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Z Zakładu Funkcji Analitycznych Zespołowej Katedry Matematyki
Wydziału Mat.-Fiz.-Chem. UMCS
Kierownik Zakładu: doc. dr Jan Krzyż

BARBARA PIŁAT

On Typically Real Functions with Montel's Normalization

O pewnych podklasach funkcji typowo rzeczywistych z unormowaniem Montela

О некоторых подклассах типично-вещественных функций с нормированием Монтеля

1. Introduction. Notation and statement of results.

In his book [6] P. Montel suggested to investigate two classes of functions regular and univalent in the unit circle $K = \{z: |z| < 1\}$. The former class should satisfy the normalization conditions

$$f(0) = 0, \tag{1.1}$$

$$f(z_0) = z_0, \tag{1.2}$$

while the latter one should satisfy

$$f'(z_0) = 1 \tag{1.2a}$$

instead of (1.2), where $z_0 \neq 0$ is a fixed point of K.

The problem of determining the domain of all possible values of $f(z)$ for the former class was treated in [4] for the general case and in [5] for the particular case of starshaped functions.

In this paper we obtain distortion theorems for functions regular and univalent in K which satisfy (1.1), (1.2) and (1.1), (1.2a) resp. and have real Taylor coefficients at the origin. The corresponding theorems are obtained as corollaries of analogous theorems concerning the more general classes of functions which satisfy

$$\operatorname{im} z \cdot \operatorname{im} f(z) > 0 \tag{1.3}$$

for any $z \in K$ with $\operatorname{im} z \neq 0$, instead of univalency. The inequality (1.3) means that any function satisfying this condition takes real values only on the real axis and such functions are called following W. W. Rogosinski

[9] typically — real. Let $T(z_0)$ be the class of functions regular in K which satisfy (1.1)-(1.3) with $0 < z_0 < 1$ and let $S(z_0)$ be the corresponding subclass of univalent functions.
Similarly, let $T'(z_0)$ be the class of functions regular in K which satisfy (1.1), (1.2a) and (1.3) and let $S'(z_0)$ be the corresponding subclass of univalent functions.
We also consider the class T_1 of functions regular in K which satisfy (1.1), (1.3) and

$$\lim_{x\to 1-} f(x) = 1, \quad x \text{ real}, \tag{1.2b}$$

as well as the corresponding subclass S_1 of univalent functions.

In this paper we find structural formulae for the classes $T(z_0)$, $T'(z_0)$, T_1 (Theorems 2.1, 4.1, 5.1) which yield in a standard manner the corresponding sets $\Omega(z, z_0)$, $\Omega'(z, z_0)$, $\Omega_1(z)$ of all possible values $f(z)$ where z, z_0 are fixed and f ranges over the given class (Theorems 2.2, 4.2, 5.2). This enables us to find the exact bounds of $|f(z)|$, $|f'(z)|$, $|\operatorname{im} f(z)|$. In the limiting case $z_0 \to 0$, $T(z_0)$ becomes the well known class T of typically-real functions with the usual normalization $f(0) = 0$, $f'(0) = 1$, and we obtain the corresponding results due to G. M. Golusin [3] and M. P. Remisova [7]. Since some parts of the boundary of the sets considered above correpond to univalent functions, we obtain at the same time distortion theorems for the corresponding subclasses of univalent functions.
In particular, the corresponding results for the class S_1 (Theorems 4.1-4.6) are generalizations of some theorems of V. Singh [11] obtained under some further restrictions by variational methods.

The results given in this paper form a part of a Ph. D. thesis written under supervision of Professor J. Krzyż.

I wish to express my sincere gratitude to Professor J. Krzyż for his interest in this work and for his remarks, and also to Professor W. Kleiner for helpful comments.

2. Structural formula for the class $T(z_0)$

We have the known lemma [13, p. 134] for Riemann-Stieltjes integrals which will be our basic tool for what follows.

Lemma 2.1. Let $g(t)$ be a continuous (real – or complex – valued) function of the real variable $t \in [0, 1]$, let $h(t)$ be a real, continuous and positive function of $t \in [0, 1]$ and let $\beta(t)$ be real and non-decreasing in $[0, 1]$. If

$$\mu(t) = \int_0^t h(\tau)\cdot d\beta(\tau),$$

then the Riemann — Stieltjes integrals

$$\int_0^1 \frac{g(t)}{h(t)}\, d\mu(t), \tag{2.1}$$

$$\int_0^1 g(t)\, d\beta(t) \tag{2.1'}$$

both exist and are equal.

We are now in a position to find a structural formula for $T(z_0)$.

Theorem 2.1. *If $f(z)$ is a function of the class $T(z_0)$, then there exists a real, non-negative and non-decreasing function $\mu(t)$, $t \in [0,1]$, satisfying*

$$\int_0^1 d\mu(t) = 1 \tag{2.2}$$

and such that

$$f(z) = \int_0^1 \frac{z[z_0^2+1+2z_0(2t-1)]}{z^2+1+2z(2t-1)}\, d\mu(t) \tag{2.3}$$

holds. On the other hand, if $f(z)$ has the representation (2.3) *with $\mu(t)$ satisfying the above stated conditions, then $f \in T(z_0)$.*

Proof. If $f \in T(z_0)$, then there exists obviously a positive number k such that $\varphi = kf \in T$. Conversely, for any $\varphi \in T$ we can find $\varkappa > 0$ such that $\varkappa\varphi \in T(z_0)$. As shown by M. S. Robertson [8], for any $\varphi \in T$ we have the representation

$$\varphi(z) = \int_0^1 \frac{z\, da(t)}{z^2+1+2z(2t-1)} \tag{2.4}$$

where $a(t)$ is non-negative and non-decreasing in $[0,1]$ and satisfies

$$\int_0^1 da(t) = 1 \tag{2.5}$$

and any φ as given by (2.4), (2.5) belongs to T.
In view of (2.4) we obtain for any $f \in T(z_0)$ the following representation

$$f(z) = \int_0^1 \frac{z\, d\beta(t)}{z^2+1+2z(2t-1)}, \tag{2.6}$$

where $\beta(t) = \varkappa a(t)$ is real, non-negative and non-decreasing in $[0,1]$ and satisfies

$$\int_0^1 \frac{d\beta(t)}{z_0^2+1+2z_0(2t-1)} = 1 \tag{2.7}$$

which is a consequence of (1.2).

Next we put

$$\mu(t) = \int_0^t \frac{d\beta(\tau)}{z_0^2+1+2z_0(2\tau-1)}. \tag{2.8}$$

In view of (2.8) $\mu(t)$ is real, non-decreasing and non-negative in $[0, 1]$ and satisfies

$$\int_0^1 d\mu(t) = \mu(1)-\mu(0) = 1 \tag{2.9}$$

which is a consequence of (2.7).

We now apply Lemma 2.1 with

$$g(t) = z[z^2+1+2z(2t-1)]^{-1},$$
$$h(t) = [z_0^2+1+2z_0(2t-1)]^{-1}$$

and we obtain in view of (2.6) and (2.8) the following equality

$$f(z) = \int_0^1 g(t)d\beta(t) = \int_0^1 \frac{g(t)}{h(t)}d\mu(t)$$

which yields the structural formula (2.3). Conversely, it is easy to verify that any functions represented by (2.3) satisfies (1.2) and has the form $\varkappa\varphi$ with $\varphi \in T$ and $\varkappa > 0$. This means that $f \in T(z_0)$. Theorem 2.1 is proved.

The set of all possible values of the Riemann — Stieltjes integral $\int_0^1 H(t)d\mu(t)$ where $H(t)$ is a fixed, continuous, complex-valued function and $\mu(t)$ is varying, can be determined by means of the following lemma which is well known (cf. e.g. [1]).

Lemma 2.2. Let $H(t)$ be a fixed, continous, complex-valued function of a real variable $t \in [0, 1]$ and let $\mu(t)$ be a variable non-negative, non-decreasing function of $t \in [0, 1]$ which satisfies $\mu(1)-\mu(0) = 1$. Then the set of all possible values of the integral

$$I(\mu) = \int_0^1 H(t)d\mu(t) \tag{2.10}$$

is the convex hull of the curve $w = H(t)$, $0 \leqslant t \leqslant 1$.

Using the Theorem 2.1 and Lemma 2.2 we can easily determine the set $\Omega(z, z_0)$. We obtain

Theorem 2.2. *Put*

$$\zeta_0 = z_0+\frac{1}{z_0}, \tag{2.11'}$$

$$\zeta = z+\frac{1}{z}. \tag{2.11}$$

The set $\Omega(z, z_0)$ of all possible values $f(z)$ for a fixed $z \in K$ and f ranging over $T(z_0)$ is the circular segment bounded by the circular arc

$$w = z_0 \frac{\zeta_0 + \tau}{\zeta + \tau}, \quad -2 \leqslant \tau \leqslant 2, \tag{2.12}$$

with and-points

$$A = \frac{(1+z_0)^2 z}{(1+z)^2} \tag{2.13}$$

$$B = \frac{(1-z_0)^2 z}{(1-z)^2}, \tag{2.14}$$

For real z the set $\Omega(z, z_0)$ reduces to the interval with end-points A, B.

Proof. In view of (2.3) the values $f(z)$ have the form (2.10) with

$$H(t) = z_0 \frac{\zeta_0 + 2(2t-1)}{\zeta + 2(2t-1)}, \quad 0 \leqslant t \leqslant 1.$$

The curve described by the point $H(t)$ is the circular arc

$$w(\tau) = z_0 \frac{\zeta_0 + \tau}{\zeta + \tau}, \quad -2 \leqslant \tau \leqslant 2,$$

with end-points determined by (2.13), (2.14). Our theorem now follows readily as a consequence of Lemma 2.2.

3. Distortion theorems for the class $T(z_0)$.

Let Γ_1 be the image of the circumference

$$|\zeta - \tfrac{1}{2}(\zeta_0 - 2)| = \tfrac{1}{2}(\zeta_0 + 2) \text{ under the mapping}$$

$$z = z(\zeta) = (\zeta - \sqrt{\zeta^2 - 4})/2 \tag{3.1}$$

where we take this branch of the square-root for which $|z(\zeta)| \leqslant 1$. Let Γ_2 be the image of the circumference

$$|\zeta - \tfrac{1}{2}(\zeta_0 + 2)| = \tfrac{1}{2}(\zeta_0 - 2) \text{ under the same mapping.}$$

Both Γ_1 and Γ_2 are Jordan curves symmetric w.r.t. the real axis which have a common tangent at z_0 and one-sided tangents at -1 and 1 resp. intersecting at right angles. Let $D_k (k = 1, 2)$ be inside domains of Γ_k and put $D_3 = K \setminus (D_1 \cup D_2)$.

We now state the following

Theorem 3.1. *If $f \in T(z_0)$, then the following exact estimations hold: if $z \in D_1$ then*

$$\frac{(1-z_0)^2 |z|}{|1-z|^2} \leqslant |f(z)| \leqslant \frac{(1+z_0)^2 |z|}{|1+z|^2}; \tag{3.2}$$

if $z \in D_2$ then

$$(3.3)\qquad \frac{(1+z_0)^2|z|}{|1+z|^2} \leqslant |f(z)| \leqslant \frac{(1-z_0)^2|z|}{|1-z|^2}.$$

Put now

$$(3.4)\qquad t_0 = \frac{|\zeta|^2 - \zeta_0 \operatorname{re}\zeta}{\operatorname{re}\zeta - \zeta_0},$$

where ζ_0, ζ are given by (2.11′), (2.11).

If $z \in D_3$, then

$$(3.5)\qquad \left|\frac{|z(z-z_0)(1-zz_0)|}{(1-z^2)^2} \operatorname{im} \frac{(1-z)^2(1+z_0)^2}{(z-z_0)(1-zz_0)}\right| \leqslant |f(z)| \leqslant \frac{z_0(\zeta_0 - t_0)|z|}{|z_0^2+1-t_0 z|}.$$

The upper bound in (3.2) *and the lower bound in* (3.3) *are attained by the function*

$$(3.6)\qquad f_1(z) = \frac{(1+z_0)^2 z}{(1+z)^2}.$$

The lower bound in (3.2) *and the upper bound in* (3.3) *are attained by the function*

$$(3.7)\qquad f_2(z) = \frac{(1-z_0)^2 z}{(1-z)^2}.$$

The upper bound in (3.5) *is attained by the function*

$$(3.8)\qquad f_3(z) = \frac{z(z_0^2+1-z_0 t_0)}{z^2+1-zt_0}$$

where t_0 is defined by (3.4).

The lower bound in (3.5) *is attained by the function*

$$(3.9)\qquad f_4(z) = \lambda f_1(z) + (1-\lambda) f_2(z)$$

where $\lambda \in [0, 1]$ satisfies

$$\left|\frac{4\lambda}{(1+z_0)^2} + \frac{(1-z)^2}{(1-zz_0)(z-z_0)}\right| = \left|\operatorname{im} \frac{(1-z)^2}{(z-z_0)(1-zz_0)}\right|.$$

Proof. The set $\Omega(z, z_0)$ of all possible values $f(z)$ is the circular segment determined by (2.12). According to the position of A, B as given by (2.13), (2.14) there are 3 possible cases.

(i) Suppose that $\sphericalangle OBA \geqslant \frac{\pi}{2}$. Since the circumference (2.12) contains the origin, we have $\sup_{f \in T(z_0)} |f(z)| = |A|$, $\inf_{f \in T(z_0)} |f(z)| = |B|$. On the other hand $\operatorname{re}\frac{B}{B-A} \leqslant 0$ in this case which implies, in view of (2.11′), (2.11), (2.13), (2.14), $\operatorname{re}\frac{\zeta+2}{\zeta-\zeta_0} \geqslant 0$ and this means that $z \in \overline{D}_1 \setminus \{-1\}$

(ii) Suppose that $\sphericalangle OAB \geqslant \frac{\pi}{2}$, or $\operatorname{re}\frac{A}{A-B} \leqslant 0$ which implies similarly as in (i) $\operatorname{re}\frac{\zeta-2}{\zeta-\zeta_0} \leqslant 0$, i.e. $z \in \overline{D}_2 \setminus \{1\}$. We have now $\sup_{f \in T(z_0)} |f(z)| = |B|$, $\inf_{f \in T(z_0)} |f(z)| = |A|$ and this yields (3.3).

(iii) Suppose now that both angles $\sphericalangle OBA$, $\sphericalangle OAB$ are less than $\frac{\pi}{2}$. This implies $\operatorname{re}\frac{B}{B-A} > 0$, $\operatorname{re}\frac{A}{A-B} > 0$, i.e. $\operatorname{re}\frac{\zeta+2}{\zeta-\zeta_0} < 0$, $\operatorname{re}\frac{\zeta-2}{\zeta-\zeta_0} < 0$ which means that $z \in D_3$. In this case the upper bound of $|f(z)|$ corresponds to a point of $\partial\Omega(z, z_0)$ lying on the open circular arc with andpoints A, B, whereas the lower bound corresponds to a point on the chord $[A, B]$. An elementary calculation yields now (3.5).

It is easy to see that the functions $f_k(z)$ $(k = 1, 2, 3)$ are univalent. This means that these functions realize the exact bounds of $|f(z)|$ also for the subclass $S(z_0)$ of univalent functions. Hence we obtain as a particular case of Theorem 3.1. the following

Theorem 3.2. *If $f(z)$ belongs to the class $S(z_0)$ of functions regular and univalent in K which satisfy* (1.1)-(1.3), *then the following exact estimations hold: if $z \in D_1$, then*

$$\frac{(1-z_0)^2|z|}{|1-z|^2} \leqslant |f(z)| \leqslant \frac{(1+z_0)^2|z|}{|1+z|^2} \tag{3.10}$$

if $z \in D_2$, then

$$\frac{(1+z_0)^2|z|}{|1+z|^2} \leqslant |f(z)| \leqslant \frac{(1-z_0)^2|z|}{|1-z|^2} \tag{3.11}$$

if $z \in D_3$, then

$$|f(z)| \leqslant \frac{z_0(\zeta_0 - t_0)|z|}{|z^2+1-t_0 z|} \tag{3.12}$$

where t_0 is given by (3.4).

We now determine the set of all possible values of the derivative $f'(z)$ of $f \in T(z_0)$.

Theorem 3.3. *The set $D(z, z_0)$ of all possible values $f'(z)$ for fixed $z \in K$ and variable f ranging over the class $T(z_0)$ is the closed convex domain whose boundary consists of the arc*

$$W(\tau) = \frac{z_0(1-z^2)(\zeta_0+\tau)}{z^2(\zeta+\tau)^2}, \quad -2 \leqslant \tau \leqslant 2 \tag{3.13}$$

and the straight line segment $[A', B']$, where

$$A' = \frac{(1-z_0)^2(1+z)}{(1-z)^3}, \tag{3.14}$$

$$(3.15)\qquad B' = \frac{(1+z_0)^2(1-z)}{(1+z)^3}.$$

Proof. Differentiating (2.3) we obtain

$$(3.16)\qquad f'(z) = (1-z^2)\int_0^1 \frac{z_0^2+1+2z_0(2t-1)}{[z^2+1+2z(2t-1)]^2}\,d\mu(t).$$

Hence the derivative $f'(z)$ has the form (2.10) with

$$H(t) = \frac{z_0(1-z^2)[\zeta_0+2(2t-1)]}{z^2[\zeta+2(2t-1)]^2}, \qquad 0 \leqslant t \leqslant 1.$$

Let $\varphi(\tau)$ be the complex function of the real variable which has a continuous derivative and does not vanish.

Then we have $\dfrac{d}{d\tau}\arg\varphi(\tau) = \operatorname{im}\dfrac{\varphi'(\tau)}{\varphi(\tau)}$.

Using this with $\varphi(\tau) = H_1'(\tau)$ where $H_1(\tau) = \dfrac{\zeta_0+\tau}{(\zeta+\tau)^2}$ we see that $\arg H_1'(\tau)$ is strictly monotonic. Moreover, the change of $\arg H_1'(\tau)$ does not surpass π (see e.g. [7]). This means that the convex hull of $H_1(\tau)$ as well as that of $H(t)$ are bounded by the corresponding arcs of $H_1(\tau)$ and $H(t)$ resp. and by the chords joining the end-points. Theorem 3.3 now follows in view of Lemma 2.2.

We now give the exact bounds of $|f'(z)|$ for $f\in T(z_0)$.

Let $\Delta_k (k = 1, 2)$ be the inside domains of γ_k where γ_k are maps of the circumferences $|\zeta-\zeta_0| = \zeta_0+2$, $|\zeta-\zeta_0| = \zeta_0+2$ resp., under the transformation (3.1) and put $\Delta_3 = K\setminus(\Delta_1 \cup \Delta_2)$.

The curves γ_1, γ_2 are disjoint Jordan curves symmetric w.r.t. the real axis which have at the points ∓ 1 one sided tangents intersecting at an angle $\dfrac{\pi}{2}$. Under the above notation the following theorem holds.

Theorem 3.4. *Suppose $f\in T(z_0)$. Then we have the sharp inequalities: if $z\in\Delta_1$, then*

$$(3.17)\qquad |f'(z)| \leqslant \frac{(1+z_0)^2|1-z|}{|1+z|^3};$$

if $z\in\Delta_2$, then

$$(3.18)\qquad |f'(z)| \leqslant \frac{(1-z_0)^2|1+z|}{|1-z|^3}$$

if $z\in\Delta_3$, then

$$(3.19)\qquad |f'(z)| \leqslant \frac{z_0^2|1-z^2|}{2|z-z_0|\,|1-zz_0|(1+\cos a)|z|}$$

where $a = \arg(z_0-z)+\arg(1-zz_0)-\arg z$.

The signs of equality in (3.17) *and* (3.18) *hold for* $f_1(z)$ *and* $f_2(z)$ *resp., where* $f_k(z)$ *are defined by the formulae* (3.6), (3.7). *The sign of equality in* (3.19) *holds for the function*

$$f(z) = \frac{z(1+z_0^2+z_0\tau^*)}{z^2+1+z\tau^*}, \tag{3.20}$$

where $\tau^* = |\zeta-\zeta_0|-\zeta_0$.

Proof. It follows from (3.16) that

$$|f'(z)| \leqslant \left|\frac{1-z^2}{z^2}\right| \int_0^1 |s(z_0, z, t)|\, d\mu(t) \tag{3.21}$$

where

$$s(z_0, z, t) = \frac{z^2[z_0^2+1+2z_0(2t-1)]}{[z^2+1+2z(2t-1)]^2}. \tag{3.22}$$

Now, it can be easily deduced that

$$\text{if } z\in\Delta_1, \text{ then } \sup_{0\leqslant t\leqslant 1} |s(z_0, z, t)| = s(z_0, z, 1),$$

$$\text{if } z\in\Delta_2, \text{ then } \sup_{0\leqslant t\leqslant 1} |s(z_0, z, t)| = s(z_0, z, 0),$$

$$\text{if } z\in\Delta_3, \text{ then } \sup_{0\leqslant t\leqslant 1} |s(z_0, z, t)| = s\big(z_0, z, (|\zeta-\zeta_0|-\zeta_0+2)/4\big)$$

Now, the well known estimation for Riemann-Stieltjes integrals implies the Theorem 3.4.

All the extremal functions are univalent and this means that the inequalities (3.17) – (3.19) are also best possible for the derivative $f'(z)$ of $f\in S(z_0)$.

The sets Δ_k defined above appear also in the bounds of imaginary parts of functions of the classes $T(z_0)$ and $S(z_0)$. We have the following

Theorem 3.5. *Suppose* $f\in T(z_0)$. *Then we have the sharp inequalities*: *if* $z\in\Delta_1$, *then*

$$|\operatorname{im} f(z)| \leqslant (1+z_0)^2 \left|\operatorname{im}\frac{z}{(1+z)^2}\right|, \tag{3.23}$$

if $z\in\Delta_2$, *then*

$$|\operatorname{im} f(z)| \leqslant (1-z_0)^2 \left|\operatorname{im}\frac{z}{(1-z_0)^2}\right|, \tag{3.24}$$

if $z\in\Delta_3$, *then*

$$|\operatorname{im} f(z)| \leqslant \frac{|z|}{|z-z_0|\,|1-zz_0|(1+\cos a)} \left|\operatorname{im}\frac{1+z^2}{z}\right| \tag{3.25}$$

where $a = \arg(z_0-z)+\arg(1-zz_0)-\arg z$.

The extremal functions are the same as in Theorem 3.4.

Proof. It follows from (2.3) that

$$\operatorname{im} f(z) = -\operatorname{im}\left(\frac{1+z^2}{z}\right)\int_0^1 \frac{z_0^2+1+2z_0(2t-1)}{|z^2+1+2z(2t-1)|^2}\,d\mu(t)$$

and this gives

$$|\operatorname{im} f(z)| \leqslant \frac{1}{|z|^2}\left|\operatorname{im}\frac{1+z^2}{z}\right|\int_0^1 |s(z_0,z,t)|\,d\mu(t) \tag{3.26}$$

The right hand sides in (3.21) and (3.26) are the same apart from a factor which does not depend on t. Using the same argument as in Theorem 3.4 we obtain the inequalities (3.23) – (3.25).

The same inequalities also hold for the class $S(z_0)$.

4. The structural formula for the class T_1 and its applications.

The formula (2.3) enables us to find the structural formula for the class T_1. We prove the following

Theorem 4.1. *If $f \in T_1$, then there exists a non-negative, non-decreasing function $\mu(t)$ statisfying (2.2) and*

$$\lim_{t\to 0+}\mu(t) = \mu(0) = 0 \tag{4.1}$$

such that

$$f(z) = \int_0^1 \frac{4tz}{z^2+1+2z(2t-1)}\,d\mu(t), \tag{4.2}$$

Conversely, if $\mu(t)$ satisfies the above stated conditions, then the function definied by the right hand side term in (4.2) belongs to T_1.

Proof. Suppose $f \in T_1$. It is easy to see that

$$g_n(z) = \left(1-\frac{1}{n}\right)\frac{f(z)}{f\left(1-\frac{1}{n}\right)},\quad n \geqslant 2,$$

belong to $T\left(1-\dfrac{1}{n}\right)$. In view of Theorem 2.1 we have

$$g_n(z) = \int_0^1 \frac{z[z_n^2+1+2z_n(2t-1)]}{z^2+1+2z(2t-1)}\,d\mu_n(t),$$

where $z_n = 1-\dfrac{1}{n}$ and $\mu_n(t)$ are non-negative, non-decreasing and satisfy (2.2). In view of a theorem of Helly (cf. e.g. [11]) we can choose a convergent subsequence $\mu_{n_k}(t)$ with the limit $\mu(t)$ subject to analogous conditions as $\mu_n(t)$. We have $\lim_n g_n(z) = \lim_k g_{n_k}(z) = f(z)$ and

$$g_{n_k}(z) = \int_0^1 \frac{z[\zeta_k^2+1+2\zeta_k(2t-1)]}{z^2+1+2z(2t-1)}\, d\mu_{n_k}(t),$$

where $\zeta_k = 1-\frac{1}{n_k}$. In the last integral the integrand tends in $[0,1]$ uniformly to $\frac{4tz}{z^2+1+2z(2t-1)}$ and this yields in the limiting case the formula (4.2).

Suppose now that $\mu(t)$ is a non-negative, non-decreasing function which satisfies (2.2) and (4.1). We shall prove $f(t)$ as defined by (4.2) belongs to T_1. We first verify that $f(z)$ takes real values only for real z. We have

$$|f(z)-\overline{f(z)}| = |z-\bar{z}|(1-|z|^2)\int_0^1 \frac{4t\,d\mu(t)}{|z^2+1+2z(2t-1)|^2}$$

$$\geqslant \frac{|z-\bar{z}|(1-|z|^2)}{[(|z|+1)^2+4|z|]^2}\int_0^1 4t\,d\mu(t).$$

We now take $\delta\epsilon(0,1)$ such that $\bigvee_\delta^1(\mu)\geqslant\frac{1}{2}$.

Then $\int_0^1 4t\,d\mu(t)\geqslant\int_\delta^1 4t\,d\mu(t)\geqslant 4\delta\cdot\frac{1}{2} = 2\delta > 0$.

Hence $|f(z)-\overline{f(z)}| = 0$ if and only if $z-\bar{z} = 0$. We next verify that (1.2b) holds.

Take an arbitrary $\varepsilon > 0$. In view of (4.1), $h(\delta) = \mu(\delta)-\mu(0) < \frac{1}{2}\varepsilon$ for δ sufficiently small. Again for real z sufficiently near 1 and for δ already chosen

$$\frac{(1-z)^2}{(1-z)^2+4z\delta} < \frac{\varepsilon}{2}.$$

Thus we have

$$|1-f(z)| = \left|(1-z)^2\int_0^\delta \frac{d\mu(t)}{(z-1)^2+4tz} + (1-z)^2\int_\delta^1 \frac{d\mu(t)}{(z-1)^2+4tz}\right|$$

$$\leqslant h(\delta)+\frac{(1-z)^2}{(1-z)^2+4z\delta} < 2\cdot\frac{1}{2}\varepsilon = \varepsilon$$

and this proves that $f\epsilon T_1$.

We have still to prove the necessity of (4.1). Suppose, on the contrary, that $\mu(0) = 0 < \lim_{t\to 0+} \mu(t) = \lambda$. Then we have $\mu(t) = \lambda + (1-\lambda)\nu(t)$, where $\bigvee_0^1(\nu) = 1$ and $\nu(t)$ is continuous at $t = 0$. It is easy to see that $\lim_{x\to 1-} f(x) = 1-\lambda < 1$. The Theorem 4.1 is proved.

Corollary 4.1. *The condition* (1.2b) *cannot be replaced by the condition* $\lim_{z\to 1} f(z) = 1$. Let D be the square $|\mathrm{re}\, w| < 1$, $|\mathrm{im}\, w| < 1$ with removed the segments: $\mathrm{re}\, w = 1-\frac{1}{n}$, $|\mathrm{im}\, w| \geqslant \frac{1}{n}$, $n = 2, 3, \ldots$. The function $w = f(z)$ mapping conformally K on D with $f(0) = 0$, $f'(0) > 0$ obviously belongs to T_1, the limit $\lim_{z\to 1} f(z)$, however, does not exist.

We now determine the set $\Omega(z)$ of values taken by $f(z)$ for fixed $z \in K$ and varying $f \in T_1$.

Theorem 4.2. *Let ζ be defined by* (2.11), *let Γ be the circular are defined by the equation*

$$w = \frac{\tau+2}{\tau+\zeta}, \quad -2 \leqslant \tau \leqslant 2, \tag{4.2}$$

and put

$$B_1 = \frac{4z}{(1+z)^2}. \tag{4.3}$$

The set $\Omega(z) = \{w\colon w = f(z), f \in T_1\}$ is the union of the open circular segment whose boundary consists of Γ and the straight-line segment $[0, B_1]$ and of the arc Γ with the origin excluded.

For real z the set $\Omega(z)$ reduces to the segment $(0, B_1]$.

Proof. Let us consider the class T^* of functions admitting the representation (4.1). Obviously $T_1 \subset T^*$. The value $f(z)$, $f \in T^*$, has the form considered in Lemma 2.2 with $h(t) = 4t[\zeta+2(2t-1)]^{-1}$, $0 \leqslant t \leqslant 1$. The curve described by the point $h(t)$ is the circular arc (4.2) with and points 0, B_1. Using the Lemma 2.2 we obtain the set $D^*(z)$ of values $f(z)$, $f \in T^*$, which shows to be the closed circular segment corresponding to Γ. We now prove that all the points of $D^*(z)$ different from those on the segment $[0, B_1)$ correspond to $f \in T_1$. If $w \in \Gamma$, $w \neq 0$, then w can be expressed by (4.1) with $\mu(t) = \mu_\eta(t)$, where $\mu_\eta(t)$ has a jump equal 1 at $\eta \in (0, 1]$. This gives obviously a function of the class T_1. Suppose now $w \in \{D^*(z) \setminus [0, B_1)\}$. There exists $F_\eta \in T_1$ corresponding to a point B_η on Γ such that $B_\eta = F_\eta(z)$ and $w \in [B_1, B_\eta]$. Then we have $w = F(z)$ with $F(z) = \lambda F_\eta(z) + (1-\lambda)F_1(z)$, where $0 < \lambda < 1$ and $F_1(z) = 4z(1+z)^{-2}$ which corresponds to $\mu(t) = \lambda\mu_\eta(t) + (1-\lambda)\mu_1(t)$. Obviously $F \in T_1$. The points on $[0, B_1)$ correspond to the functions $\lambda F_1(z)$ with $0 \leqslant \lambda < 1$ which do not belong to T_1. This proves our theorem.

Put now $G_1 = K_1 \cap K_2$, where $K_1 = \{z: |z-i| < \sqrt{2}\}$, $K_2 = \{z: |z+i| < \sqrt{2}\}$ and suppose that

$$t^* = \frac{|\zeta|^2 + 2\,\mathrm{re}\,\zeta}{\mathrm{re}\,\zeta - 2} \tag{4.4}$$

where ζ is defined by (2.11).

Under this notation the following theorem holds.

Theorem 4.3. *Suppose $f \in T_1$. Then we have the sharp inequalities: if $z \in G_1$, then*

$$|f(z)| \leqslant \frac{4\,|z|}{|1+z|^2}\,; \tag{4.5}$$

if $z \in K \setminus G_1$, then

$$|f(z)| \leqslant \frac{(2-t^*)\,|z|}{|z^2+1-t^* z|} \tag{4.6}$$

The signs of equality in (4.5) *and* (4.6) *hold for*

$$F_1(z) = 4z(1+z)^{-2} \quad \textit{and} \quad F_2(z) = \frac{(2-t^*)z}{z^2+1-zt^*} \quad \textit{resp.}$$

Proof. In view of (4.1), $|f(z)| \leqslant \sup_{0 \leqslant t \leqslant 1} |F(z,t)|$, where $F(z,t) = \dfrac{4tz}{z^2+1+2z(2t-1)}$. An elementary calculation shows that $\sup_{0 \leqslant t \leqslant 1} |F(z,t)| = |F(z,1)|$ for $z \in G_1$, whereas $\sup_{0 \leqslant t \leqslant 1} |F(z,t)| = |F(z,t_0)|$, where $t_0 = \frac{1}{4}(2-t^*)$ and t^* is defined by (4.4). This gives (4.5) and (4.6).

Corollary 4.2. The greatest lower bound of $|f(z)|$, $f \in T_1$, is equal 0. If $f_n(z) = \dfrac{\varepsilon_n^2 z}{(1-z+\varepsilon_n z)^2}$, where $0 < \varepsilon_n < 1$, then obviously $f_n \in T_1$. Besides, for any fixed $z \in K$, $f_n(z) \to 0$, if $\varepsilon_n \to 0$.

Corollary 4.3. The extremal functions in (4.5) and (4.6) are univalent. This implies that the estimations (4.5) and (4.6) also hold for the subclass S_1 of functions univalent in K which satisfy $f(0) = 0$, $\lim_{z \to 1-} f(z) = 1$ and have real Taylor coefficients at $z = 0$.

We now find the bounds of $|f'(z)|$ for the class T_1. Let Ω_1 be the Jordan domain with the boundary C_1 where C_1 is the map of the circumference $\{\zeta: |\zeta - 2| = 4\}$ under the transformation $z = z(\zeta)$ defined by (3.1). We now prove

Theorem 4.4. *Suppose $f \in T_1$. Then we have the sharp inequalities: if $z \in \Omega_1$, then*

$$|f'(z)| \leqslant 4|1-z|/|1+z|^3; \tag{4.7}$$

if $z\in K\setminus\Omega_1$, *then*

$$|f'(z)| \leqslant \frac{|1+z|}{2|z||1-z|(1+\cos\alpha)}, \text{ where} \tag{4.8}$$

$\alpha = 2\arg(1-z)-\arg z$.

The signs of equality in (4.7) *and* (4.8) *hold for*

$$F_1(z) = 4z(1+z)^{-2} \text{ and } F(z) = \frac{4t_1 z}{z^2+1+2z(2t_1-1)},$$

where $t_1 = \frac{1}{4}|\zeta-2|$, *resp.*

Proof. Differentiating both sides in (4.1) we obtain

$$f'(z) = \int_0^1 \frac{4t(1-z^2)}{[z^2+1+2z(2t-1)]^2}\, d\mu(t) \tag{4.9}$$

Put $G(z,t) = 4t[z^2+1+2z(2t-1)]^{-2}$. An elementary calculation shows that $\sup\limits_{0\leqslant t\leqslant 1}|G(z,t)|$ is equal $|G(z,1)|$ and $|G(z,t_1)|$ for $z\in\Omega_1$ and $z\in K\setminus\Omega_1$ resp. The estimations (4.7), (4.8) follow now readily from (4.9).

The set Ω_1 is also involved in the estimation of the imaginary part of $f\in T_1$. We have

Theorem 4.5. *Suppose* $f\in T_1$. *Then we have the sharp inequalities*: *if* $z\in\Omega_1$, *then*

$$|\operatorname{im} f(z)| \leqslant 4\left|\operatorname{im}\frac{z}{(1+z)^2}\right|; \tag{4.10}$$

if $z\in K\setminus\Omega_1$, *then*

$$|\operatorname{im} f(z)| \leqslant \frac{|z|}{|1-z|^2(1+\cos\alpha)}\left|\operatorname{im}\frac{1+z^2}{z}\right|, \tag{4.11}$$

where $\alpha = 2\arg(1-z)-\arg z$.

The signs of equality in (4.10) *and* (4.11) *hold for* $F_1(z)$ *and* $F(z)$ *resp., where* $F_1(z)$, $F(z)$ *are defined as in the statement of Theorem* 4.4.

Proof. It follows from (4.1) that for any $f\in T_1$ we have

$$\operatorname{im} f(z) = -\operatorname{im}\frac{1+z^2}{z}\int_0^1 \frac{4t}{|z^2+1+2z(2t-1)|^2}\, d\mu(t).$$

Using the notation of the Theorem 4.4 we have $|\operatorname{im} f(z)| \leqslant \left|\operatorname{im}\frac{1+z^2}{z}\right| \sup\limits_{0\leqslant t\leqslant 1}|G(z,t)|$, and an analogous reasoning as that used in proving the Theorem 4.4 yields (4.10) and (4.11).

5. The structural formula for the class $T'(z_0)$

We now derive the structural formula for $f \in T'(z_0)$.

If $f \in T'(z_0)$ and $a = \frac{1}{f'(0)}$, then obviously, $af \in T$. Conversely, if $\varphi \in T$, then $f(z) = \varphi(z)/\varphi'(z_0)$ belongs to $T'(z_0)$. This implies, in view of (2.4) that any $f \in T'(z_0)$ has the form

$$f(z) = \int_0^1 \frac{z\,da(t)}{z^2+1+2z(2t-1)} \tag{5.1}$$

where $a(t)$ is non-negative and non-decreasing in $[0,1]$ and satisfies

$$\int_0^1 \frac{(1-z_0^2)\,da(t)}{[z_0^2+1+2z_0(2t-1)]^2} = 1. \tag{5.2}$$

Conversely, any f, as given by (5.1), with $a(t)$ satisfying the conditions just stated, belongs to $T'(z_0)$. We now apply the Lemma 2.1 with $g(t) = \frac{z}{z^2+1+2z(2t-1)}$,

$$h(t) = \frac{1-z_0^2}{[z_0^2+1+2z_0(2t-1)]^2}.$$

This gives the

Theorem 5.1. *Suppose $f \in T'(z_0)$. Then there exists a function $\mu(t)$ non-negative and non-decreasing in $[0,1]$ with $\int_0^1 d\mu(t) = 1$ such that*

$$f(z) = \frac{z}{1-z_0^2}\int_0^1 \frac{[z_0^2+1+2z_0(2t-1)]^2}{z^2+1+2z(2t-1)}\,d\mu(t) \tag{5.3}$$

Conversely, any function of the form (5.3) with $\mu(t)$ satisfying just stated conditions belongs to $T'(z_0)$.

Since the arc

$$w = \frac{z}{1-z_0^2}\,\frac{(z_0^2+1+\tau z_0)^2}{z^2+1+z\tau}, \qquad -2 \leqslant \tau \leqslant 2 \tag{5.4}$$

is not convex, the set of all posible values $f(z)$ for a fixed $z \in K$ and f ranging over $T'(z_0)$ which is the convex hull of the arc (5.4) has a complicated form and the estimation of $|f(z)|$ involves elementary but tedious calculations.

On the other hand, the set $D'(z, z_0)$ of all possible values $f'(z)$, $z \in K$ being fixed and f ranging over $T'(z_0)$, can be found more easily.

Theorem 5.2. *Suppose $f \in T'(z_0)$. Then the set $D'(z, z_0) = \{w:\ w = f'(z), f \in T'(z_0)\}$ is the closed convex domain with the boundary consisting of the arc of the cardioid*

$$w = \frac{z_0^2(1-z^2)}{z^2(1-z_0^2)}\,\frac{(\zeta_0+\tau)^2}{(\zeta+\tau)^2}, \qquad -2 \leqslant \tau \leqslant 2 \tag{5.5}$$

where ζ_0, ζ are given by (2.10), (2.11) *and of the straight line segment* $[A^*, B^*]$, *where*

$$A^* = \frac{(1+z)(1-z_0)^3}{(1+z_0)(1-z)^3} \tag{5.6}$$

$$B^* = \frac{(1+z_0)^3(1-z)}{(1+z)^3(1-z_0)} \tag{5.7}$$

Proof. Differentiating both sides of (5.3) w.r.t. z we obtain

$$f'(z) = \frac{1-z^2}{1-z_0^2}\int_0^1\left[\frac{z_0^2+1+2z_0(2t-1)}{z^2+1+2z(2t-1)}\right]^2 d\mu(t) \tag{5.8}$$

with $\mu(t)$ satisfying the usual conditions. The formula (5.8) has the form occurring in Lemma 2.2 with

$$H(t) = \frac{z_0^2(1-z^2)}{z^2(1-z_0^2)}\left[\frac{\zeta_0+2(2t-1)}{\zeta+2(2t-1)}\right]^2, \qquad 0 \leqslant t \leqslant 1.$$

The curve

$$\omega = \left(\frac{\zeta_0+\tau}{\zeta+\tau}\right)^2; \tag{5.9}$$

is similar to the curve described by $H(t)$ and it is the map of the circumference $Z = \dfrac{\zeta_0+\tau}{\zeta+\tau}$ containing the origin under the transformation $\omega = Z^2$. Hence $H(t)$ also describes an arc of the cardioid

$$W(\tau) = \frac{z_0^2(1-z^2)}{z^2(1-z_0^2)}\left(\frac{\zeta_0+\tau}{\zeta+\tau}\right)^2; \qquad -2 \leqslant \tau \leqslant 2$$

with end-points A^*, B^*. Using the Lemma 2.2 we obtain the Theorem 5.2.

We next give the exact upper bounds of $|f'(z)|$ and $|\mathrm{im} f(z)|$ for $f \in T'(z_0)$. Using the notation of Theorems 2.2 and 3.1 we obtain

Theorem 5.3. *Suppose* $f \in T'(z_0)$.
if $z \in D_1$, *then*

$$(5.10)\qquad |f'(z)| \leqslant \frac{(1+z_0)^3}{1-z_0} \frac{|1-z|}{|(1+z)|^3};$$

if $z \in D_2$, *then*

$$(5.11)\qquad |f'(z)| \leqslant \frac{(1-z_0)^3}{1+z_0} \frac{1+z}{(1-z)^3};$$

if $z \in D_3$, *then*

$$(5.12)\qquad |f'(z)| \leqslant \frac{z_0^2(\zeta_0-t_0)^2}{1-z_0^2} \left|\frac{1-z^2}{(z^2+1-t_0 z)^2}\right|.$$

The signs of equality in (5.10 — (5.13) *are attained for the functions*

$$(5.13)\qquad f(z) = \frac{(1+z_0)^3}{1-z_0} \frac{z}{(1+z)^2},$$

$$(5.14)\qquad f(z) = \frac{(1-z_0)^3}{1+z_0} \frac{z}{(1-z)^2},$$

$$(5.15)\qquad f(z) = \frac{(z_0^2+1-z_0 t_0)^2}{1-z_0^2} \frac{z}{z^2+1-t_0 z},$$

respectively. The real number t_0 *is defined by* (3.4).

Proof. Using (5.8) we obtain

$$|f'(z)| \leqslant \frac{z_0^2}{|z|^2} \frac{|1-z^2|}{1-z_0^2} \sup_{-2\leqslant\tau\leqslant 2} |\omega(\tau)|,$$

where $\omega(\tau)$ is defined by (5.9). Moreover, $|\omega(\tau)| = \frac{1}{z_0^2}|w(\tau)|^2$ where $w(\tau)$ is given by (2.12). Clearly $\sup|\omega(\tau)| = \frac{1}{z_0^2}\sup|w(\tau)|^2$.

Now, the same calculations as those used in proving Theorem 3.1 yield Theorem 5.3.

Corollary 5.1. All the functions (5.13) — (5.15) are univalent in K. Hence the inequalities (5.10) — (5.12) are also best possible for the class $S'(z_0)$ of functions $\varphi(z)$ univalent in K which satisfy $\varphi(0) = 0$, $\varphi'(z_0) = 1$ and have real Taylor coefficients at the origin.

In the same notation we have

Theorem 5.4. *Suppose $f \in T'(z_0)$.*
If $z \in D_1$, then

$$|\operatorname{im} f(z)| \leqslant \frac{|z|^2(1+z_0)^3}{(1-z_0)\,|1+z|^4}\left|\operatorname{im}\frac{1+z^2}{z}\right| \tag{5.16}$$

if $z \in D_2$, then

$$|\operatorname{im} f(z)| \leqslant \frac{(1-z_0)^3|z|^2}{(1+z_0)\,|1-z|^4}\left|\operatorname{im}\frac{1+z^2}{z}\right| \tag{5.17}$$

if $z \in D_3$, then

$$|\operatorname{im} f(z)| \leqslant \frac{(z_0^2+1-z_0t_0)^2|z|^2}{(1-z_0^2)\,|z^2+1-zt_0|^2}\left|\operatorname{im}\frac{1+z^2}{z}\right|. \tag{5.18}$$

The signs of equality in (5.16) — (5.18) *are attained for the functions* (5.13) — (5.15) *respectively.*

Proof. The formula (5.3) gives

$$\operatorname{im} f(z) = -\frac{|z|^2}{1-z_0^2}\operatorname{im}\frac{1+z^2}{z}\int_0^1 \frac{[z_0^2+1+2z_0(2t-1)]^2}{|z^2+1+2z(2t-1)|^2}\,d\mu(t).$$

Hence

$$|\operatorname{im} f(z)| \leqslant \frac{z_0^2}{1-z_0^2}\left|\operatorname{im}\frac{1+z^2}{z}\right| \sup_{\tau\in[-2,2]} |\omega(\tau)|$$

where $\omega(\tau)$ is given by (5.9). Now, the same reasoning as that used in proving Theorem 5.3 yields Theorem 5.5.

Theorem 5.5. *Since the extremal functions* (5.13) — (5.15) *are univalent, the inequalities* (5.16) — (5.18) *are also best possible for the class $S'(z_0)$.*

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Streszczenie

Praca dotyczy klasy $T(z_0)$ funkcji holomorficznych w kole jednostkowym K spełniających warunki:

(1) $$f(0) = 0$$

(2) $$\operatorname{im} f(z)\cdot \operatorname{im} z > 0 \quad \text{dla} \quad \operatorname{im} z \neq 0$$

(3) $$f(z_0) = z_0, \quad 0 < z_0 < 1,$$

klas $T'(z_0)$ i T_1, które otrzymujemy zastępując odpowiednio warunek (3) warunkiem:

(4) $$f'(z_0) = 1, \quad 0 < z_0 < 1,$$

względnie warunkiem

(5) $$\lim_{x\to 1-} f(x) = 1$$

oraz ich podklas $S(z_0)$, $S'(z_0)$ i S_1 utworzonych z funkcji jednolistnych.

Ze znanego wzoru Robertsona wyprowadzono przedstawienia parametryczne tych klas i w oparciu o nie wyznaczono obszar $\Omega(z, z_0)$ wartości $f(z)$ przy z ustalonym w kole K i funkcji f zmieniającej się w klasie $T(z_0)$ oraz obszar $\Delta(z, z_0)$ wartości pochodnej $f'(z)$. Analogiczne wyniki uzyskano dla klas $T'(z_0)$ i T_1. Stąd otrzymano ostre oszacowanie $|f|$, $|f'|$ i $|\operatorname{im} f|$ w tych klasach, przy czym prawie wszystkie są ostre w podklasach funkcji jednolistnych.

Резюме

В работе исследован класс $T(z_0)$ функций, голоморфных в единичном круге K, отвечающих условиям:

$$f(0) = 0; \tag{1}$$

$$\operatorname{im} f(z)\cdot \operatorname{im} z > 0 \quad \text{для} \quad \operatorname{im} z \neq 0; \tag{2}$$

$$f(z_0) = z_0, \quad 0 < z_0 < 1, \tag{3}$$

классы $T'(z_0)$ и T_1, получаемые заменой условия (3) на условие

$$f'(z_0) = 1, \quad 0 < z_0 < 1, \tag{4}$$

или

$$\lim_{x \to 1-} f(x) = 1, \tag{5}$$

а также их подклассы $S(z_0)$, $S'(z_0)$ и S_1, образованные из однолистных функций.

Из известной формулы Робертсона выведены параметрические представления этих классов и, опираясь на них, найдена область $\Omega(z, z_0)$ значений $f(z)$ с постоянным z в круге K и функциии f меняющейся в классе $T(z_0)$. Найдена также область значений производной $f'(z)$. Аналогичные результаты получены для классов $T'(z_0)$ и T_1. Отсюда найдены точные оценки $|f|$, $|f'|$ и $|\mathrm{im} f|$ в указанных классах. При этом почти все оценки в подклассах однолистных функций точны.