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Isoperimetrical defect and conformal mapping

Defekt izoperymetryczny i odwzorowania konforemne

Изопериметрический дефект и конформные отображения

1. Introduction. Notations.

Let $f(z)$ be a function regular for $|z| < R$ and let C_r denote the map of the circle $|z| = r$ $(r < R)$ by $f(z)$. Besides, let $L(r)$ and $S(r)$ denote the length of C_r and the area of Riemann surface of $f(z)$ enclosed by C_r respectively. If $f'(0) \neq 0$, then C_r are simple closed Jordan curves for small r and the isoperimetrical inequality holds:

$$\delta(r) = L^2(r) - 4\pi S(r) \geqslant 0, \tag{1.1}$$

or

$$q(r) = \frac{L^2(r)}{4\pi S(r)} - 1 \geqslant 0. \tag{1.2}$$

The left-hand sides of (1.1) and (1.2) may be called isoperimetrical defects of first and of second kind respectively. In a previous paper [1] (due to the former of both authors and to M. Biernacki) a hypothesis was announced that the isoperimetrical defect of second kind be an increasing function of $r \in (0, R)$. In the same paper a weaker result concerning (1.1) has been proved: the isoperimetrical defect of the first kind is either a strictly increasing function of $r \in (0, R)$ or it vanishes identically (for $f(z)$ being a bilinear function). This statement was proved under the assumption: $f'(z) \neq 0$ for $|z| < R$. The above mentioned hypothesis and its conclusion concerning $\delta(r)$ are two different statements of the fact that the curves C_r monotonically deviate from the circular shape as r increases.

In this paper we give an example of a function regular and univalent in the unit circle for which $q(r)$ decreases strictly for $r \in (r_0, 1)$, $0 < r_0 < 1$. In this counter-example, however, the map of $|z| < 1$ is not a convex domain, so that the question concerning the monotonity of $q(r)$ remains still unanswered for functions representing the unit circle on convex domains.

2. A formula for $q'(r)$.

We now prove that

$$\frac{dL(r)}{dr} = r\int_0^{2\pi} \varkappa(r,\Theta)\,|f'(re^{i\Theta})|^2\,d\Theta, \tag{2.1}$$

the curvature $\varkappa(r,\Theta)$ of C_r at $w=f(re^{i\Theta})$ is to be taken positive (negative) if the centre of curvature of C_r lies on the interior (exterior) normal.

Since

$$L(r) = r\int_0^{2\pi} |f'(re^{i\Theta})|\,d\Theta,$$

so

$$L'(r) = \int_0^{2\pi} |f'(re^{i\Theta})|\,d\Theta + \int_0^{2\pi} r\,|f'(re^{i\Theta})|\,\frac{\partial \log|f'|}{\partial r}\,d\Theta =$$

$$= \int_0^{2\pi} |f'|\left(1+\frac{\partial \arg f'}{\partial \Theta}\right)d\Theta = r\int_0^{2\pi} |f'|^2\,\frac{1+\dfrac{\partial \arg f'}{\partial \Theta}}{r\,|f'|}\,d\Theta.$$

But

$$\frac{1+\dfrac{\partial \arg f'}{\partial \Theta}}{r\,|f'|} = \frac{\Re\left[1+\dfrac{zf''(z)}{f'(z)}\right]}{|zf'(z)|} \qquad (z = re^{i\Theta}) \tag{2.2}$$

and the right-hand side is a well known expression for the curvature $\varkappa(r,\Theta)$ of C_r (see [3], p. 105).

Besides,

$$S(r) = \int_0^{r} \varrho\,d\varrho \int_0^{2\pi} |f'(\varrho e^{i\Theta})|^2\,d\Theta$$

and a differentiation gives

$$q'(r) = \frac{1}{4\pi}\,\frac{rL^2(r)}{S^2(r)} \int_0^{2\pi} \left|\frac{2S(r)\varkappa(r,\Theta)}{L(r)} - 1\right| |f'(re^{i\Theta})|^2\,d\Theta. \tag{2.3}$$

This formula for $q'(r)$ helps us to construct the desired counter-example. It suffices to find a function $f(z)$ such that for a value $r < R$ there is

$$\frac{2S(r)\max\limits_{\Theta}\varkappa(r,\Theta)}{L(r)} < 1.$$

We now prove the

Lemma. *If the univalent function $w = f(z)$ regular for $|z| < 1$ represents the unit circle on the domain G being the interior of a simple, closed and rectifiable Jordan curve C with a continuous curvature $\varkappa = \varkappa(s)$ (s is the length of the arc of C), then the curvature $\varkappa(r,\Theta)$ of C_r tends to the curvature of C at $f(e^{i\Theta})$ uniformly as $r \to 1$ (Θ being fixed).*

Proof. The lemma can be easily proved by using the following, well known result, due to W. Seidel [4].

„Let $w = f(z)$ be a univalent (schlicht) function, regular for $|z| < 1$, which represents the unit circle on the interior of a simple, closed and rectifiable Jordan curve C with a continuous tangent, i. e. the angle $\psi = \psi(s)$ between the tangent of C and the real axis is a continuous function of the length of arc s on C. If, moreover, $\psi(s)$ is Lipschitzian:

$$|\psi(s+h) - \psi(s)| < Kh \qquad (K = \text{const.}),$$

then $f(z)$ and $f'(z)$ are continuous in the closed circle $|z| \leqslant 1$. Besides, $f'(z)$ does not vanish in the closed circle and is absolutely continuous on $|z| = 1$".

Since $\varkappa(s) = d\psi(s)/ds$ is continuous, so $\psi(s)$ is Lipschitzian and all the conditions of Seidel's theorem are fulfilled. Let $\psi(r,\Theta)$ denote the angle between the tangent of C_r at $w = f(re^{i\Theta})$ and the real axis. We have

$$\varkappa(r,\Theta) = \frac{\partial\psi}{\partial s} = \frac{\partial\psi(r,\Theta)}{\partial\Theta}\,|z f'(z)|^{-1}. \tag{2.4}$$

Since

$$\varkappa(s) = \frac{d\psi(s)}{ds} \quad \text{and} \quad \frac{ds}{d\Theta} = |f'(e^{i\Theta})|$$

exist and are continuous, the derivative

$$\frac{d\psi}{d\Theta} = \frac{\partial\psi(1,\Theta)}{\partial\Theta} = \frac{d\psi}{ds}\cdot\frac{ds}{d\Theta} = \varkappa(s)\cdot|f'(e^{i\Theta})| \tag{2.5}$$

exists for all Θ and is continuous on $|z| = 1$ as a function of Θ. In view of continuity and non-vanishing of $f'(z)$ in the closed circle and by (2.4) and (2.5) it suffices to prove that

$$\lim_{r\to 1}\frac{\partial\psi(r,\Theta)}{\partial\Theta}=\frac{\partial\psi(1,\Theta)}{\partial\Theta}. \tag{2.6}$$

We have

$$\psi(r,\Theta)=\arg\{izf'(z)\}=\frac{\pi}{2}+\Theta+\arg f'(z)$$

and therefore

$$\psi(r,\Theta)-\Theta=\frac{\pi}{2}+\Im\{\log f'(z)\}.$$

We see that $\psi(r,\Theta)-\Theta$ being the imaginary part of a function regular for $|z|<1$ and continuous in the closed circle may be expressed as the Poisson integral of its boundary values. The boundary values are obviously $\psi(1,\Theta)-\Theta$ since the angles on the boundary are preserved and we have

$$\psi(r,\Theta)-\Theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\Theta-\alpha)}\{\psi(1,\alpha)-\alpha\}\,d\alpha. \tag{2.7}$$

Since $\partial\psi(1,\Theta)/\partial\Theta$ exists and is continuous, the differentiation and then an integration by parts give

$$\frac{\partial\psi(r,\Theta)}{\partial\Theta}-1=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\Theta}\left\{\frac{1-r^2}{1+r^2-2r\cos(\Theta-\alpha)}\right\}\{\psi(1,\alpha)-\alpha\}\,d\alpha=$$

$$=-\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\alpha}\left\{\frac{1-r^2}{1+r^2-2r\cos(\Theta-\alpha)}\right\}\{\psi(1,\alpha)-\alpha\}\,d\alpha=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\Theta-\alpha)}\left[\frac{\partial\psi(1,\alpha)}{\partial\alpha}-1\right]d\alpha$$

and hence

$$\frac{\partial\psi(r,\Theta)}{\partial\Theta}=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\Theta-\alpha)}\,\frac{\partial\psi(1,\alpha)}{\partial\alpha}\,d\alpha.$$

The continuity of $\partial\psi(1,\Theta)/\partial\Theta$ as a function of Θ and the well known behaviour of Poisson integral of a function continuous on the boundary imply the uniform convergence:

$$\frac{\partial\psi(r,\Theta)}{\partial\Theta}\rightarrow\frac{\partial\psi(1,\Theta)}{\partial\Theta}$$

as $r \to 1$. The continuity of $|zf'(z)|^{-1}$ in the closed ring $0 < \delta \leqslant |z| \leqslant 1$ and thus the uniform convergence:

$$[r\,|f'(re^{i\Theta})|]^{-1} \rightrightarrows |f'(e^{i\Theta})|^{-1}$$

and the relations (2.4) and (2.5) prove the lemma.

We now define the univalent function $w = X + iY = f(z)$ as an arbitrary univalent function representing the unit circle $|z| < 1$ on the interior

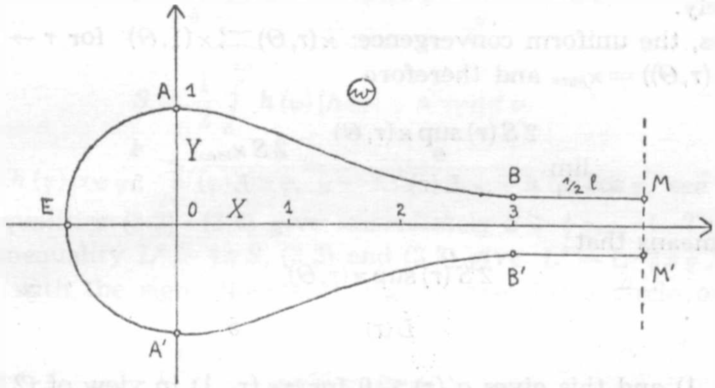

Fig. 1

of a simple, closed and rectifiable curve C with continuous curvature, C being defined as follows. The arc AEA' of C is a semicircle: $X = -\sqrt{1 - Y^2}$ $(-1 \leqslant Y \leqslant 1)$, the arc AB is an arc of a quartic:

$$Y = Y(X) = -\frac{1}{36}X^4 + \frac{2}{9}X^3 - \frac{1}{2}X^2 + 1 \qquad (0 \leqslant X \leqslant 3),$$

the arc $A'B'$ is symmetric to AB with respect on the real axis OX. From the points $B\,(3, {}^1/_4)$ and $B'\,(3, -{}^1/_4)$ we draw two straight line segments BM and $B'M'$ parallel to the real axis and of the length ${}^1/_2\, l$ (the value l shall be determined below). Now, the entire curve C is composed of two parts symmetric to each other with respect on the straight line MM'. It is easy to verify that the curvature $\varkappa = \varkappa(s)$ of C is continuous and that $|\varkappa(s)| \leqslant 1$. This inequality is an immediate consequence of the inequality

$$\sup_{X \in [0,3]} |Y''(X)| = \sup_{X \in [0,3]} \frac{1}{3}|(X-1)(X-3)| = 1.$$

The area of the part $AA'B'B$ of the interior of C is equal to 3,3. Let a_0 be the length of the arc AB and let L and S denote the length of C and the area of the interior of C respectively. If we put $\sup \varkappa(s) = \varkappa_{max}$, we obtain

$$\frac{2 S \varkappa_{max}}{L} = \frac{2\left(\pi + \frac{33}{5} + \frac{1}{2} l\right)}{2\pi + 4 a_0 + 2 l} < \frac{4}{5}$$

for $l > 9$ (since $a_0 > 3$). Now, $f'(z)$ is a continuous function in the closed circle $|z| \leqslant 1$, $S(r)$ and $L(r)$ are continuous, too, and tend to S and L respectively.

Besides, the uniform convergence: $\varkappa(r,\Theta) \rightrightarrows \varkappa(1,\Theta)$ for $r \to 1$ implies: $\lim_{r\to 1} (\sup_{\Theta} \varkappa(r,\Theta)) = \varkappa_{max}$ and therefore

$$\lim_{r\to 1} \frac{2 S(r) \sup_{\Theta} \varkappa(r,\Theta)}{L(r)} = \frac{2 S \varkappa_{max}}{L} < \frac{4}{5}$$

and this means that

$$\frac{2 S(r) \sup_{\Theta} \varkappa(r,\Theta)}{L(r)} < \frac{4}{5}$$

for $r \in (r_0, 1)$ and this gives $q'(r) < 0$ for $r \in (r_0, 1)$ in view of (2.3). Therefore $L^2(r)/4\pi S(r)$ decreases strictly for $r \in (r_0, 1)$,

3. An inequality for convex domains.

The foregoing counter-example is based on the construction of a simple, closed and rectifiable curve C with continuous curvature for which $2 S \varkappa_{max}/L < 1$. We now show that the construction of a convex curve for which such an inequality holds, is impossible. In other words, for convex curves we have always $S \geqslant 1/2\, L \varrho_{min}$, $\varrho_{min} = 1/\varkappa_{max}$ being the least radius of curvature. A similar upper bound for S can be also given.

Suppose, the closed convex curve C with a continuous and non-vanishing curvature $\varkappa(\psi)$ may be defined by the parametric equations $x = x(\psi)$, $y = y(\psi)$, the parameter ψ being the angle of the tangent with the Ox axis. Denote L and S the length of C and the area enclosed by C respectively. Then

$$\frac{1}{2} L \varrho_{min} \leqslant S \leqslant \frac{1}{2} L \bar{\varrho}, \quad \bar{\varrho} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\varkappa(\psi)} d\psi, \tag{3.1}$$

with the sign of equality in any case for a circle only.

P r o o f. Put $h(\psi) = x(\psi)\cos\psi + y(\psi)\sin\psi$. This means $h(\psi)$ is the so called function of support, see [2], p. 24 or [3], p. 106. We can suppose that the origin lies in the interior of C, and then $h(\psi) > 0$. We have

$$\varrho(\psi) = \frac{1}{\varkappa(\psi)} = h(\psi) + h''(\psi) \geqslant 0, \tag{3.2}$$

$$L = \int_0^{2\pi} |h(\psi) + h''(\psi)|\, d\psi = \int_0^{2\pi} h(\psi)\, d\psi, \tag{3.3}$$

$$S = \frac{1}{2}\int_0^{2\pi} h(\psi)\,|h(\psi) + h''(\psi)|\, d\psi, \tag{3.4}$$

since $x = h(\psi)\cos\psi - h'(\psi)\sin\psi$, $y = h(\psi)\sin\psi + h'(\psi)\cos\psi$, see [2], p. 65.

The equalities (3.2) - (3.4) give immediately $S \geqslant \frac{1}{2}\varrho_{min}\cdot L$. The isoperimetrical inequality $L^2 \geqslant 4\pi S$, (3.2) and (3.3) give $L^2 = L\cdot 2\pi\bar{\varrho} \geqslant 4\pi S$, or $\frac{1}{2}\bar{\varrho} L \geqslant S$ with the sign of equality in any case for a circle only.

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S t r e s z c z e n i e

Niech $w = f(z)$ będzie funkcją holomorficzną w kole $|z| < R$ i niech C_r będzie obrazem okręgu $|z| = r$ $(r < R)$, określonym przez funkcję $w = f(z)$. Niech $L(r)$ oznacza długość krzywej C_r, zaś $S(r)$ pole obszaru powierzchni Riemanna funkcji $f(z)$, ograniczonego krzywą C_r.

Jeśli $f'(0) \neq 0$, wówczas C_r są krzywymi Jordana bez punktów wielokrotnych dla r dostatecznie małego. W pracy [1] pierwszy z autorów wykazał, że „defekt izoperymetryczny 1-go rodzaju": $\delta(r) = L^2(r) - 4\pi S(r)$ jest bądź funkcją ściśle rosnącą od r, bądź też $\delta(r) = \text{const}$ (dla funkcji ułamkowo liniowej).

W pracy niniejszej podajemy przykład funkcji odwzorowującej jednolistnie koło $|z| \leqslant 1$ na pewien obszar domknięty, ograniczony krzywą o ciągłej krzywiźnie $\varkappa$ przy czym „defekt izoperymetryczny 2-go rodzaju": $q(r) = L^2(r)/4\pi S(r) - 1$ nie jest funkcją monotoniczną od r. Dowód opiera się na konstrukcji pewnej krzywej, dla której zachodzi nierówność: $2S(1)\max\varkappa/L(1) < 1$.

W dalszym ciągu dowodzimy, że nie istnieje krzywa wypukła, dla którejby taka nierówność miała miejsce.

Резюме

Пусть $w = f(z)$ есть функция голоморфная в круге $|z| < R$, и пусть C_r есть образ окружности $|z| = r$ $(r < R)$, определённый функцией $w = f(z)$. Пусть $L(r)$ обозначает длину кривой C_r, а $S(r)$ — площадь ограниченной кривою C_r области римановой поверхности функции $f(z)$. Если $f'(0) \neq 0$, то C_r суть кривые Жордана без многократных точек для достаточно малого r. В работе [1] первый из авторов показал, что „изопериметрический дефект 1-го рода": $\delta(r) = L^2(r) - 4\pi S(r)$ есть или функция от r строго возрастающая или же $\delta(r) = \text{const.}$ (для дробно-линейной функции).

В этой работе мы даём пример функции, отображающей однолистно круг $|z| \leqslant 1$ на некоторую замкнутую область, ограниченную кривой с непрерывною кривизною $\varkappa$, причём „изопериметрический дефект 2-го рода" $q(r) = L^2(r)/4\pi S(r) - 1$ не является монотонною функцией от r. Доказательство опирается на построении некоторой кривой, для которой имеет место неравенство $2S(1)\max\varkappa/L(1) < 1$.

Далее мы доказываем, что не существует выпуклая кривая, для которой такое неравенство имело бы место.