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On the Relative Growth of Subordinate Functions

O względnym wzroście funkcji podporządkowanych

Об относительном росте подчиненных функций

1. Introduction

Suppose f, F are functions regular in the unit disk K, both vanishing at the origin. The function f is said to be subordinate to F in K, if there exists a function ω regular in K such that $\omega(0) = 0$, $|\omega(z)| < 1$ in K and $f(z) \equiv F(\omega(z))$. Then we write: $f \dashv F$. As a rule it happens that in all sufficiently small disks $K_r = \{z: |z| < r\}$ the most important functionals corresponding to r and f are dominated by the relevant functionals corresponding to r and F, whenever $f \dashv F$. Many authors were concerned with the problem of determinating the largest disk where such a domination takes place. E.g. Golusin [2] showed that, if $f \dashv F$ and $a(r)$, $A(r)$ denote the area of Riemann surfaces being the maps of K_r under f and F resp., then $a(r) \leqslant A(r)$ for any $r \in \left(0, \frac{1}{\sqrt{2}}\right]$. It seems that E. Reich was first to investigate a problem of more general type. Instead of evaluating the radius where $a(r)$ is dominated by $A(r)$ he was concerned with the estimates of the ratio $a(r)/A(r)$ in the whole unit disk under the assumption $f \dashv F$. He showed [4] that $a(r)/A(r) \leqslant mr^{2m-2}$, for $\frac{m-1}{m} \leqslant r^2 \leqslant \frac{m}{m+1}$, $m = 1, 2, \ldots$, which implies Golusin's result in case $m = 1$.

Now, we can similarly consider functionals others than the area. The most interesting case is perhaps the absolute value at corresponding points. Thus we are led to the following problem.

Suppose A_n, $n = 1, 2, \ldots$, is the class of functions f regular in K

such that $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots, a_n \geq 0$. Suppose, moreover, S_0 is a fixed subclass of the class S of functions regular and univalent in K subject to the usual normalization. Find for a given positive integer n and given $r \in (0,1)$ the l.u.b. $\varkappa(r, n, S_0)$ of the ratio $|f(z)/F(z)|$ for all $f \in A_n$ and all $F \in S_0$ satisfying $f \dashv F$, $|z| \leq r$.

In this paper we find the solution for the class S_0 such that: $f \in S_0$, $|\eta| \leq 1$, implies $\eta^{-1} f(\eta z) \in S_0$.

Suppose B_n, $n = 1, 2, \ldots$, is the class of functions ω, $\omega(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, which are regular in K and satisfy $a_n \geq 0$ and $|\omega(z)| < 1$ for all $z \in K$.

A reasoning similar to that used in [3] shows that for a fixed $z_1 \in K$ and ω ranging over B_n the set of all possible values of $\omega(z_1)$ is the closed domain $H_n(z_1)$ (generalized Rogosinski's domain) whose boundary consists of three arcs:

$$z_0(\theta) = |z_1|^{n+1} e^{i\theta}, \arg z_1^n + \frac{1}{2}\pi \leq \theta \leq \arg z_1^n + \frac{3}{2}\pi, \tag{1.1}$$

$$z_1(a) = z_1^n (a + i|z_1|)/(1 + ia|z_1|), \ 0 \leq a \leq 1, \tag{1.2}$$

$$z_2(a) = z_1^n (a - i|z_1|)/(1 - ia|z_1|), \ 0 \leq a \leq 1. \tag{1.3}$$

Suppose $Q_n(z_1, S_0) = \{u\colon u = F(z_2)/F(z_1)\}$, where z_1 is a fixed point of K and z_2, F range over $H_n(z_1)$ and S_0 resp. Under our assumptions on S_0, the set Q_n has the following properties (cf. [1]):

$$Q_1(z_1, S_0) = Q_1(|z_1|, S_0), \tag{1.4}$$

$$\text{if } 0 < r < R < 1, \text{ then } Q_n(r, S_0) \subset Q_n(R, S_0). \tag{1.5}$$

We shall also be concerned with the set $\Omega_n(z_1, S_0) = \{w\colon w = f(z_1)/F(z_1)\}$, where z_1 is a fixed point of K, $f \in A_n$, $F \in S_0$, and $f \dashv F$.

2. Main results

Theorem 1. *Under the notations of sect. 1 we have:*

$$\Omega_n(z_1, S_0) = Q_n(z_1, S_0)$$

Proof. Suppose $u \in \Omega_n(z_1, S_0)$. This means that there exist $f \in A_n$, $F \in S_0$ and $z_1 \in K$ such that $f \dashv F$ and $u = f(z_1)/F(z_1)$. Now, the condition $f \dashv F$ implies that there exists $\omega \in B_n$ such that $f(z) = F(\omega(z))$ and also $f(z_1) = F(\omega(z_1))$. If $z_2 = \omega(z_1)$, then $z_2 \in H_n(z_1)$, cf. [3]. We can write: $u = f(z_1)/F(z_1) = F(\omega(z_1))/F(z_1) = F(z_2)/F(z_1)$ and this means that $u \in Q_n(z_1, S_0)$ by (1.4), i.e. $\Omega_n(z_1, S_0) \subset Q_n(z_1, S_0)$.

Suppose now $q \in Q_n(z_1, S_0)$. Hence $q = F(z_2)/F(z_1)$ with $z_1 \in K$ and $z_2 \in H_n(z_1)$. Since $z_2 \in H_n(z_1)$, there exists $\omega \in B_n$ such that $z_2 = \omega(z_1)$, cf. [3]. Consequently, $q = F(\omega(z_1))/F(z_1) = f(z_1)/F(z_1)$, where $F(\omega(z)) = f(z) \in A_n$. This means that $q \in \Omega_n(z_1, S_0)$, i.e. $Q_n(z_1, S_0) \subset \Omega_n(z_1, S_0)$, or, finally,

$$Q_n(z_1, S_0) = \Omega_n(z_1, S_0).$$

Corollary. *If* $z \in \bar{K}_r, f \in A_n, F \in S_0$ *and* $f \dashv F$, *then*

$$\sup |f(z)/F(z)| = \sup\{|w| : w \in \Omega_n(z, S_0)\}.$$

Hence we have the solution of the problem

$$\varkappa(r, n, S_0) = \sup\{|w| : w \in Q_n(z, S_0)\}$$

Theorem 2. *Suppose* S_0 *is the class* S^c *of convex functions. Then*

$$\varkappa(r, 1, S^c) = \max\{1, r(1-r)^{-1}\},$$

$$\varkappa(r, n, S^c) = r^{n-1}\frac{1+r}{1-r^n} \quad \text{for} \quad n \geqslant 2.$$

Theorem 3. *For the class* S^* *of starlike functions we have:*

$$\varkappa(r, 1, S^*) = \max\{1, r(1-r)^{-2}\},$$

$$\varkappa(r, n, S) = r^{n-1}\left(\frac{1+r}{1-r^n}\right)^2 \quad \text{for} \quad n \geqslant 2.$$

The proofs of Theorems 2, 3 will be published in vol. 18 of the Michigan Mathematical Journal.

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STRESZCZENIE

Niech A_n, $n = 1, 2, \ldots$, oznacza klasę funkcji $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, $a_n \geqslant 0$, regularnych w kole jednostkowym K. Oznaczmy przez S_0 ustaloną podklasę klasy S funkcji regularnych i jednolistnych w kole K, takich, że:

*) $\quad f \in S_0, \; |\eta| \leqslant 1$, implikuje $\eta^{-1} f(\eta z) \in S_0$.

Niech $\varkappa(r, n, S_0) = \sup |f(z)/F(z)|$, gdzie

$$f \in A_n, \; F \in S_0, \; f \dashv F, \; |z| \leqslant r.$$

Twierdzenie 1 i wniosek po nim następujący mówi, że

**) $\varkappa(r, n, S_0) = \sup\{|w|\colon w \in Q_n(z, S_0)\}$,

gdzie $Q_n(z, S_0) = \left\{w\colon w = \frac{f(\zeta)}{f(z)}\right\}$, gdy ζ, f przebiegają odpowiednio: obszar Rogosinskiego $H_n(z)$ i S_0.

Korzystając z warunku **) można wyznaczyć efektywnie funkcję $\varkappa$. Jeśli S_0 jest klasą funkcji odpowiednio: wypukłych S^c i gwiaździstych S^*, wówczas (twierdzenia 2,3):

$$\varkappa(r, 1, S^c) = \max\{1, r(1-r)^{-1}\}, \varkappa(r, n, S^c) = r^{n-1}\frac{1+r}{1-r^n}, n \geqslant 2,$$

$$\varkappa(r, 1, S^*) = \max\{1, r(1-r)^{-2}\}, \varkappa(r, n, S^*) = r^{n-1}\left(\frac{1+r}{1-r^n}\right)^2, n \geqslant 2.$$

Dowody twierdzeń 2,3, ukażą się w pracy oddanej do druku w periodyku Michigan Mathematical Journal, vol. 18.

РЕЗЮМЕ

Пусть A_n, $n = 1, 2, 3, \ldots$, обозначает класс функций $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, $a_n \geqslant 0$, регулярных в единичном круге K. Через S_0 обозначим фиксированный подкласс класса S регулярных и однолистных функций в круге K, которые удовлетворяют условию:

*) $f(z) \in S_0, |\eta| \leqslant 1 \Rightarrow \eta^{-1} f(\eta z) \in S_0$

Пусть $\varkappa(r, n, S_0) = \sup|f(z)/F(z)|$,, при

$$f \in A_n, F \in S_0, f \prec F, |z| \leqslant r.$$

Из теоремы 1 этой работы вытекает, что

**) $\varkappa(r, n, S_0) = \sup\{|w|\colon w \in Q_n(z, S_0)\}$,

при $Q_n(z, S_0) = \{w\colon w = f(\xi)/f(z)\}$, $\xi \in H_n(z)$, $f \in S_0$.

Из условия **) вытекает, что

$$\varkappa(r, 1, S^c) = \max\{1, r(1-r)^{-1}\}, \varkappa(r, n, S^c) = r^{n-1}\frac{1+r}{1-r_n}, n \geqslant 2$$

$$\varkappa(r, 1, S^*) = \max\{1, r(1-r)^{-2}\}, \varkappa(r, n, S^*) = r^{n-1}\left(\frac{1+r}{1-r_n}\right)^2, n \geqslant 2$$

(теоремы 2, 3).

Доказательства теорем 2, 3 находятся в печати в журнале Michigan Mathematical Journal, vol. 18.