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On a certain converse of the mean value theorem

O pewnym odwróceniu twierdzenia o wartości średniej

0. It is well known that, if a plane Jordan curve has a tangent vector everywhere, any arc of the curve contains a point at which the tangent is parallel to the chord closing the arc. For the plane curves admitting the representation $x=t$, $y=f(t)$, $t \in [a,b]$, where $f(t)$ is a function continuous in the closed interval $[a,b]$ and differentiable in the open interval (a,b), this is the contents of the mean value theorem.

C. Ryll-Nardzewski has put the following problem: is the converse of the above mentioned property true, i. e., consider the Jordan curve in three dimensional Euclidean space which has the tangent vector everywhere, such that any arc of the curve contains a point at which the tangent is parallel to the chord closing the arc, is such a curve plane? I shall prove that in case of sufficiently regular curves the answer is affirmative. For the suggestions concerning the proof I am much indebted to Prof. A. Bielecki.

1. Under the usual assumptions of differential geometry this converse can be proved as follows. The projection of the curve Γ on the plane normal to the curve at A which is not a stationary point is a curve Γ_1 which has a cusp at A with tangent being the principal normal of Γ at A. Let BC be a chord of the curve corresponding to the intersection of the curve Γ with a plane parallel to the rectifying plane at A and situated at distance δ from it. It can be easily proved by using the Taylor's expansion of the coordinates of a point of the curve in the neighbourhood of A according to the powers of arc that, if δ is small enough, there is no point on the arc $\widehat{BAC}$ at

which the tangent would be parallel to BC. For otherwise there would exist on the projection Γ_1 of the arc $\widehat{BAC}$ on the normal plane a point at which the tangent should be parallel to the projection of BC, which is generally impossible for sufficiently small δ.

2. In connection with this A. Denjoy remarked that the problem can be considered as a particular case of a more general one. Let us consider the curve which has a tangent vector everywhere and has moreover the following property: to every chord there corresponds at least one point (lying not necessarily on the arc closed by this chord), such that the tangent at this point is parallel to the given chord. Or, in other words, let us consider the curve, such that the cone of the directions of all chords is contained in the cone of the directions of tangents. Is such a curve plane? In order to answer this question we shall prove at first the following lemma.

LEMMA 1. *Let Γ be a Jordan curve in three dimensional Euclidean space which has continuous tangent vector everywhere in the interval (a, b), and is defined by the continuous and differentiable functions $x = x(t)$, $y = y(t)$, $z = z(t)$, $t \in [a, b]$, and suppose that the cone of chords is contained in the cone of tangents. Then, if the curve Γ' formed by moving the end point of the unit tangent vector (with origin at 0) is of measure zero on the unit sphere, any chord of Γ and two tangents of Γ at its ends are coplanar.*

Proof. Let $P(t)$ be the point of the curve Γ corresponding to the value t of the parameter. To every point (u, v) of the square $a \leqslant u, v \leqslant b$, such that $P(u) \neq P(v)$, there corresponds at least one point of the curve Γ', namely the one which is the end point of a unit tangent vector of Γ (with origin at 0) parallel to the chord $P(u)P(v)$. If $P(u) = P(v)$, then 0 corresponds to (u, v). Thus, the point with coordinates

$$X = [x(u) - x(v)]\,w, \quad Y = [y(u) - y(v)]\,w, \quad Z = [z(u) - z(v)]\,w, \tag{1}$$

where

$$a \leqslant u, v \leqslant b, \quad -\infty < w < +\infty, \tag{2}$$

lies on the cone surface S with vertex at 0 and directrix Γ'. The transformation (1) maps a three dimensional region (2) into a region lying on a cone surface and being of three dimensional measure zero. Therefore the Jacobian of this transformation is identically zero:

$$\frac{D(X, Y, Z)}{D(u, v, w)} = 0\,,$$

$$\begin{vmatrix} w\,x'(u), & -w\,x'(v), & x(u)-x(v) \\ w\,y'(u), & -w\,y'(v), & y(u)-y(v) \\ w\,z'(u), & -w\,z'(v), & z(u)-z(v) \end{vmatrix} = 0 \, , \quad \text{or}$$

(3) $$\begin{vmatrix} x'(u)\,, & x'(v)\,, & x(u)-x(v) \\ y'(u)\,, & y'(v)\,, & y'u)-y(v) \\ z'(u)\,, & z'(v)\,, & z(u)-z(v) \end{vmatrix} = 0 \, ,$$

or, in vector notation,

(3′) $$\left[\vec{r}'(u)\,,\ \vec{r}'(v)\,,\ \vec{r}(u)-\vec{r}(v)\right] = 0 \, ,$$ (see e. g. Haupt, Aumann: Differential- und Integralrechnung, vol. II, p. 150, „Volumverzerrungssatz").

We can suppose without loss of generality that the parametric representation of Γ does not contain the intervals of constancy of all three coordinates. Then we have on a dense set D of t : $|x'(t)| + |y'(t)| + |z'(t)| > 0$.

If u and v belong to D, then (3′) implies

(4) $$\left[\vec{t}(u)\,,\ \vec{t}(v)\,,\ \vec{r}(u)-\vec{r}(v)\right] = 0 \, ,$$

where $\vec{t}$ denotes the tangent vector. Since (4) is valid on a dense set of u and v, so the continuity of $\vec{t}$ implies (4) for all values of u and v.

LEMMA 2. *The Jordan curve Γ in three dimensional Euclidean space which has everywhere a tangent vector (except possibly at $P(a)$ and $P(b)$) fulfilling the condition* (4) *is a plane curve.*

Proof. It is easy to see that the fulfilled condition (4) implies the equation

(5) $$\left[\vec{r}'(u)\,,\ \vec{t}(v)\,,\ \vec{r}(u)-\vec{r}(v)\right] = 0$$

(because for $\vec{r}'(u) \neq \vec{0}$ we have $\vec{t}(u) = \dfrac{\vec{r}'(u)}{|\vec{r}'(u)|}$) .

Let the point A of Γ with $t = v < b$ be the origin of a new coordinate system $A\,\xi\eta\zeta$, such that the $A\zeta$ axis coincides with the tangent vector $\vec{t}(v)$ at A. Then the vector equation (5) will take the form

$$\begin{vmatrix} \xi'(u), 0, \xi(u) \\ \eta'(u), 0, \eta(u) \\ \zeta'(u), 1, \zeta(u) \end{vmatrix} = 0 .$$

or, writing t instead of u:

(6)
$$\begin{vmatrix} \xi'(t), \xi(t) \\ \eta'(t), \eta(t) \end{vmatrix} = 0 .$$

If we introduce the polar coordinates ϱ, Θ in the (ξ, η) plane, the equation (6) is reduced to

(7)
$$\varrho^2 \frac{d\Theta}{dt} = 0 .$$

Now, it is sufficient to show that the arc $\widehat{P(v)P(b)}$ is plane, since v can be any value of the interval (a, b). The formula (7) shows that, if a point $M = P(m)$, $m > v$, does not lie on the $A\zeta$ axis, then the whole arc $\widehat{P(t_1)P(t_2)}$, where $t_1 = \sup\limits_{\substack{t<m \\ \varrho(t)=0}} t$, $t_2 = \inf\limits_{\substack{t>m \\ \varrho(t)=0}} t$ (or $t_2 = b$ if no such t exist), is plane. So if the arc $\widehat{P(v)P(b)}$ is not plane, then there exist at least two plane arcs, $\widehat{P(t_1)P(t_2)}$ and $\widehat{P(t_3)P(t_4)}$, where $v \leqslant t_1 < t_2 \leqslant t_3 < t_4 \leqslant b$, that do not lie in the same plane and such that $\varrho(t) > 0$ in the intervals $(t_1, t_2), (t_3, t_4)$, and, moreover, $\varrho(t_1) = \varrho(t_2) = \varrho(t_3) = 0$. On the arc $\widehat{P(t_1)P(t_2)}$ there is a point with $t = \tau'$ at which the continuous function $\varrho(t)$ has a maximum. It is easy to see that $\vec{t}(\tau')$ is parallel to $A\zeta$ axis. Similarly on the arc $\widehat{P(t_3)P(t_4)}$ there is a point $P(\tau'')$, such that the tangent $\vec{t}(\tau'')$ is not parallel to $A\zeta$ axis. Now, $[\vec{t}(\tau'), \vec{t}(\tau''), \vec{r}(\tau') - \vec{r}(\tau'')] \neq 0$ because the plane containing the points $P(\tau')$, $P(\tau'')$ and the vector $\vec{t}(\tau')$ is parallel to $A\zeta$ and intersects the plane of the arc $\widehat{P(t_3)P(t_4)}$ on a straight line being parallel to $A\zeta$, and thus, it cannot contain $\vec{t}(\tau'')$. This proves the lemma. The lemmas 1. and 2. give immediately the

Theorem. *Let Γ be a Jordan curve in three dimensional space which is defined by the continuous and differentiable functions $x(t)$, $y(t), z(t)$, $t \in [a, b]$, and has in the interval (a, b) a continuous tangent* *vector. Suppose the cone of chords is contained in the cone of tangents. If moreover the curve formed by moving the end point of the unit tangent vector (with constant origin) is of measure zero on the unit sphere, then Γ is a plane curve.*

Streszczenie

Dla krzywych Jordana płaskich, posiadających wszędzie wektor styczny, na każdym łuku istnieje punkt, w którym styczna jest równoległa do cięciwy, zamykającej łuk. Własność ta, dla krzywych posiadających przedstawienie parametryczne $x=t$, $y=f(t)$, jest treścią twierdzenia o wartości średniej. Wykazuję w pracy niniejszej, że gdy krzywa Jordana przestrzenna posiada wszędzie ciągły wektor styczny o następujących własnościach: 1° indykatrysa sferyczna wektora stycznego jest miary zero na kuli jednostkowej, 2° każdej cięciwie odpowiada conajmniej jedna równoległa do niej styczna, wówczas krzywa jest płaska.

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