned explicitly. We shall assume the origin to be the centroid of the zeros of $p(z)$. This will not involve any loss of generality since for any given a the product $p(z+a)p^{(1)}(z+a) \ldots p^{(n-1)}(z+a)$ has the same number of distinct zeros as $p(z)p^{(1)}(z) \ldots p^{(n-1)}(z)$.

Definition. A polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, will be said to belong to the class $\mathscr{P}_n$ if $a_{n-1} = 0$, i. e. the origin is the centroid of the zeros of $p(z)$. The subclass $\mathscr{P}_n^{(C)}$ will consist of those polynomials whose zeros are collinear whereas the polynomials with noncollinear zeros will form the subclass $\mathscr{P}_n^{(NC)}$.

In the sequel we shall make extensive use of the following corollary of Theorem 1.

Corollary 1. *If* $p(z) \notin \mathscr{P}_n^{(C)}$ *then* $P(z) = p(z)p^{(1)}(z)\dots p^{(n-1)}(z)$ *has*

i) 1 *distinct zero if* $p(z) \approx z^n$,

ii) $n+1$ *distinct zeros if* $p(z) \approx (z-a)\{z+a/(n-1)\}^{n-1}$ *or* $p(z) \approx (z-a)^2(z+a)^2$ *or* $p(z) \approx (z-a)^3(z+a)^3$,

iii) $n+2$ *distinct zeros if* $p(z) \approx z(z^2-a^2)$ *or* $p(z) \approx z(z^2-a^2)^2$,

iv) *at least* $n+3$ *distinct zeros in any other case.*

Lemma 1. *If* $f(z)$ *is a polynomial of degree* $m+1$ *such that* $f^{(1)}(z) \approx z^m - a^m$ *for some* $a \neq 0$ *then the product* $f(z)f^{(1)}(z)f^{(2)}(z)$ *has at least* $2m$ *distinct zeros.*

Proof. The polynomial $f(z)$ is a constant multiple of $z^{m+1}-(m+1)a^m z + b$ for some b. Hence if $f(z)$, $f^{(1)}(z)$ have a common zero it is necessary but not sufficient that it be equal to $b/(ma^m)$. In fact, there is no common zero if $b=0$ for the simple reason that $f^{(1)}(0) \neq 0$, i. e. $f(z)$, $f^{(1)}(z)$ have no common zero if the only zero of $f^{(2)}(z)$ is a zero of $f(z)$. Except possibly for one double zero all the zeros of $f(z)$ are therefore simple and we readily see that the product $f(z)f^{(1)}(z)f^{(2)}(z)$ has at least $2m$ distinct zeros.

Remark 1. Given a polynomial $p(z)$ of degree n the zeros of $p^{(k)}(z)$ are coincident if and only if the zeros of $p^{(k-1)}(z)$ are coincident or form a regular $(n-k+1)$-gon. Hence if $p(z) \not\approx z^n - a^n$ and $p^{(k)}(z) \approx z^{n-k}$ then $p(z)p^{(1)}(z)\dots p^{(k)}(z)$ has at least $2(n-k+1)$ distinct zeros.

Lemma 2. *Let* $p(z)$ *be a polynomial of degree* n. *If for some* k $(1 \leqslant k \leqslant n-2)$

$$p^{(k)}(z) \approx (z-a)^{n_1}(z-b)^{n_2},\ a \neq b,\ n_1 \geqslant 1,\ n_2 \geqslant 1,\ n_1+n_2 = n-k,$$

then $p^{(k-1)}(z)$ *can vanish at most once on the straight line segment joining the points* a, b. *The product* $p^{(k-1)}(z)p^{(k)}(z)\dots p^{(n-1)}(z)$ *has at least* n_1+2n_2+1 *distinct zeros if* $p^{(k-1)}(a) = 0$, *at least* $2n_1+n_2+1$ *distinct zeros if* $p^{(k-1)}(b) = 0$ *and at least* $2n_1+2n_2+1$ *distinct zeros in any other case. If the zeros of* $p(z)$ *are not collinear and* $p^{(k)}(z) \approx (z-a)(z-b)^{n_2}$ *then* $P(z) = p(z)p^{(1)}(z)\dots p^{(n-1)}(z)$ *has at least* $2(n_2+1)$ *distinct zeros.*

Proof. We may clearly assume a, b to be real and $a < b$. If $p^{(k-1)}(c) = 0$ for some $c \in [a, b]$ then $p^{(k-1)}(x)$ is real for real x. Now if $d \neq c$ is another point of the interval $[a, b]$ such that $p^{(k-1)}(d) = 0$ then by Rolle's theorem $p^{(k)}(x)$ must vanish at least once in the open interval I with c, d as end points. But by hypothesis $p^{(k)}(x) \neq 0$ in (a, b). Hence $p^{(k-1)}(x)$ cannot vanish more than once on $[a, b]$.

According to Corollary 1 the product $p^{(k)}(z)p^{(k+1)}(z)\dots p^{(n-1)}(z)$ has at least n_1+n_2+1 distinct zeros which of course lie on $[a, b]$. Since any point other than a, b cannot be a multiple zero of $p^{(k-1)}(z)$ the product $p^{(k-1)}(z)p^{(k)}(z)\dots p^{(n-1)}(z)$ has at least n_1+2n_2+1 distinct zeros if

$p^{(k-1)}(a)=0$, at least $2n_1+n_2+1$ distinct zeros if $p^{(k-1)}(b)=0$, and at least $2n_1+2n_2+1$ distinct zeros in any other case.

Now let us suppose that the zeros of $p(z)$ are not collinear and $p^{(k)}(z) \approx (z-a)(z-b)^{n_2}$. If $p^{(k-1)}(b)=0$ then $p^{(k-1)}(z)$ has only one other zero which must lie at $\{(n_2+2)a-b\}/(n_2+1)$. Thus $p^{(k-1)}(z)$ is of the same form as $p^{(k)}(z)$. If again $p^{(k-2)}(b)=0$ then $p^{(k-2)}(z)$ is also of the same form as $p^{(k)}(z)$ and $p^{(k-1)}(z)$. Since the zeros of $p(z)$ are not collinear $p^{(j)}(b)$ cannot be zero for every j such that $0 \leqslant j \leqslant k-1$ $(p^{(0)}(z) \equiv p(z))$. Now if $p^{(i)}(b) \neq 0$ whereas $p^{(j)}(b)=0$ for $i<j \leqslant k$ then except possibly for one double zero all the zeros of $p^{(i)}(z)$ are simple and we readily conclude that $p^{(i)}(z)p^{(i+1)}(z) \ldots p^{(n-1)}(z)$ has at least $2(n_2+1)$ distinct zeros.

Lemma 2′. *If $p(z)$ is a polynomial of degree 8 such that $p^{(2)}(z) \approx (z-a)^3(z+a)^3$ then the product $P(z)=p(z)p^{(1)}(z)\ldots p^{(7)}(z)$ has at least 12 distinct zeros.*

Proof. Without loss of generality we may assume a to be real and positive. If $p^{(1)}(z) \neq 0$ at $z=\pm a$ then according to Lemma 2 the product $P(z)$ has at least 13 distinct zeros. However, if $p^{(1)}(z)$ vanishes at a (the case $p^{(1)}(-a)=0$ is analogous) then

$$p^{(1)}(z) \approx (20z^3+80az^2+116a^2z+64a^3)(z-a)^4 \equiv 20(z-a)\{z-(\beta+ \\ +i\gamma)\}\{z-(\beta-i\gamma)\}(z-a)^4, \gamma \neq 0$$

where $a<-a$ since according to Rolle's theorem $p^{(2)}(z)$ has at least one zero in the open interval joining a, a. It is clear that if $p(z)$ has no real zeros then at least four of its zeros are simple and are not zeros of the product $p^{(1)}(z)p^{(2)}(z)\ldots p^{(7)}(z)$. According to Corollary 1 the product $p^{(2)}(z)p^{(3)}(z)\ldots p(z)^{(7)}$ has 7 distinct zeros which all lie on the interval $[-a, a]$. Hence $P(z)$ has at least 14 distinct zeros. If $p(z)$ has a real zero then it $(p(z))$ must be a polynomial with real coefficients. Since a is a zero of $p^{(1)}(z)$ of multiplicity 4 and $H_p \subseteq H_{p^{(1)}}$ the polynomial $p(z)$ being of degree 8 can vanish at a only if

$$p(z) \approx (z-a_1)\big(z-(\beta_1+i\gamma_1)\big)\big((z-(\beta_1-i\gamma_1)\big)(z-a)^5, \gamma_1 \neq 0$$

where $a_1<a$. Hence $P(z)$ has at least 13 distinct zeros. If $p(a)=0$ then $p(z) \neq 0$ on the interval $(a, a]$. Even if $p(\beta+i\gamma)=0$, $p(\beta-i\gamma)=0$ there are two simple zeros of $p(z)$ which are not zeros of $p^{(1)}(z)p^{(2)}(z)\ldots p^{(7)}(z)$. Hence $P(z)$ has at least 12 distinct zeros. If $p(z)$ has a real zero which does not lie at a or a then $P(z)$ has at least 13 distinct zeros.

Lemma 2″. *If $p(z)$ is a polynomial of degree 8 such that $p^{(3)}(z) \approx z(z_2-a^2)^2$ then $P(z)$ has at least 12 distinct zeros.*

Proof. There is no loss of generality in assuming a to be real and positive. We may use Rolle's theorem to conclude that $p^{(2)}(z)$ can vanish at most once on each of the intervals $[-a, 0], [0, a]$. Note that $p^{(2)}(z)$ vanishes at $-a$ or a if and only if $p^{(2)}(z) \approx (z-a)^3(z+a)^3$ and then by Lemma 2 the product $P(z)$ has at least 12 distinct zeros. If $p^{(2)}(\pm a) \neq 0$ but $p^{(2)}(0) = 0$ then $p^{(2)}(z)$ has four non-real zeros which form a rectangle. Since $p^{(1)}(z)$ is of degree 7 all the vertices of $H_{p^{(2)}}$ cannot be zeros of $p^{(1)}(z)$, i.e. $H_{p^{(2)}}$ is a proper subset of $H_{p^{(1)}}$. Hence $p^{(1)}(z)p^{(2)}(z)\ldots p^{(7)}(z)$ has at least 12 distinct zeros. If $p^{(2)}(\pm a) \neq 0$ and also $p^{(2)}(0) \neq 0$ then at least five simple zeros of $p^{(2)}(z)$ do not lie on $[-a, a]$ and the product $p^{(2)}(z)p^{(3)}(z)\ldots p^{(7)}(z)$ has at least 12 distinct zeros.

Lemma 3. *If $f(z) \approx (z-A)^2(z-B)^2(z-C)^3$ and C is a root of the equation $3z^2+15z+10 = 0$, then $f^{(2)}(z)$ cannot be a constant multiple of $z^3(3z^2+15z+29)$.*

Proof. According to hypothesis $f(C) = 0$, $f^{(1)}(C) = 0$. Hence if $f^{(2)}(z) \approx z^3(3z^2+15z+20)$ then $f(z)$ must be a constant multiple of

$$z^7+7z^6+14z^5-(7C^6+42C^5+70C^4)z+6C^7+35C^6+56C^5$$

which is easily seen to be different from $(z-A)^2(z-B)^2(z-C)^3$ whatever A, B may be.

Lemma 4. *If $p(z) \approx (z-v_1)^2(z-v_2)^3(z-v_3)^3 \in \mathscr{P}_8^{(NC)}$ then $p^{(3)}(z)$ cannot be a constant multiple of $(z-a)^3(3z^2+9az+8a^2)$.*

Proof. Let $f(z) \equiv p(az+a)$. If $p^{(3)}(z) \approx (z-a)^3(3z^2+9az+8a^2)$ then $f^{(3)}(z)$ is a constant multiple of $z^3(3z^2+15z+20)$ and

$$f(z) \approx z^8+8z^7+(56/3)z^6+\lambda z^2+\mu z+\nu$$

for some λ, μ, ν. It can be directly verified that $z^8+8z^7+(56/3)z^6+\lambda z^2+ +\mu z+\nu$ is never of the form $(z-\alpha)^2(z-\beta)^3(z-\gamma)^3$ whatever λ, μ, ν may be. This contradicts the fact that $p(z)$ is a constant multiple of $(z-v_1)^2 (z-v_2)^3(z-v_3)^3$.

The next lemma is trivial.

Lemma 5. *If a vertex v of H_p is a zero of $p(z)$ of multiplicity k then $H_{p^{(k)}}$ is a proper subset of H_p. The point v does not belong to $H_{p^{(j)}}$ for $k \leq j \leq n-1$.*

If $p(z) \approx z^n - a^n$ for some $a \neq 0$ then $P(z) = \prod_{j=0}^{n-1} p^{(j)}(z)$ has $n+1$ distinct zeros. In future we shall exclude these polynomials from our consideration.

2.1 Let $p(z) \in \mathscr{P}_3^{(NC)}$, $p(z) \not\approx z^3 - a^3$. Since the zeros of $p(z)$ are not collinear they must be simple and different from 0. Now let us note that $p^{(1)}(z)$ has two distinct zeros d, $-d$ whereas $p^{(2)}(z)$ vanishes at the origin. Hence $P(z)$ has 6 distinct zeros.

2.2 Now let $p(z) \in \mathscr{P}_4^{(NC)}$, $p(z) \not\approx z^4 - a^4$. Since $p^{(1)}(z) \not\approx z^3$ the product $p^{(1)}(z)p^{(2)}(z)p^{(2)}(z)$ has at least 4 distinct zeros. Hence if three or more of the vertices of H_p are simple zeros of $p(z)$ the product $P(z) = p(z) \times p^{(1)}(z)p^{(2)}(z)p^{(3)}(z)$ has at least 7 distinct zeros. Only those polynomials which have one double and two simple zeros are not covered. So let $p(z) \approx (z - v_1)^2(z - v_2)(z - v_3)$. We observe that $p^{(1)}(z)$ cannot be a constant multiple of $z(z^2 - a^2)$. For otherwise v_1 must be equal to a or $-a$, i. e. $p^{(1)}(z) \approx z(z^2 - v_1^2)$ and $p(z) \approx (z^2 - v_1^2)^2$ which is a contradiction. Hence $P(z)$ has at least 8 distinct zeros unless $p^{(1)}(z)$ is a constant multiple of $z^3 - a^3$ or of $(z-a)(z+a/2)^2$. In the latter two cases $p(z) \approx z^4 - 4b^3z + 3b^4$ (b is one of the cube roots of a^3), $p(z) \approx 2z^4 - 3a^2z^2 - 2a^3z + 3a^4$ respectively, and $P(z)$ has 6 distinct zeros.

We therefore have the following theorem.

Theorem 2. *If $p(z) \in \mathscr{P}_4$ then $P(z)$ has*

i) *1 distinct zero if $p(z) \approx z^4$,*

ii) *5 distinct zeros if $p(z) \approx (z-a)(z+a/3)^3$ or $p(z) \approx (z-a)^2 \times (z+a)^2$ or $p(z) \approx z^4 - a^4$,*

iii) *6 distinct zeros if $p(z) \approx z^4 - 4a^3z + 3a^4$ or $p(z) \approx 2z^4 - 3a^2z^2 - 2a^3z + 3a^4$,*

iv) *at least 7 distinct zeros in any other case.*

2.3 Let $p(z) \in \mathscr{P}_5^{(NC)}$, $p(z) \not\approx z^5 - a^5$. Since $p^{(1)}(z) \not\approx z^4$ the product $p^{(1)}(z)p^{(2)}(z)p^{(3)}(z)p^{(4)}(z)$ has at least 5 distinct zeros. Hence if three or more of the vertices of H_p are simple zeros of $p(z)$ the product $P(z) = p(z)p^{(1)}(z) \dots p^{(4)}(z)$ has at least 8 distinct zeros. This is surely the case if H_p is a pentagon or a quadrilateral.

If H_p is a triangle and two of its vertices are double zeros of $p(z)$ then $H_{p(1)}$ can neither be a square nor a straight line segment. Hence $P(z)$ has at least 8 distinct zeros unless $p^{(1)}(z)$ happens to be a constant multiple of $z^4 - 4a^3z + 3a^4 = (z-a^2)(z^2 + 2az + 3a^2)$ or of

$$2z^4 - 3a^2z^2 - 2a^3z + 3a^4 = (z-a)^2(2z^2 + 4az + 3a^2)$$

for some $a \neq 0$. It is clear that the two simple zeros of $p^{(1)}(z)$ must come from the two double zeros of $p(z)$. Thus we respectively have

i) $p(z) \approx (z-b)(z^2 + 2az + 3a^2)^2$, $p^{(1)}(z) \approx (z-a)^2(z^2 + 2az + 3a^2)$

ii) $p(z) \approx (z-b)(2z^2 + 4az + 3a^2)^2$, $p^{(1)}(z) \approx (z-a)^2(2z^2 + 4az + 3a^2)$

for some b. But, neither (i) nor (ii) can hold whatever b may be.

Let only one vertex of H_p be a multiple zero of $p(z)$. If $p^{(1)}(z)$ is a constant multiple of z^4-a^4 or of $(z-a)(z+a/3)^3$ for some $a\neq 0$ then by Lemma 1, Lemma 2 respectively $P(z)$ has at least 8 distinct zeros. It is readily seen that $p^{(1)}(z)\approx(z-a)^2(z+a)^2$ if and only if $p(z)\approx(z\mp a)^3(3z^2\pm 9az+8a^2)$ and then $P(z)$ has only 7 distinct zeros.

We therefore have the following theorem.

THEOREM 3. *If $p(z)\in\mathscr{P}_5$ then $P(z)$ has*

i) 1 *distinct zero if* $p(z)\approx z^5$,

ii) 6 *distinct zeros if* $p(z)\approx(z-a)(z+a/4)^4$ *or* $p(z)\approx z^5-a^5$,

iii) 7 *distinct zeros if* $p(z)\approx z(z^2-a^2)^2$ *or* $p(z)\approx(z-a)^3(3z^2+9az+$ $+8a^2)$,

iv) *at least* 8 *distinct zeros in any other case*

2.4 Let $p(z)\in\mathscr{P}_6^{(NC)}$, $p(z)\not\approx z^6-a^6$. If $p^{(1)}(z)$ is a constant multiple of z^5-a^5 or of $(z-a)(z+a/4)^4$ then by Lemmas 1, 2 respectively $P(z)$ $=p(z)p^{(1)}(z)\dots p^{(5)}(z)$ has at least 10 distinct zeros. Theorem 3 says that in any other case $p^{(1)}(z)p^{(2)}(z)\dots p^{(5)}(z)$ has at least 7 distinct zeros. Hence if two or more of the vertices of H_p are simple zeros of $p(z)$ then $P(z)=p(z)p^{(1)}(z)\dots p^{(5)}(z)$ has at least 9 distinct zeros. This is certainly the case if H_p is a hexagon, a pentagon, or a quadrilateral.

Let H_p be a triangle $v_1v_2v_3$ and suppose two of the vertices (say v_2, v_3) are multiple zeros of $p(z)$. It is clear that $H_{p^{(1)}}$ cannot be a straight line segment. Now suppose, if possible, that $p^{(1)}(z)\approx(z-a)^3$ $(3z^2+9az+$ $+8a^2)$. According to hypothesis $p^{(1)}(z)$ vanishes at v_2, v_3. Hence $(3z_2+$ $+9az+8a^2)=3(z-v_2)(z-v_3)$ and $p(z)\approx(3z^2+9az+8a^2)^2A_2(z)$ where $A_2(z)$ vanishes at v_1 but not at v_2 or v_3. However, it is readily seen the derivative of $(3z^2+9az+8a^2)^2A_2(z)$ can never be a constant multiple of $(z-a)^3(3z^2+9az+8a^2)$. Hence in the case under consideration $p^{(1)}(z)$ $\not\approx(z-a)^3(3z^2+9az+8a^2)$. By Theorem 3 the product $p^{(1)}(z)p^{(2)}(z)\dots$ $\dots p^{(5)}(z)$ has at least 8 distinct zeros. Since v_1 is not a zero of this product $P(z)$ has at least 9 distinct zeros.

If none of the vertices of H_p is a simple zero of $p(z)$ then each vertex must be a double zero of $p(z)$. It can be directly verified that if $p(z)\approx(z^3+$ $-a^3)^2$ then $P(z)$ has 10 distinct zeros. So let $p(z)\approx(z-v_1)^2(z-v_2)^2(z-$ $-v_3)^2$, $p(z)\not\approx(z^3-a^3)^2$. It is clear that the zeros of $p^{(1)}(z)$ are all simple and $p^{(2)}(z)\not\approx z^4$. Hence the product $p^{(1)}(z)p^{(2)}(z)$ has 9 distinct zeros if $p^{(2)}(z)\approx z^4-a^4$. Since $p^{(1)}(z), p^{(2)}(z)$ cannot have any common zeros we may use Lemma 2 to conclude that if $p^{(2)}(z)\approx(z-a)(z+a/3)^3$ or $\approx(z-a)^2(z+a)^2$ the product $P(z)=p(z)p^{(1)}(z)\dots p^{(5)}(z)$ has at least 9 distinct zeros. In any other case the same conclusion can be drawn from Theorem 2.

We therefore have the following theorem.

Theorem 4. *If $p(z) \notin \mathscr{P}_6$ then $P(z)$ has*

i) 1 *distinct zero if* $p(z) \approx z^6$,

ii) 7 *distinct zeros if* $p(z) \approx (z-a)(z+a/5)^5$ *or* $p(z) \approx (z-a)^3(z+a)^3$ *or* $p(z) \approx z^6 - a^6$,

iii) *at least* 9 *distinct zeros in any other case.*

2.5 Let $p(z) \in \mathscr{P}_7^{(NC)}$, $p(z) \not\approx z^7 - a^7$. If $p^{(1)}(z)$ is a constant multiple of $(z-a)(z+a/5)^5$ or of $(z-a)^3(z+a)^3$ then by Lemma 2 the product $P(z) = p(z)p^{(1)}(z) \ldots p^{(6)}(z)$ has at least 10 distinct zeros. In view of Lemma 1 the same can be said about the number of distinct zeros of $P(z)$) if $p^{(1)}(z) \approx z^6 - a^6$. Theorem 4 says that in any other case $p^{(1)}(z) \times p^{(2)}(z) \ldots p^{(6)}(z)$ has at least 9 distinct zeros. Hence if at least one of the vertices of H_p is a simple zero of $p(z)$ the product $P(z)$ has at least 10 distinct zeros. This is certainly the case if H_p has four or more vertices. So let H_p be a triangle and suppose that all its vertices are multiple zeros of $p(z)$. If $p^{(2)}(z)$ is a constant multiple of $z^5 - a^5$ or of $(z-a)(z+a/4)$ then by Lemmas 1, 2 respectively $P(z)$ has at least 10 distinct zeros. According to Theorem 3 the product $p^{(2)}(z)p^{(3)}(z) \ldots p^{(6)}(z)$ has at least 7 distinct zeros in any other case. Thus if all the vertices of H_p are double zeros of $p(z)$ the product $P(z) = p(z)p^{(1)}(z) \ldots p^{(6)}(z)$ has at least 10 distinct zeros. If on the other hand, $p(z) \approx (z-a)^2(z-\beta)^2(z-\gamma)^3$ the $p^{(2)}(z)$ has a simple zero at γ and cannot therefore be a constant multiple of $z(z^2-a^2)^2$. Lemma 3 applied to $p(z+a)$ says that $p^{(2)}(z)$ cannot be a constant multiple of $(z-a)^3(3z^2+9az+8a^2)$ either. Hence by Theorem 3 the product $p^{(2)}(z)p^{(3)}(z) \ldots p^{(6)}(z)$ has at least 8 distinct zeros in this case. Since a and β are not zeros of this product, $P(z) = p(z)p^{(1)}(z) \ldots p^{(6)}(z)$ has at least 10 distinct zeros.

We therefore have the following theorem.

Theorem 5. *If $p(z) \in \mathscr{P}_7$ then $P(z)$ has*

i) 1 *distinct zero if* $p(z) \approx z^7$,

ii) 8 *distinct zeros if* $p(z) \approx (z-a)(z+a/6)^6$, *or* $p(z) \approx z^7 - a^7$,

iii) *at least* 10 *distinct zeros in any other case.*

2.6 Let $p(z) \in \mathscr{P}_8^{(NC)}$, $p(z) \not\approx z^8 - a^8$. If $p^{(1)}(z)$ is a constant multiple of $z^7 - a^7$ or of $(z-a)(z+a/6)^6$ then by Lemmas 1, 2 respectively $P(z) = p(z)p^{(1)}(z) \ldots p^{(7)}(z)$ has at least 14 distinct zeros. By Theorem 5 the product $p^{(1)}(z)p^{(2)}(z) \ldots p^{(7)}(z)$ has at least 10 distinct zeros in any other case. Hence if there exists a vertex of H_p which is a simple zero of $p(z)$ then $P(z) = p(z)p^{(1)}(z) \ldots p^{(7)}(z)$ has at least 11 distinct zeros. If all the vertices of H_p are multiple zeros of $p(z)$ out of which at least two are double zeros of $p(z)$ we may apply Theorem 4 to $p^{(2)}(z)$ and conclude that $P(z) = p(z)p^{(1)}(z)p^{(2)}(z) \ldots p^{(7)}(z)$ has at least 11 distinct zeros except possibly when $p^{(2)}(z)$ is a constant multiple of z^6-a^6, of $(z-a)(z+a/5)^5$ or of $(z-a)^3(z+a)^3$. However, according to Lemmas 1, 2, 2′ respectively $P(z)$ has at least 12 distinct zeros in these exceptional cases. Finally, let $p(z)\approx(z-v_1)^2(z-v_2)^3(z-v_3)^3$. If $p^{(3)}(z)\approx z^5$ or $p^{(3)}(z)\approx z(z^2-a^2)^2$ then according to Lemmas 1, 2″ respectively the product $P(z)$ has at least 12 distinct zeros. Besides, if $p^{(3)}(z)$ is a constant multiple of z^5-a^5 or of $(z-a)(z+a/4)^4$ then by Lemmas 1, 2 respectively the product $p^{(2)}(z)p^{(3)}(z)\dots p^{(7)}(z)$ has at least 10 distinct zeros and since v_1 is not a zero of this product $P(z)$ has at least 11 distinct zeros. Since Lemma 4 says that $p^{(3)}(z)$ cannot be a constant multiple of $(z-a)^3(3z^2+9az+8a^2)$ the product $p^{(3)}(z)p^{(4)}(z)\dots p^{(7)}(z)$ has at least 8 distinct zeros. It is clear that v_1, v_2, v_3 are not zeros of this product. Hence $P(z)$ has at least 11 distinct zeros.

We therefore have the following theorem.

Theorem 6. *If $p(z)\in\mathcal{P}_8$ then $P(z)$ has*

i) 1 *distinct zero if $p(z)\approx z^8$,*

ii) 9 *distinct zeros if $p(z)\approx(z-a)(z+a/7)^7$ or $p(z)\approx z^8-a^8$,*

iii) *at least* 11 *distinct zeros in any other case.*

2.7 Let $p(z)\in\mathcal{P}_9^{(NC)}$, $p(z)\not\approx z^9-a^9$. There are four possibilities:

1. At least one of the vertices of H_p is a simple zero of $p(z)$.
2. At least two of the vertices of H_p are double zeros of $p(z)$.
3. All the vertices of H_p are zeros of $p(z)$ of multiplicity $\leqslant 3$.
4. $p(z)\approx(z-v_1)^2(z-v_2)^3(z-v_3)^4$.

If $p^{(1)}(z)$ is a constant multiple of z^8-a^8 or of $(z-a)(z+a/7)^7$ then by Lemmas 1, 2 respectively $P(z)$ has at least 16 distinct zeros. By Theorem 6 the product $p^{(1)}(z)p^{(2)}(z)\dots p^{(8)}(z)$ has at least 11 distinct zeros in any other case. Hence if there exists a vertex of H_p which is a simple zero of $p(z)$ then $P(z)$ has at least 12 distinct zeros.

If two or more of the vertices of H_p are double zeros of $p(z)$ we may use Theorem 5 in conjunction with Lemmas 1, 2 to conclude that $P(z)$ has at least 12 distinct zeros.

If $p^{(3)}(z)$ is a constant multiple of z^6-a^6, of $(z-a)(z+a/5)^5$, or of $(z-a)^3(z+a)^3$ then by Lemmas 1, 2, 2′ respectively the product $P(z)=p(z)p^{(1)}(z)\dots p^{(8)}(z)$ has at least 12 distinct zeros. Theorem 4 implies that in any other case $p^{(3)}(z)p^{(4)}(z)\dots p^{(8)}(z)$ has at least 9 distinct zeros. Hence if all the vertices of H_p are zeros of $p(z)$ of multiplicity $\leqslant 3$ the product $P(z)=p(z)p^{(1)}(z)\dots p^{(8)}(z)$ has at least 12 distinct zeros. Besides, if $p(z)\approx(z-v_1)^2(z-v_2)^3(z-v_3)^4$ then $P(z)$ has at least 11 distinct zeros.

We therefore have the following theorem.

Theorem 7. *If $p(z) \in \mathscr{P}_9$ then $P(z)$ has*
i) 1 *distinct zero if* $p(z) \approx z^9$,
ii) 10 *distinct zeros if* $p(z) \approx (z-a)(z+a/8)^8$ *or* $p(z) \approx z^9 - a^9$,
iii) *at least* 11 *distinct zeros if* $p(z) \approx (z-v_1)^2(z-v_2)^3(z-v_3)^4$,
iv) *at least* 12 *distinct zeros in any other case.*

2.8. Let $p(z) \in \mathscr{P}_{10}^{NC}$, $p(z) \not\approx z^{10} - a^{10}$. If $p^{(2)}(z)$ is a constant multiple of $z^8 - a^8$ or of $(z-a)(z+a/7)^7$ then by Lemmas 1, 2 respectively $P(z)$ has at least 16 distinct zeros. Theorem 6 says that in any other case $p^{(2)}(z)p^{(3)}(z) \ldots p^{(9)}(z)$ has at least 11 distinct zeros. Hence if one or more of the vertices of H_p is a zero of $p(z)$ of multiplicity $\leqslant 2$ then $P(z)$ has at least 12 distinct zeros. If none of the vertices of H_p is a zero of $p(z)$ of multiplicity $\leqslant 2$ then at least two of the vertices of H_p must be triple zeros of $p(z)$. We may use Theorem 5 along with Lemmas 1, 2 to conclude that $P(z)$ has at least 12 distinct zeros.

We therefore have the following theorem.

Theorem 8. *If $p(z) \in \mathscr{P}_{10}$ then $P(z)$ has*
i) 1 *distinct zero if* $p(z) \approx z^{10}$,
ii) 11 *distinct zeros if* $p(z) \approx (z-a)(z+a/9)^9$ *or* $p(z) \approx z^{10} - a^{10}$,
iii) *at least* 12 *distinct zeros in any other case.*

2.9 Let $p(z) \in \mathscr{P}_{11}^{NC}$, $p(z) \not\approx z^{11} - a^{11}$. If $p(z)^{(3)}$ is a constant multiple of $z^8 - a^8$ or of $(z-a)(z+a/7)^7$ then by Lemmas 1, 2 respectively $P(z) = p(z)p^{(1)}(z) \ldots p^{(10)}(z)$ has at least 14 distinct zeros. Theorem 6 says that in any other case $p^{(3)}(z)p^{(4)}(z) \ldots p^{(10)}(z)$ has at least 11 distinct zeros. Since there always exists a vertex of H_p which is a zero of $p(z)$ of multiplicity $\leqslant 3$ the product $P(z) = p(z)p^{(1)}(z) \ldots p(z)^{(10)}$ has at least 12 distinct zeros.

We therefore have the following theorem.

Theorem 9. *If $p(z) \in \mathscr{P}_{11}$ then $P(z)$ has*
i) 1 *distinct zero if* $p(z) \approx z^{11}$,
ii) *at least* 12 *distinct zeros in any other case.*

2.10. Let $p(z) \in \mathscr{P}_{12}^{(NC)}$, $p(z) \not\approx z^{12} - a^{12}$. There are two possibilities:

1. At least one of the vertices of H_p is a zero of $p(z)$ of multiplicity $\leqslant 2$.

2. At least two of the vertices of H_p are zeros of $p(z)$ of multiplicity $\leqslant 4$.

In the first case we may apply Theorem 8 along with Lemmas 1, 2 to $p^{(2)}(z)$ and in the second case Theorem 6 together with Lemmas 1, 2 to $p^{(4)}(z)$ to conclude that $P(z)$ has at least 13 distinct zeros.

We therefore have the following theorem.

Theorem 10. *If* $p(z) \in \mathscr{P}_{12}$ *then* $P(z)$ *has*

i) 1 *distinct zero if* $p(z) \approx z^{12}$,

ii) *at least* 13 *distinct zeros in any other case.*

Conclusion. We have shown in particular that if $p(z)$ is a polynomial of degree $n \leqslant 12$ then the product $P(z) = p(z)p^{(1)}(z) \ldots p^{(n-1)}(z)$ has at least $n+1$ distinct zeros unless $p(z) = c(z-a)^n$. It has been conjectured by Popoviciu that the same holds for polynomials of all degree.

STRESZCZENIE

T. Popoviciu postawił hipotezę, że jeżeli $p(z)$ jest wielomianem różnym od $c(z-a)^n$, to wielomian $P(z) = p(z)p'(z) \ldots p^{(n-1)}(z)$ ma conajmniej $n+1$ różnych zer.

Autor uzyskuje kilka rezultatów, dotyczących ilości różnych zer wielomianu $P(z)$, z których wynika prawdziwość hipotezy Popoviciu dla wielomianów $p(z)$ stopnia $n \leqslant 12$. W przypadku wielomianów $p(z)$ o zerach kolinearnych autor uzyskał dokładniejsze oszacowanie ilości różnych zer wielomianu $P(z)$.

РЕЗЮМЕ

Т. Поповичю поставил гипотезу: если $p(z)$ — многочлен, отличающийся от $c(z-a)^n$, то многочлен $P(z)=p(z)p'(z) \ldots p^{(n-1)}(z)$ имеет по меньшей мере $n+1$ различных нулей.

Получено несколько результатов, касающихся числа различных нулей многочлена $P(z)$, из которых вытекает справедливость гипотезы Поповичю для многочленов $p(z)$ степени $n \leqslant 12$. В случае многочленов $p(z)$ с колинеарными нулями получена лучшая оценка числа различных нулей мночочлена $P(z)$