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Properties of Pseudo-Conformal Images of Reinhardt Circular Domains

Własności obrazów obszarów kołowych Reinhardta przy odwzorowaniach pseudo-konforemnych

Свойства образов круговых областей Рейнгардта при псевдоконформных отображениях

1. Introduction

A one-to-one transformation t of the domain of $\mathscr{C}^2$(1) by a pair of analytic functions of two complex variables z_1, z_2,

$$z_k^* = f_k(z_1, z_2), \tag{1}$$

is called a PCT (pseudo-conformal transformation). In a previous paper the author considered a complete set $\{\varphi_\nu(z_1, z_2)\}$, $\nu = 1, 2, \ldots$, of functions which are orthonormal in the (bounded) domain B, $B \subset \mathscr{C}^2$.

$$\sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(t)}, \quad z = (z_1, z_2), \quad t = (t_1, t_2) \tag{2}$$

converges for $z \in B$, $t \in B$ and is independent of the choice of the set $\{\varphi_\nu\}$ of functions orthonormal in B. It is denoted as the *kernel function* $K_B(z, \bar{t})$ of B. See [8], p. 178, [2], [4], pp. 31, 32.

The hermitian differential form

(2) $ds_B^2(z_1, z_2)$

$$= J(z_1, z_2; \bar{z}_1, \bar{z}_2) \sum_{m,n=1}^{2} T_{m\bar{n}}(z_1, z_2; \bar{z}_1, \bar{z}_2)\, dz_m d\bar{z}_n, \qquad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},$$

(1) $\mathscr{C}^2$ denotes the space of two complex variables $z_k = x_k + iy_k$; here and in the following $k = 1, 2$.

defines in B a metric which is invariant with respect to PCT's *of B. J is an arbitrary function which is invariant with respect to* PCT's; *in particular one can set $J = 1$*. See [8], p. 183. See also [2], [4], p. 53, [5].

A domain C which admits the one parameter group

$$z_k^* = z_k e^{i\varphi}, \quad 0 \leqslant \varphi \leqslant 2\pi, \tag{3}$$

of PCT's onto itself (automorphism) is called a circular domain with the center at the origin. A circular domain which admits the two parameter group

$$z_k^* = z_k e^{i\varphi_k}, \quad 0 \leqslant \varphi_k \leqslant 2\pi \tag{4}$$

is called a Reinhardt circular domain. See [8], p. 168, [4], p. 14 ff.

The theory of PCT's differs in many respects from the theory of CT's (conformal transformations). A simply connected bounded domain of $\mathscr{C}^2$ cannot in general be mapped by a PCT onto another domain of $\mathscr{C}^2$.

One of the questions of the theory is to determine whether a given domain, say B, can be mapped by a PCT onto a Reinhardt circular domain, and, if this is the case, to find the pair of analytic functions $\{v^{10}, v^{01}\}$ which realize this mapping. This question has been discussed in [11] and in other papers (to appear) by the author. In the following we shall give a short survey of the results of these papers.

2. Multiply connected domains of $\mathscr{C}^1$

It should be stressed that the situation in the case of PCT's of simply connected domains of $\mathscr{C}^2$ is similar to the situation of conformal mapping of n-ply connected domains $2 \leqslant n < \infty$. Therefore it will be instructive

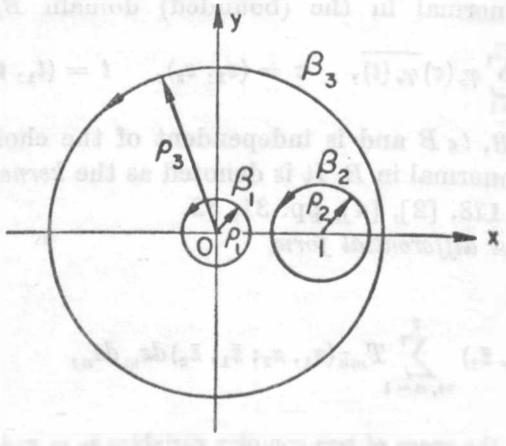

Fig. 1. The domain Ω

to consider at first an example of mappings of 3-ply connected domains of $\mathscr{C}^1$. Following [10] we consider the domain Ω bounded by three circles. See Fig. 1.

In this case the line element of the invariant metric is

$$ds_{\Omega}(z) = \sqrt{K_{\Omega}(z, \bar{z})}\,[dx^2 + dy^2]^{1/2}, \tag{2a}$$

while

$$J_{\Omega}(z, \bar{z}) = \frac{1}{K_{\Omega}} \frac{\partial^2 \log K_{\Omega}}{\partial z \partial \bar{z}} \tag{5}$$

is the curvature of the invariant metric multiplied by $-\frac{1}{2}$. The function $J_{\Omega}(z, \bar{z})$ is in general not constant. In Fig. 2 the level lines

$$J_{\Omega}(z, \bar{z}) = \text{const} = c \tag{6}$$

in the upper half of Ω are indicated. When we approach the boundary (along a path which lies completely in Ω), $\lim J_{\Omega}(z, \bar{z}) = 2\pi$, see [8], p. 39.

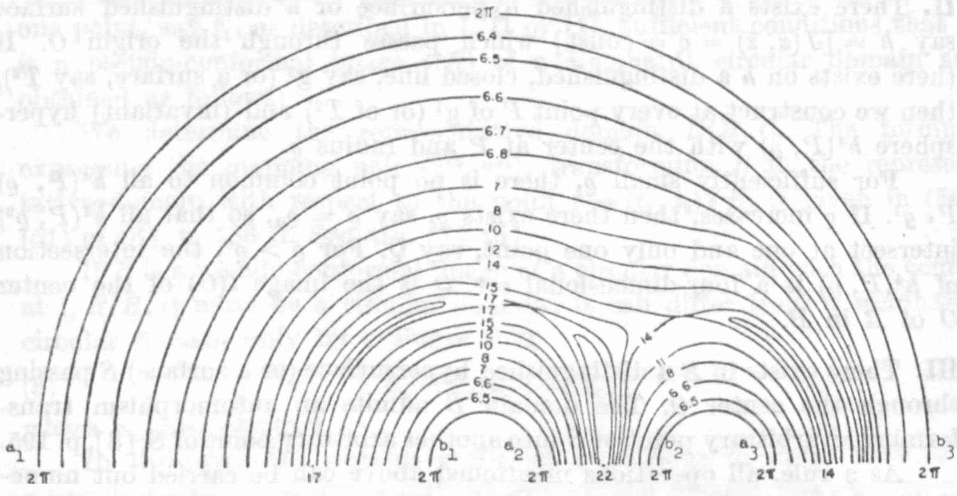

Fig. 2. The level lines of $J_{\Omega}(z, \bar{z}) = c = \text{const.}$ in the upper half $(y > 0)$ of the domain Ω, with $p_1 = 1/4$, $p_2 = 1/4$, $p_3 = 2$

$J_{\Omega}(z, \bar{z})$ has in Ω critical points, which we call *interior distinguished points*. See [5], p. 48. Since $J_{\Omega}(z, \bar{z})$ is an invariant these points are preserved in CT's of Ω. As the case of doubly connected domains shows, $J(z, \bar{z})$ can have a distinguished line. For details see the paper by K. Zarankiewicz [18].

3. The behavior at the center of the invariant J in the case of a Reinhhardt circular domain

In the case of simply connected domains of $\mathscr{C}^2$ as a rule the inverse of the (scalar) curvature

$$(5a)\qquad J(z,\bar z) = \frac{K}{T_{11}T_{22}-|T_{12}|^2},\quad T_{m\bar n} = \frac{\partial^2 \log K}{\partial z_m \partial \bar z_n},\quad z=(z_1,z_2)$$

(see [37a], p. 183 of [8]) is not constant.

In order to decide whether a domain, say B, is a pseudo-conformal image of a Reinhardt circular domain R in the paper by Bergman and Hahn [11], the following invariant characterization of the center O of R is derived.

Three possibilities are considered.

I. The level hypersurfaces (or a segment) of

$$(7)\qquad 1/J(z,\bar z) = \text{const},\qquad z=(z_1,z_2)$$

degenerate at O to a point.

II. There exists a distinguished hypersurface or a distinguished surface say $h = [J(z,\bar z) = c = \text{const}]$ which passes through the origin O. If there exists on h a distinguished, closed line, say g^1 (or a surface, say T^2), then we construct at every point P of g^1 (or of T^2) and (invariant) hypersphere $h^4(P,\varrho)$ with the center at P and radius ϱ.

For sufficiently small ϱ, there is no point common to all $\bar h^4(P,\varrho)$ $P\in g^1$. If ϱ increases, then there exists ϱ, say $\varrho=\varrho_0$, so that all $\bar h^4(P,\varrho^0)$ intersect at one and only one point, say Q. For $\varrho>\varrho^0$, the intersection of $h^4(P,\varrho)$ is a four-dimensional set. Q is the image $t(O)$ of the center O of R in B.

III. There exists in B a distinguished hypersurface (or a surface) S passing through the center O. The domain B admits an automorphism transforming an arbitrary point of S into another arbitrary point of S, [8], p. 195.

As a rule, all operations mentioned above can be carried out numerically, and we obtain a necessary condition in order that a domain B is a pseudoconformal image of a Renhardt circular domain possesing the properties indicated above. Suppose $J_B(z,\bar z)$ is not constant. If B is a pseudoconformal image of a Reinhardt circular domain, then there exists in B a point, say t_1, in which the level lines of the invariant $J_B(z,\bar z)$ behave as indicated in I, II, or III.

Remark. It should be noted that using the kernel function one can determine various quantities, invariant with respect to PCT's. Skwarczyński in [15] introduced an invariant distance

$$\varrho(t^{(1)}, t^{(2)}) = \left[2-2\left(\frac{K(t^{(1)}, t^{(2)})K(t^{(2)}, t^{(1)})}{K(t^{(1)}, t^{(1)})K(t^{(2)}, t^{(2)})}\right)^{1/2}\right]^{1/2}$$

between two points $t^{(k)} = (t_1^{(k)}, t_2^{(k)})$. In [11] Skwarczyński investigated the relations between ϱ and the representative domains. See also [5] p. 48 and [16].

4. The mapping pair $\{v^{10}, v^{01}\}$ transforming B into R

In order to determine the pair of functions mapping B onto a circular domain R (if this is possible) we use the following results.

Theorem 4.1. *Let B be a domain possessing the kernel function $K(z, \bar{z})$ To every point $t \in B \subset \mathscr{C}^2$, there exists a representative (domain) $R(B, t)$. The mapping pair $\{v^{10}, v^{01}\}$ of B onto $R(B, t)$ can be expressed in terms of K_B and its derivatives.* See (50) and (51) of [8], p. 188 ff, [1], [5], p. 29.

Theorem 4.2. *If the domain B can be mapped onto B^* by a* PCT *so that the point $t \in B$ goes into $t^* \in B^*$, then a linear* PCT *maps $R(B, t)$ onto $R(B^*, t^*)$.* See [8], p. 190; [5], p. 29; [1].

Once we established that in the domain there exists one and only one point, say t_1, as described in I, II or III, sufficient conditions that B is a pseudo-conformal image $t(R)$ of a Reinhardt circular domain are obtained as follows:

We determine the representative domain $R(B, t)$. The formula expressing the mapping pair $\{v^{10}, v^{01}\}$, transforming B in the representative domain with respect to the point $t = (t_1, t_2) \in B$, is given in (50), (51) of [8], p. 188 ff. See also [1], [5].

If B is a pseudo-conformal image of a circular domain with the center at t, $R(B, t)$ must be a circular domain; it can differ from a Reinhardt circular domain only by a linear PCT

(8) $$z_k^* = a_{k1}z_1 + a_{k2}z_2,$$

where $a_{k\nu}$ are constant.

In this way one obtains sufficient conditions in order that B is a pseudo-conformal image of a Reinhardt circular domain. See also [6], [7],

For further details see [11].

5. Series developments of K_R and J_R at the center O in terms of certain geometric quantities (*)

The possibility of explicit computations indicated above is connected with the determination of the kernel function $K_B(z, \bar{t})$. One can prove

(*) Geometric quantities = quantities which can be computed if the domain given.

that for a large class of simply connected domains (including bounded domains), every function regular in $B \subset \mathscr{C}^2$ can be approximated by polynomials. Domains of this kind are called Runge's domains. See [12], p. 129. In B the system of orthonormal polynomials is complete. See also [17]. Another procedure to obtain a complete set of orthonormal functions for analytic polyhedra is based on an integral representation for analytic functions of two complex variables.

In order to derive the desired results, one has to approximate (3.7), p. 861 of [9] by a finite sum. In this case one obtains the system

$$\frac{B_{sk}(t_1, t_2, \lambda_k^{(\nu)}, \lambda_s^{(\mu)})}{\Phi_k(t_1, t_2, \lambda_k^{(\nu)}) \Phi_s(t_1, t_2, \lambda_s^{(\mu)})}, \quad k, s = 1, 2, \ldots, n.$$

Here $\lambda_k^{(\nu)}$ and $\lambda_s^{(\mu)}$ is a sufficiently dense set of points for every k, s. For the notation see [9].

Once a development for the kernel function is established, the determination of $J_B(z, \bar{z})$ and of $T_{m\bar{n}}$ is a computational task.

A further method to compute the kernel function for certain domains (Siegel domains) is based on the use of a generalized β-function (see [14]).

In addition to the characterizations **I, II, III**, see p. 12, we can use the following property of the hypersurface (7) in the case of a domain B which is a pseudo-conformal image of a circular domain C. Let t^* be a point of (7) and a the intersection of (7) and a conveniently chosen ball (relative to the metric (2)) with the center at t^*. If we make the assumption that a is connected but $(a-t^*)$ is no longer connected, then (under some further simple conditions) t^* is the image in B of the center of C.

As indicated in [11] the formulas expressing the invariant J in terms of geometric quantities become comparatively simple for Reinhardt circular domains. For instance, the coefficients $E_{mn}^{(1)}$ of the development of

$$(9) \qquad 1/J_R = K^{-1} \det\left(\frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n}\right) = \sum_{m,p} E_{mp}^{(1)} |z_1|^{2m} |z_2|^{2p},$$

at the center O of R can be expressed in terms of polynomials in B_{00}^{-1}, B_{mp},

$$(10) \qquad B_{mp}^{-1} = \int_R |z_1|^{2m} |z_2|^{2p} d\omega, \ (mp) = (00), (10), (01), \ldots,$$

$$d\omega = \text{volume element}.$$

It holds

$$(11) \qquad E_{00}^{(1)} = B_{10} B_{10} B_{00}^{-3}, \ E_{10}^{(1)} = (B_{10} B_{11} + 4 B_{01} B_{20}) B_{00}^{-3} - 4 B_{10}^2 B_{01} B_{00}^{-4}, \ldots$$

for further relations of similar type see [11].

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STRESZCZENIE

Niech B będzie obszarem w przestrzeni $\mathscr{C}^2$ dwóch zmiennych zespolonych z_1, z_2, mającym funkcję jądrową K_B. Przy pomocy K_B możemy określić metrykę M_B, przy czym element liniowy ds_B wyraża się wzorem

$$ds_B^2 = \sum T_{m\bar{n}} dz_m d\bar{z}_n, \quad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n}$$

Jest ona niezmiennicza ze względu na PCT (przekształcenie pseudokonforemne, tzn. homeomorfizmy określone przez parę $z_k^* = f_k(z_1, z_2)$ funkcji analitycznych dwóch zmiennych).

Obszar posiadający dwuparametrową grupę przekształceń $z_k^* = z_k e^{i\varphi_k}$, $k = 1, 2$, na siebie, nazywa się obszarem kołowym Reinhardta R. W pracy tej bada się rozmaite własności metryki M_R. Znajduje się warunki na to by dany obszar B, $B \subset \mathscr{C}^2$, był pseudokonforemnym obszarem R oraz wyznacza się w tym przypadku parę funkcji (v^{10}, v^{01}) określającą odwzorowanie B na R.

РЕЗЮМЕ

Пусть B — область в двухмерном комплексном пространстве $\mathscr{C}^2$ переменных z_1, z_2, имеющая кернфункцию K_B. При помощи K_B можно определить метрику M_B, при этом элемент длины ds_B имеет вид

$$ds_B^2 = \sum T_{m\bar{n}} dz_m d\bar{z}_n; \quad T_{m\bar{n}} = \frac{\partial_2 \log K}{\partial z_m \partial \bar{z}_n}$$

Эта метрика обладает свойством инвариантности относительно псевдоконформных отображений PCT (т. е. гомеоморфизмов, определенных парами $z_k^* = f_k(z_1, z_2)$ аналитических функций переменных z_1, z_2).

Область R, допускающая двухпараметрическую группу преобразований $z_k^* = z_k e^{i\varphi_k}$, $k = 1, 2$, на себя называется круговой областью Рейнгардта. В этой работе исследованы разные свойства метрики M_R, которые использованы для получения условий, чтобы заданная область B, $B \subset C^2$ являлась псевдоконформной областью R. В этом случае определена пара функций $\{v^{10}, v^{01}\}$, отображающая B на R.