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On Certain Method of Constructing Sets of Mutually Orthogonal Comparisons

O pewnej metodzie konstrukcji zbioru ortogonalnych porównań

Об одном методе построения множества ортогональных сравнений

1. R. A. Fisher [3] presents the following theorem: If $x_1, x_2, \dots, x_n$ are independent random variables and all normal (0,1) and if

$$y_i = \sum_{j=1}^{n} b_{ij} x_j \tag{1}$$

where $\{b_{ij}\}$ $(i, j = 1, 2, \dots, n)$ is orthogonal and normalized matrix, then the variates y_i are independent and normal (0,1) and the following relationship holds:

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 . \tag{2}$$

A more general theorem, which is often used, says: If $x_1, x_2, \dots, x_n$ are independent random variables and all normal $(m\ \sigma)$, and if

$$y_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i, \qquad y_k = \sum_{j=1}^{n} b_{kj} x_j, \qquad (k = 2, 3, \dots, n) \tag{3}$$

where $\{b_{ij}\}$ is orthogonal and normalized matrix with

$$b_{11} = b_{12} = \dots = b_{1n} = 1/\sqrt{n} ,$$

then y_1 is normal $(m\sqrt{n}, \sigma)$ and y_k are normal $(0, \sigma)$ for $k = 2, 3, \ldots, n$, and also

$$\sum x_i^2 = \sum y_i^2 \tag{4}$$

or

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=2}^{n} y_i^2 \tag{4'}$$

where

$$n\bar{x} = \sum_{i=1}^{n} x_i .$$

Each of n-1 components of the right side of identity (4') is based upon a single degree of freedom, since the variable y_k^2/σ $(k = 2, 3, \ldots, n)$ is distributed as chi-square with one degree of freedom. It is evident that by orthogonality of martrix $\{b_{ij}\}$ and by the form of the variable y_1 given in (3) we have

$$\sum_{j=1}^{n} b_{ij} = 0 \qquad (i = 2, 3, \ldots, n).$$

A linear combination of x's for which

$$\sum_{j=1}^{n} b_{ij} = 0$$

will be called *comparison*.

Hence, the right side of identity (4') is the sum of squares of n-1 mutually orthogonal comparisons, each based upon a single degree of freedom. Since the orthonormalized matrix $\{b_{ij}\}$ given in (3) may be defined in an infinite number of ways, the left side of identity (4') may be also defined in the same number of ways.

J. O. Irwin [4] gives one of these ways determining the matrix $\{b_{ij}\}$ by means of the following relations:

$$b_{11} = b_{12} = \cdots = b_{1n} = \frac{1}{\sqrt{n}}, \qquad b_{r1} = b_{r2} = \cdots = b_{r,\,r-1} = \frac{+1}{\sqrt{r(r-1)}},$$

$$b_{rr} = \frac{-(r-1)}{\sqrt{r(r-1)}}$$

$(r = 2, 3, \ldots, n)$. This autor finds (1930) an indentity related to this matrix, which may be found in Burnside's textbook [1].

R. E. A. C. Paley [5] gives some methods of constructing the orthogonal matrices with elements equal to $+1$ and -1.

In this paper I present a simple method of constructing sets of mutually orthogonal comparisons; this method may be also used to construct orthogonal matrices; it is based upon the theorem of the number theory concerning the systematic expansion of natural numbers to any integer numeration base, [**6**], and is graphically illustrated in Fig. 1, 2 and 3. Moreover, using relation (4′) I deduce identities (**14**) and (14′). The method may be used in the analysis of variance (cf. sec. 3).

2*a*. First, we are going to deal with the construction of sets of mutually orthogonal comparisons taking two values as the base of comparison. This may be reached by means of the following graphical construction.

Let the values of variates defined in sec. 1 be designated as $x_1, x_2, \dots x_n$. Moreover, let n points determined by these observation be plotted along the horizontal axis in the order given above. The points will be named sets of the zero order, the pairs of points will be named sets of the first order, the four points-sets of the second order, and in general sets consisting of 2^i points will be named sets of the i-th order $(i = 0, 1, 2, \dots)$.

Proceeding from the left to the right side of the sequence of x's let us link the successive pairs of points by means of single arcs i.e. x_1 with x_2, x_3 with x_4, etc. i.e. the sets of the first order are formed by linking the sets of the zero order. To each such conjunction its strictly determined comparison is assigned. Thus, the following comparisons of the first order will be assigned to the above-given conjunctions:

$$y_1^{(1)} = x_1 - x_2, \qquad y_2^{(1)} = x_3 - x_4, \dots$$

where the upper index denotes the order of the set obtained from linking and the lower index denotes the ordinal number of the set (i.e. of the comparison). The observations x_i corresponding to points x_i $(i = 1, 2, \dots)$ and appearing at the left end of the arc of conjunction will be multiplied by $+1$, and those occuring at the right end of this arc will be multiplied by -1, unless some restrictions are taken.

It is evident that after all conjunctions between the sets of the zero order have been carried out, there remains at most one point unlinked; this fact is noted as $c_0 = 0, 1$. It is also clear that the number of the sets of the first order (i.e. the number of pairs or number of conjunctions) will be equal to $n_1 = E(n/2)$, where the symbol E denotes *entier*. Thus

$$n = 2\,n_1 + c_0. \tag{5}$$

Further, let us form the sets of the second order (i.e. the sets including four points) by linking successively the sets of the first order from the left to the right side, and let us assign to these conjunctions the comparisons of the second order:

$$y_1^{(2)} = (x_1 + x_2) - (x_3 + x_4), \qquad y_2^{(2)} = (x_5 + x_6) - (x_7 + x_8), \ldots .$$

To distinguish the comparisons of the second order from those of the first order the corresponding conjunctions are marked in the figure by means of double arcs (cf. Fig. p. 12). The number of these conjunctions is $n_2 = E(n_1/2)$ and at most one set of the first order may be left unlinked. In this way we obtain n_2 sets of the second order and $c_1 = 0, 1$ unlinked set of the first order; we have therefore

$$n_1 = 2n_2 + c_1. \tag{6}$$

This process of linking is continued; the sets of the third order are formed by linking the sets of the second order, the sets of the fourth order are obtained by linking the sets of the third order, and so on; to each conjunction its corresponding comparison is assigned with the observations x having the coefficients defined on page 7.

The procedure of linking and of forming corresponding comparisons must evidently have its end. Let the last conjunction be the conjunction of the p-th order of the form

$$y^{(p)} = (x_1 + x_2 + \cdots + x_d) - (x_{d+1} + x_{d+2} + \cdots + x_{2d})$$

where d stands for 2^{p-1}.

Since $2d = 2^p$ of points are included in this comparison, $n_p = 1$, and

$$n_{p-1} = 2n_p + c_{p-1} = 2 + c_{p-1} \tag{7}$$

where the number of unlinked sets of the $(p-1)$-th order is determined by $c_{p-1} = 0, 1$. The first phase of linking and forming the comparisons is finished. We have obtained the finite sequence of relations (5) - (7) which in the number theory is known as the result of the applicability of the algorithm permitting the expansion of the natural number n to the base two (cf. [6]). By this sequence the expansion of the number n is obtained in the form:

$$n = 2n_1 + c_0 = 2(2n_2 + c_1) + c_0 = \cdots = 2^p + c_{p-1} \cdot 2^{p-1} + \cdots + c_1 2^1 + c_0 2^0 \tag{8}$$

where $c_{p-1}, c_{p-2}, \ldots, c_0 = 0,1$.

In this manner we have obtained $N = n_1 + n_2 + \ldots + n_{p-1} + n_p$ conjunctions (or comparisons) and among them n_i conjunctions of the i-th order ($i = 1, 2, \ldots, p$). These conjunctions (comparisons) will be named conjunctions (comparisons) of the *first type*.

If in the expansion (8) the coefficients c which are equal to zero are omitted, then this expression may be written in a simpler form:

$$n = 2^{u_1} + 2^{u_2} + \cdots + 2^{u_t}, \tag{9}$$

where $p = u_1 > u_2 > \ldots > u_t \geqslant 0$, and

$$t = c_0 + c_1 + \cdots + c_{p-1} + 1. \tag{10}$$

The graphical configuration obtained from performing conjunctions is uniquely determined by expansion (9). In fact, it results from (9) that the configuration includes t disjoint sets: the first of these sets is of the $p = u_1$-th order, the second of the u_2-th order, etc., and the last is of the u_t-th order. It is also evident that the comparisons of the first type corresponding to the above-given conjunctions constitute the comparisons within mentioned t sets.

Now, we are going to prove that these comparisons are mutually orthogonal.

Proof: Note that two arbitrary comparisons chosen from t disjoint sets are mutually orthogonal, since they include different observations. Thus it remains to prove that the comparisons formed within the set of the k-th order (each of the disjoint sets is of this form as it is seen in (9)) constitute a set of mutually orthogonal comparisons.

To attain this it is sufficient to prove that two arbitrary comparisons within the set of 2^k points are orthogonal. The i-th comparison of the w-th order belonging to the set of the k-th order has the form:

$$y_i^{(w)} = (x_{s+1} + x_{s+2} + \cdots + x_{s+h}) - (x_{s+h+1} + x_{s+h+2} + \cdots + x_{2hi}) \tag{11}$$

where for brevity it is noted $s = (i-1)\,2^w$ and $h = 2^{w-1}$ $(i = 1, 2, \ldots, 2^{k-w};$ $w = 1, 2, \ldots, k)$.

Consider here two cases: (a) when the comparisons belong to the sets of the same order, and (b) when they belong to sets of different orders.

In the first case the comparisons are orthogonal since they include disjoint sets (i.e. without common observations x). In the second case let us suppose that the sets have some common points. The comparisons corresponding to these sets are also orthogonal. In fact, since all observations occurring in the comparison of a lower order have in the comparison of a higher order the coefficients equal to $+1$ or to -1, then the sum of products of these coefficients of corresponding observations in both comparisons must be equal to zero. This proves the proposition.

Since the i-th set represents 2^{k-i} comparisons ($i = 1, 2, ..., k$) in a set of the k-th order, the number of all comparisons in this set is

$$\sum_{i=1}^{k} 2^{k-i} = 2^k - 1.$$

This fact proves that all comparisons determined in a set of the k-th order constitute a complete set of mutually orthogonal comparisons, i.e. they provide all orthogonal comparisons for 2^k observations.

Thus, it is proved that the sets of the first type represent a complete set of comparisons which includes

$$(12) \qquad N = n_1 + n_2 + \cdots + n_p = \sum_{i=1}^{t} (2^{u_i} - 1) = \sum_{i=1}^{t} 2^{u_i} - t$$

mutually orthogonal comparisons.

Now let us perform the conjuctions and determine corresponding orthogonal comparisons between t sets not yet linked; to them belong the sets of the u_i-th order ($i = 1, 2, ..., t$). The mentioned conjunctions and comparisons will be called conjunctions and comparisons of the *third type*. The comparisons of the *second type* appear only to the base $a > 2$; these will be defined in sec. 2*b*.

A simple method of linking sets of the u_i-th order is as follows. The set of the u_1-th order is linked by means of one conjunction with the set of the u_2-th order, then the total set obtained from this linking, denoted by (u_1, u_2), is linked with the set of the u_3-th order; then the total set (u_1, u_2, u_3), which includes all preceding sets, is linked with the set of the u_4-th order, and so on. In this manner we obtain t-1 conjunctions.

The comparison in the form:

$$(13) \qquad y_i^{(\mathrm{III})} = (x_1 + x_2 + \cdots + x_{r_i}) - \frac{r_i}{s_{i+1}} (x_{r_i+1} + x_{r_i+2} + \cdots + x_{r_i+s_{i+1}})$$

where $r_i = 2^{u_1} + 2^{u_2} + ... + 2^{u_i}$ and $s_{i+1} = 2^{u_{i+1}}$, ($i = 1, 2, ..., t$-1) corresponds conjuction of the total set $(u_1, u_2, ..., u_i)$ with the set of the u_{i+1}-th order. The number of type of comparison is given by the upper index (Roman numeral) of y.

We will prove that the comparison $y_i^{(\mathrm{III})}$ is orthogonal to each of N comparisons of the first type.

Proof: Since the comparisons including different observations are evidently orthogonal, it is sufficient to consider only comparisons which have some observations in common. The latter are mutually orthogonal on the basis given in the preceding proof (let this basis be called B basis).

In fact, the coefficients of all the observations appearing in the comparison of the first type are identical with those in the comparison of the third type (this follows from the definition), i.e. they are equal to $+1$ if the comparison of the first type includes the observations belonging to the left end of the conjunction of the third type, and they are equal to $-r_i/s_{i+1}$, if the comparison of the first type includes the observations belonging to right end of the conjunction of the third type. Thus the sum of products of the coefficients of the corresponding observations of the comparisons in question is equal to zero; this proves that the comparisons of the first are orthogonal to those of the third type.

Now we will prove that the comparisons of the third type are mutually orthogonal.

Proof: Let us give two comparisons of the third type:

$$y_i^{(III)} \quad \text{and} \quad y_k^{(III)}, \quad \text{where} \quad i < k \quad (i = 1, 2, ..., t-2;\ k = 2, 3, ..., t-1).$$

Note that the coefficients of the comparisons $y_i^{(III)}$ and $y_k^{(III)}$ included in the first r_i observations are identical and equal to 1. On the other hand, when the comparison $y_i^{(III)}$ has in the remaining s_{i+1} observations the coefficients which are equal to $-r_i/s_{i+1}$ then in the corresponding observations the comparison $y_k^{(III)}$ has its coefficients also equal to 1. Hence it is evident that the sum of products of the coefficients of the observations in the two examined comparisons is equal to zero. The proof is finished.

Now we are going to prove that the total number of comparisons of the first and of the third type obtained so far is n-1.

In fact, on account of (10) we have:

$$N + (t-1) = (n_1 + n_2 + \cdots + n_{p-1} + n_p) + t - 1 =$$
$$= (2n_1 + c_0) + (2n_2 + c_1) + \cdots + (2n_p + c_{p-1}) - (n_1 + n_2 + \cdots + n_{p-1} + n_p)$$

and using $n_p = 1$ and equalities (5) - (7) we obtain n-1.

In this way the complete set of mutually orthogonal comparisons has been determined. The graphical construction (cf. Fig. 1) presented above constitutes a simple method of obtaining set of mutually orthogonal comparisons and may be found from the expansion of the natural number n to the base two. It also presents a method of constructing an orthogonal matrix which can be found directly from the configuration of conjunctions in the graph. The coefficients of observations included in comparisons are the elements of a row of an orthogonal matrix; all the elements of the last i.e. of the n-th row of a matrix are equaly to 1. As usual, the *orthonormalized* matrix is obtained by dividing all numbers of every row of this matrix by the square root of the sum of squares of these numbers.

If the orthogonal matrix (which is constructed according to the above-given method) is assumed to be normalized, we obtain on account of (4′), (11), (13) and expansion (9) the identity in the form:

$$(14)\qquad \sum_{i=1}^{n}(x_i-\bar{x})^2=\sum_{h=1}^{t}\sum_{i=1}^{u_h}\frac{1}{f_i}\sum_{j=1}^{2^{u_h-i}}\left[\sum_{k=g_{ij}+1}^{g_{ij}+f_{i-1}}x_k-\sum_{k=g_{ij}+f_{i-1}+1}^{g_{ij}+2f_{i-1}}x_k\right]^2+$$
$$+\sum_{w=2}^{t}\frac{2^{u_w}}{r_w\cdot r_{w-1}}\left[\sum_{s=1}^{r_{w-1}}x_s-\frac{r_{w-1}}{2^{u_w}}\sum_{s=r_{w-1}+1}^{r_w}x_s\right]^2$$

where

$r_m=2^{u_1}+2^{u_2}+\dots+2^{u_m}$, $g_{ij}=(j-1)\cdot 2^i$ and $f_i=2^i$ $(m=1,2,\dots,t)$.

In particular, when $n=2^p$ the identity is of simpler form:

$$(14')\qquad \sum_{i=1}^{n}(x_i-\bar{x})^2=\sum_{i=1}^{p}\frac{1}{f_i}\sum_{j=1}^{f_{p-i}}\left[\sum_{k=g_{ij}+1}^{g_{ij}+f_{i-1}}x_k-\sum_{k=g_{ij}+f_{i-1}+1}^{g_{ij}+f_i}x_k\right]^2.$$

Example 1. The configuration of $n-1=13$ conjunctions between $n=14=2^3+2^2+2$ points is illustrated by Fig. 1.

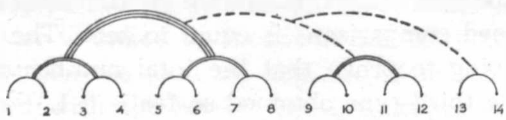

Fig. 1

Conjunctions corresponding to the comparisons of the third type are presented in Fig. 1 by means of the broken arc and conjunctions of the first type are presented by means of a single, double or triple continuous arc if the corresponding comparison is of the first, of the second or of the third order respectively. The expansion of number 14 to the base two includes three components: $2^3, 2^2$ and 2. According to (9) and to the method described above the graph in Fig. 1 represents three sets of comparisons (or conjunctions) of the first type of the $u_1=3$-rd, $u_2=2$-nd, and $u_3=1$-st orders respectively, and $t-1=2$ conjunctions of the third type linking the conjunctions of the first type.

2*b*. Now we are going to present a method of constructing sets of mutually orthogonal comparisons for n observations in the case when we are interested in forming group comparisons including a observations $(2<a<n)$.

The construction presented in sec. $2a$ to the base 2 will be generalized here for the arbitrary base $a > 2$. Now the fundamental conjunction includes not two observations as previously but the set of $a-1$ conjunctions between a observations, according to the method of sec. $2a$.

The proposed configurations of fundamental conjunctions for different values of a are presented in Fig. 2.

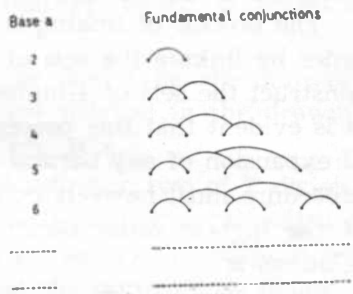

Fig. 2

Before the graphical method of linking n points to the base a is presented we will introduce certain definitions generalizing those of sec. $2a$. Each of the n points (observations) $x_1, x_2, \ldots, x_n$ ordered on the axis of coordinates will be called, as in the preceding section, a set of the zero order, moreover, the sets including a points (fixed fundamental sets) will be called sets of the first order, and in general sets including a^i points will be called sets of the i-th order ($i = 0, 1, 2, \ldots$).

We link the points within each set of a points and construct the corresponding mutually orthogonal comparisons to the base 2 exactly as in sec. $2a$ (cf. Fig. 1 and 2). Thus we have $a-1$ mutually orthogonal comparisons in such a set. It remains to present the method of constructing the comparisons between sets including a points each.

This method is similar to that given in sec. $2a$, where $a = 2$. It is as follows: In the ordered set of n points we separate the subsets of a points, proceeding from the left to the right side and we link the points by means of single arcs (cf. Fig. 3). This process gives $E(n/2) = n_1$ sets of the first order and $c_0 = 0, 1, 2, \ldots, a-1$ unlinked points. We obtain the relation similar to (5) in the form:

$$n = an_1 + c_0. \tag{5'}$$

Moreover, linking sets of the first order (single arcs are linked by means of double arcs; cf. Fig. 3) to the chosen base a provides $E(n_1/2) = n_2$ sets of the second order. To each conjunction we assign a comparison whose observations included at the left end of the arc should have identical, positive, and minimal integers; the coefficients of the observations at the right end of the arc should be identical and negative integers satisfying the condition that the sum of all coefficients of comparison should be equal to zero.

The relation between the numbers of conjunctions (or comparisons) of the first and of the second order is

$$(6') \qquad n_1 = an_2 + c_1,$$

where $c_1 = 0, 1, 2, \ldots, a-1$ denotes that in forming the sets of the second order there may remain at most $a-1$ unlinked sets of the first order.

The process of linking is continued. We construct the sets of the third order by linking the sets of the second order to the base 2; similarly we construct the sets of a higher order from the sets of the preceding order. It is evident that this process is identical with the well-known algorithm of expansion of any natural number n to the base a. Hence, the discussed procedure must have its end. Assume that a set of the p-th order is a set of the highest order obtained in the process of linking. Thus the last relations is

$$(7') \qquad n_{p-1} = an_p + c_{p-1},$$

where

$$c_{p-1} = 0, 1, 2, \ldots, a-1 \quad \text{and} \quad c_p = n_p = 1, 2, \ldots, a-1 \quad \text{(cf. [6])}.$$

On the account of the sequence of equalities (5') - (7') the expansion of the number n to the base a is obtained by simple calculations in the form:

$$(8') \qquad n = c_p a^p + c_{p-1} a^{p-1} + \cdots + c_1 a^1 + c_0 a^0,$$

where

$$c_i = 0, 1, 2, \ldots, a-1 \quad (i = 0, 1, 2, \ldots, p-1) \quad \text{and} \quad c_p = 1, 2, \ldots, a-1.$$

Omitting in expansion (8') the coefficients c equal to zero and taking $p = u_1$, the number n, as in sec. 2a, can be presented as

$$(9') \qquad n = c_{u_1} a^{u_1} + c_{u_2} a^{u_2} + \cdots + c_{u_t} a^{u_t},$$

where

$$c_{u_1}, c_{u_2}, \ldots, c_{u_t} = 1, 2, \ldots, a-1 \quad \text{and} \quad p = u_1 > u_2 > \cdots > u_t \geqslant 0.$$

Let us note that in the process of constructing sets of the first, second and higher orders to the base a a set of the u_i-th order includes a^{u_i} points, and that the coefficients c_{u_i} given in the expansion (9') indicate the number of obtained sets of the u_i-th order ($i = 1, 2, \ldots, t$). Thus in our sequence of conjunctions, which will be called the conjunctions of the first type, c_{u_i} sets of the u_i-th order are obtained ($i = 1, 2, \ldots, t$). According to the presented method the conjunctions are constructed within these sets only.

Since $a-1$ mutually orthogonal comparisons are obtained within the set of a points. the number of the mutually orthogonal comparisons (conjunctions) within n_1 sets of the first order is evidently equal to $(a-1)n_1$. Generally, the number of mutually orthogonal comparisons within the n_i sets of the i-th order is $(a-1)n_i$. All comparisons constructed so far constitute $(a-1)$ $(n_1+n_2+...+n_p)$ comparisons of the first type.

It is easy to see that in the graphical configuration there are c_{u_i} unlinked sets of the u_i-th order $(i=1, 2, ..., t)$, the total of which is equal to $T=c_{u_1}+c_{u_2}+c_{u_t}$ sets (cf. (9′)).

We will prove that all sets of points of the first type are mutually orthogonal; their corresponding comparisons are marked in the drawing (cf. Fig. 3) by means of single, double, triple, etc. arcs.

Proof: Since two arbitrary comparisons belonging to two of T disjoint sets are evidently orthogonal (as they include different observations) it is sufficient to consider the comparisons within the set T. The disjoint sets include a^{u_i} points, and it remains to prove that the comparisons obtained from linking a^{u_i} points are mutually orthogonal.

To prove this let us consider a^k points where k is an arbitrary natural number. It is evident that the present method of constructing the comparisons gives a^{k-i} sets (conjunctions) of the i-th order $(i=0, 1, 2, ..., k)$. Since the number of mutually orthogonal comparisons between a sets is $a-1$ (cf. sec. 2a), the sets of the i-th order include $(a-1)\,a^{k-i}$ mutually orthogonal comparisons $(i=1, 2, ..., k)$. The sum of these comparisons is equal to

$$\sum_{i=1}^{k}(a-1)\,a^{k-i}=(a-1)\sum_{i=1}^{k}a^{k-i}=a^k-1.$$

Comparisons including different observations are naturally orthogonal. Comparisons which are of the same order and have certain observations x in common are orthogonal, which follows from sec. 2a.

Let us then consider two arbitrary comparisons of different orders but with some number of common observations. For the sake of brevity let the comparisons including the greater and the smaller number of observations be noted by letters W and M respectively. It follows from the definition of comparison that all observations x included in comparison M have identical coefficients in comparison W. This property will be marked by letter B. Let us note that the two comparisons M and W with the property B are orthogonal. In fact, the sum of products of coefficients of corresponding observations in comparisons M nad W is equal to zero since the sum of coefficiens of comparison M (by the definition of comparison) is equal to zero, and the coefficients of comparison W are identical.

We have proved that the comparisons formed within set of a^k points are mutually orthogonal; thus all comparisons of the first type are mutually orthogonal.

Let us continue our construction. Let us introduce comparisons of *the second type*. These are comparisons corresponding to conjunctions obtained by linking the disjoint sets of the same order to the base 2. Thus c_{u_i} sets of the u_i-th order are linked by means of the c_{u_i}—1 conjunctions of the second type ($i = 1, 2, ..., t$).

According to property B all comparisons of the second type (there are $\sum_{i=1}^{t}(c_{u_i} - 1)$ comparisons of this type) are mutually orthogonal and are also orthogonal to all comparisons of the first type.

Finally we present the last group of conjunctions. It includes the conjunctions of the third type obtained by linking each of t sets of

$$c_{u_1} a^{u_1},\ c_{u_2} a^{u_2}, \ldots, c_{u_t} a^{u_t} \tag{15}$$

points (cf. expansion (9′)) with the total set including the sum of all sets preceding it.

Let the i-th total set including the first i sets of sequence (15) be noted by the symbol G_i. The comparison of the third type (marked in Fig. 3 by a dotted arc) in the form

$$y_i^{(\mathrm{III})} = c_{u_{i+1}}(x_1 + x_2 + \cdots + x_{z_i}) - \frac{z_i}{a^{u_{i+1}}}(x_{z_i+1} + x_{z_i+2} + \cdots + x_{z_{i+1}}) \tag{16}$$

where

$$z_i = \sum_{k=1}^{i} c_{u_k} a^{u_k} \qquad (i = 1, 2, ..., t-1)$$

is assigned to the linking of the $(i+1)$-th set of sequence (15) with the set G_i.

Note that the coefficient $z_i/a^{u_{i+1}}$ given in (16) is an integer since $u_1 > u_2 > ... > u_t \geqslant 0$ (cf. (9′)), which is in accordance with the requirement of the construction of comparisons.

The number of comparisons of the third type is evidently equal to t—1. Now we will prove that arbitrary comparison of the third type is orthogonal to each of the previously considered comparisons of the first and second type.

Proof: Let arbitrary comparison of the third type be denoted by the symbol R and let arbitrary comparison of the second or of the first type be denoted by the symbol P. According to the definitions of comparisons R and P all observations included in comparison P have coefficients identical with those occurring in comparison R and equal to c_{u_i+1} (cf. (16)) if conjunction P appears at the left end of the conjunction R, or equal to $-z_i a^{-u_i+1}$ if comparison P appears at the right end of conjunction R. Thus comparisons R and P as having property B are orthogonal. The proof is completed.

Now let us prove that comparisons of the third type are mutually orthogonal.

Proof: Let us take two arbitrary comparisons of the third type: $y_i^{(\mathrm{III})}$ and $y_k^{(\mathrm{III})}$ (cf. (16)), where let $i<k$ ($i=1,2,\ldots,t-2$; $k=2,3,\ldots,t-1$). It is easily seen that when comparison $y_i^{(\mathrm{III})}$ has the coefficients of z_i observations equal to c_{u_i+1}, then comparison $y_k^{(\mathrm{III})}$ has the coefficients of these observations equal to c_{u_k+1}. On the other hand, when the coefficients in comparison $y_i^{(\mathrm{III})}$ of their remaining $c_{u_i+1}a^{u_i+1}$ observations are equal to $-z_i a^{-u_i+1}$, then the corresponding coefficients of observations in comparison $y_k^{(\mathrm{III})}$ are equal to c_{u_k+1}. The coefficients of the remaining observations in comparison $y_k^{(\mathrm{III})}$ are not examined, because their coefficients in $y_i^{(\mathrm{III})}$ are equal to zero.

Consequently, the sum of products of coefficients of corresponding observations in the i-th and k-th comparisons of the third type is equal

$$c_{u_i+1}\cdot c_{u_k+1}\cdot z_i - c_{u_i+1}\, a^{u_i+1}\cdot\frac{z_i}{a^{u_i+1}}\, c_{u_k+1}=0.$$

The proof that all presented comparisons of the first, second and third types constitute a set mutually orthogonal comparisons is completed. To show that they present also a complete set of comparisons, i.e. $n-1$ comparisons, it is sufficient to sum $(a-1)\sum_{i=1}^{p} n_i$ comparisons of the first type, then $\sum_{i=1}^{t} c_{u_i}-t$ comparisons of the second type, and finally $t-1$ comparisons of the third type. In fact, using equalities (5′) - (8′) and the relation

$$\sum_{i=1}^{t} c_{u_i}=\sum_{i=0}^{p} c_i \quad \text{and} \quad n_p=c_p$$

we obtain

$$(a-1)\sum_{i=1}^{p} n_i+\left(\sum_{i=1}^{t} c_{u_i}-t\right)+(t-1)=(a-1)\sum_{i=1}^{p} n_i+\sum_{j=0}^{p} c_j-1=$$

$$=\sum_{i=1}^{p}(an_i+c_{i-1})-\sum_{i=1}^{p} n_i-1+c_p=n-1.$$

The following example illustrates the method of constructing sets of mutually orthogonal comparisons presented in sec. 2*b*.

Example 2. The graph in Fig. 3 indicates the manner of linking $n=17$ points when the base of the expansion of the number 17 is equal to $a=3$. Then $17=1\cdot 3^2+2\cdot 3+2$. Three types of conjunctions which result directly from the expansion of number n to the base a are clearly seen in the graph.

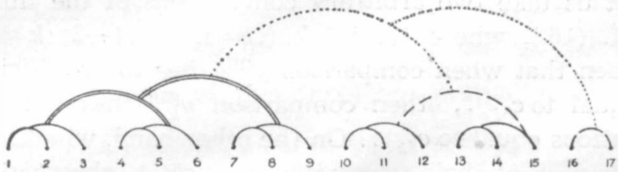

Fig. 3

———	conjunctions forming sets of the first order	conjunctinons of the first type
═══	conjunctions forming sets of the second order	
- - - - -	conjunctions of k sets of 'dentical orders ($1\leqslant k\leqslant a-1$)	conjunctions of the second type
······	conjunctions of disjoint sets separated by signs "plus" in the expansion of the number n to the base a	conjunctions of the third type

Using the configuration presented in Fig. 3 and applying the method indicated in sec. 2*b* we can immediately construct all $n-1=16$ mutually orthogonal comparisons. The coefficients appearing in these comparisons constitute also elements of rows of the orthogonal matrix with 17 rows and 17 columns (all elements of the last row are equal to 1). The matrix is of the form given in Table 1, where non-specified elements are equal to zero. The normalized matrix can be obtained like in sec. 2*a*, p. 11.

3. *Applications*. As we have shown in sec. 2*a* and 2*b* the method of constructing mutually orthogonal comparisons can be illustrated graphically. The obtained configuration makes it possible to determine immediately and explicitly the orthogonal matrix (then also the orthonormal matrix) or to determine explicitly the corresponding comparisons.

It can be applied in working out the numerical data obtained from the experiment based upon one of the known mathematical models (for instance upon the model of randomized complete blocks). If the comparisons obtained according to the described method represent the degrees of freedom which are the matter of interest to the experimenter, he can use them in the analysis of results of experiments by means of the analysis of variance; he can perform the breakdown of the sum of squares for the treatments into the sums of squares with single degrees of freedom. This breakdown constitutes the basis to obtain the particular conclusions on the existence of significant differences between the treatments.

For instance, in problems connected with the interpretation of experimental data obtained from the factorial design of type 2^p (where each of p factors appears at two levels 0 and 1; cf. [2]) an important role is played by such mutually orthogonal comparisons which determine the main effects and the interactions of the investigated factors. In these comparisons the coefficients of observations are equal to 1 or —1. It may be proved that each of these comparisons can be presented as a linear combination of simple effects, [2], which are also comparisons. In the case of the significance of interactions the above-mentioned simple factorial effects can be compared by the experimenter. Therefore the partition of the sum of squares for n treatments into simple mutually orthogonal comparisons appears to be necessary. If the experimenter is interested in comparisons between groups of treatments of the same magnitude $a > 2$ he may use the comparisons suggested in sec. 2b.

Now we shall present an example of the application of the comparisons given in sec. 2a and of the identity (14′). Consider a factorial experiment based on the model of randomized blocks when each of the $p = 3$ factors "A", "B" and "C" appears at two levels 0 and 1. Then there are in all $n = 2^3$ combinations arranged in the standard order: abc, bc, ab, b, ac, c, a, (1) (cf. [7]). According to the method described in sec. 2a let us form $n - 1 = 7$ mutually orthogonal comparisons:

$$y_1 = abc - bc, \qquad y_5 = (abc + bc) - (ab + b),$$

$$y_2 = ab - b, \qquad y_6 = (ac + c) - (a + 1),$$

$$y_3 = ac - c, \qquad y_7 = (abc + bc + ab + b) - (ac + c + a + 1).$$

$$y_4 = a - (1),$$

Comparisons y_1, y_2, y_3 and y_4 represent the simple effects of factor „A", the next three comparisons correspond to the effects of factor "C", and the last comparison constitutes the total effect of factor "B".

TABLE 1

Orthogonal matrix of order 17 · 17

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17		
1	1	—1																comparisons of the first order	comparisons of the first type
2	1	1	—2																
3				1	—1														
4				1	1	—2													
5							1	—1											
6							1	1	—2										
7										1	—1								
8										1	1	—2							
9													1	—1					
10													1	1	—2				
11	1	1	1	—1	—1	—1												comparisons of the 2-nd order	
12	1	1	1	1	1	1	—2	—2	—2										
13										1	1	1	—1	—1	—1			comparisons of the 2-nd type	
14																1	—1		
15	2	2	2	2	2	2	2	2	2	—3	—3	—3	—3	—3	—3			comparisons of the 3-rd type	
16	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	—15	—15		
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		

Let each of the eight considered combinations be equal to the total obtained by summing the individual results in r replications (blocks) of the experiment. Then, on account of identity (14′), we obtain

$$\sum (\bar{x}_i - \bar{x})^2 = \frac{y_1^2 + y_2^2 + y_3^2 + y_4^2}{2r} + \frac{y_5^2 + y_6^2}{2^2 \cdot r} + \frac{y_7^2}{2^3 \cdot r}$$

where $\bar{x}_i$ is the mean of the i-th combination ($i = 1, 2, ..., 2^3$), x is the general mean and the symbol Σ denotes the summation of all $nr = 7r$ observations. The significance of each of the seven terms of the identity can be easily verified by means of the well-known test F.

Consider that the method of constructing the orthogonal matrix presented in this paper may be also used in the case when the set of n points is partitioned into the disjoint subsets, or in more complex cases.

As it is known, breakdown of the sum of squares into the sum of squares with single degrees of freedom is particularly desirable in the computational procedure of the experiments with one replication, in which the error is determined by the mean square of higher order interactions (it is assumed that the components of this mean square are homogeneous). If it is shown in such experiment that some mean squares with single degrees of freedom constructed according to the method presented in this paper are homogeneous, their sum can be also considered as a correct estimate of error on the condition that the comparisons corresponding to these mean squares are non-significant.

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Streszczenie

W niniejszej pracy przedstawiam graficzną metodę konstrukcji zbiorów porównań wzajemnie ortogonalnych między n obserwacjami (por. rys. 1, 2 i 3), która stanowi zarazem metodę konstrukcji macierzy ortogonalnej o n wierszach i n kolumnach (por. Tab. 1).

Nadto zaznaczam ścisłe powiązanie tej metody ze znanym z teorii liczb twierdzeniem o systematycznych rozwinięciach liczb naturalnych przy dowolnej zasadzie numeracji $a \geqslant 2$. Ze względu na to, iż podstawowe porównanie określa się jako różnicę między dwiema obserwacjami, metoda konstrukcji jest oparta na zasadzie $a = 2$ (paragraf 2a). Uogólnienie na przypadek dowolnej naturalnej zasady $a > 2$ znajduje się w paragrafie 2b.

W przypadku $a = 2$ podaję explicite tożsamość wyrażającą podział sumy kwadratów odchyleń pojedynczych obserwacyj od średniej arytmetycznej na n—1 składników wyznaczonych zgodnie z przedstawioną metodą konstrukcji porównań wzajemnie ortogonalnych.

Pewne zastosowania podanej metody są omówione w paragrafie 3.

Резюме

В этой работе я предлагаю графический метод построения взаимно ортогональных множеств сравнений между n наблюдениями (ср. рис. 1, 2 и 3), который является вместе с тем методом конструирования ортогональной матрицы с n строками и n столбцами (ср. Табл. 1).

Сверх того я отмечаю тесную связь метода с известной из теории чисел теоремой о разложении натуральных чисел при произвольном основании счёта $a \geqslant 2$. Ввиду того, что основное сравнение определяется, как разность между двумя наблюдениями, метод конструирования опирается на основании $a = 2$ (§ 2a). Обобщение на случай произвольного основания (натурального числа) $a > 2$ находится в § 2b.

В случае $a = 2$ я привожу явно выраженное тождество, представляющее разложение суммы квадратов отклонений от среднего арифметического на n — 1 компонентов, определенных согласно с представленным методом конструирования взаимно ортогональных сравнений.

Некоторые применения предложенного метода оговорены в § 3.