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The Sendov–Conjecture and the Maximum Principle

Abstract. If Δ^n is the closed polycylinder then any $z=(z_1,\dots,z_n)\in\Delta^n$ determines a polynomial $p_z(\xi)$ with leading coefficient equal 1 and zeros z_k. Let $C(z)\subset\Delta$ be the set of critical points of $p_z(\xi)$. The author deals with the properties of the function $d(z):=\max\{d_k(z)\colon k=1,\dots,n\}$ where $d_k(z):=\operatorname{dist}(z_k,C(z))$. Note that according to Sendov's conjecture $\max\{d(z)\colon z\in\Delta^n\}=1$.

1. A reformulation of the Sendov–Conjecture

Let $\Delta := \{z \in \mathbb{C} : \ |z| \leq 1\}$ be the closed unit disk and $\Delta^n := \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : |z_k| \leq 1\}$ the closed unit polycylinder in $\mathbb{C}^n$. The subset

$$S^n := \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_k| = 1, \ k = 1, \cdots, n\} \subset \partial(\Delta^n)$$

is called the skeleton of Δ^n. For $z = (z_1, ..., z_n) \in \mathbb{C}^n$ let $p_z(\zeta)$ be the normalized polynomial with roots at z_k:

$$p_z(\zeta) = \prod_{k=1}^{n} (\zeta - z_k) \ .$$

Then clearly $p_z'(\zeta) = n\prod_{j=1}^{n-1}(\zeta - w_j)$, where the w_j are the critical points of $p_z(\zeta)$. The critical set of $p_z(\zeta)$ is defined by $C(z) = \{w_j \in \mathbb{C} : p_z'(w_j) = 0\}$. If $z \in \Delta^n$, then $C(z) \subset \Delta$ due to the well known Theorem of Gauß–Lucas. In the following we investigate the function $d : \mathbb{C}^n \to \mathbb{R}$, defined by

$$d(z) := \max d_k(z) \ ,$$

where

$$d_k(z) := \text{dist}\,(z_k, C(z)) = \min\{|z_k - w_j| : j = 1, \dots, n-1\} .$$

The Conjecture of Sendov can now be formulated as follows:

$$(\star) \qquad \max_{z \in \Delta^n} d(z) = 1 .$$

From a result of Goodman–Rahman–Ratti [1] on boundary zeros we have

$$\max_{z \in S^n} d(z) = 1 ,$$

where S^n is the skeleton of Δ^n defined as above. Using this result, $(\star)$ can be reformulated as

$$(\star\star) \qquad \max_{z \in \Delta^n} d(z) = \max_{z \in S^n} d(z) .$$

It should be noted that there is a stronger conjecture of Phelps–Rodriguez [2], namely

$$\max_{z \in \Delta^n} d(z) = d(z^\star) \Longleftrightarrow p_{z^\star}(\zeta) = \zeta^n - e^{i\alpha} .$$

Up to now this conjecture has been proved for $n = 2, 3, 4$ ([2], Theorem 5).

2. A maximum principle for the function d(z)

The formulation $(\star\star)$ of the Sendov–Conjecture involves a maximum principle of some kind for the function d. As is well known, each continuous and plurisubharmonic (p.s.h.) function on Δ^n must attain its maximum value at the skeleton S^n. Unfortunately, $\log d(z)$ turns out to be only piecewise p.s.h. on Δ^n but fails to be p.s.h. over the whole polycylinder Δ^n. This becomes clear from figure 1, where the graph of $d(z_1, 1, e^{2\pi i/3})$ is plotted over the unit disk $|z_1| \leq 1$. Note the boundary maximum with $d = 1$ at $z_1 = e^{4\pi i/3}$, which corresponds to the extremal polynomial $p(\zeta) = \zeta^3 - 1$ of Phelps–Rodriguez.

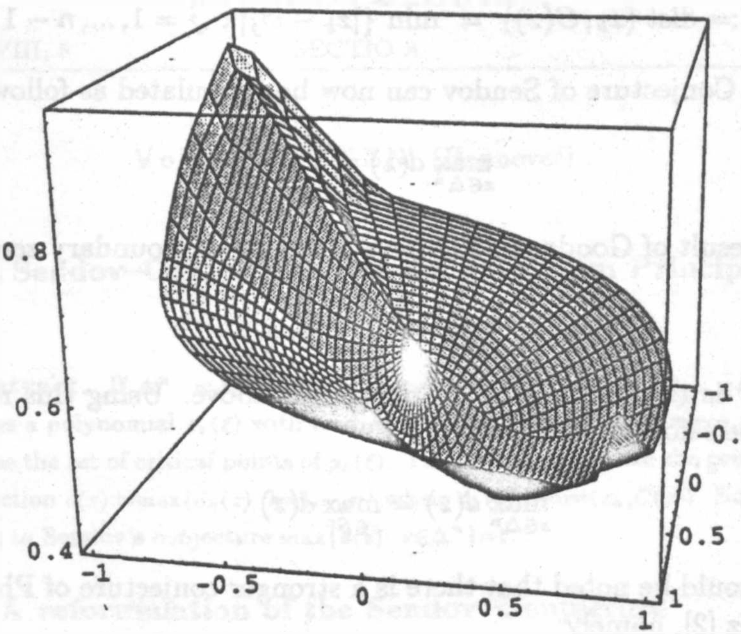

Fig.1

Despite of the fact that $d(z)$ is not plurisubharmonic, it fullfills the following maximum principle, which is essentially due to Phelps and Rodriguez ([2], Theorem 2):

Theorem 1. *The function* $d(z)$ *cannot attain a local maximum in* $\mathbb{C}^n$.

Sketch of the proof. For $z^* = (z_1^*, \cdots, z_n^*)$ let $C(z^*) = \{w_1^*, \cdots, w_{n-1}^*\}$ and $d(z^*) = \text{dist}\,(z_1^*, C(z^*))$. Then one can move the critical points w_j^* away from z_1^*, holding z_1^* fixed. This can be seen by considering the polynomial

$$q(\zeta) := \int_{z_1^*}^{\zeta} \prod_{j=1}^{n-1} (z - w_j)\, dz \quad .$$

If the critical points $w_1, \ldots, w_{n-1}$ of $q(\zeta)$ are chosen near $w_1^*, \ldots, w_{n-1}^*$, then the roots $z_1, \ldots, z_n$ of $q(\zeta)$ are near $z_1^*, \ldots, z_n^*$.

Moreover, if the critical points w_j^* are moved away from z_1^*, we have $d(z) > d(z^*)$.

Theorem 1 implies

$$\max_{z\in\Delta^n} d(z) = \max_{z\in\partial(\Delta^n)} d(z) .$$

Therefore, in order to prove the Sendov–Conjecture (⋆⋆) it would be sufficient to verify that the function $d|_{\Delta^n}$ cannot attain a local maximum at points of $\partial(\Delta^n) \setminus S^n$. Formulated in another way, we have the following

Problem 1. *Given a polynomial $p_{z^*}(\zeta)$ such that all its roots $z_1^*, \cdots, z_n^*$ are in Δ, at least one root is on $|z_k| = 1$ and at least one root has $|z_k| < 1$. Can you always find roots $z_1, \ldots, z_n$ near $z_1^*, \ldots, z_n^*$ but within Δ, such that $d(z) > d(z^*)$?*

3. The restriction of d(z) to lower dimensional planes

In order to tackle Problem 1 and also to get a refinement of Theorem 1, it may be helpful to look at the behavior of the function $d(z)$ on lower dimensional planes. It becomes clear from the proof of Theorem 1, that we have in fact the following

Theorem 1′. *Let $z^* = (z_1^*, \ldots, z_n^*) \in \mathbb{C}^n$ and A^{n-1} be the $(n-1)$-dimensional plane through z^* given by*

$$A^{n-1} = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : z_1 = z_1^*\} .$$

Then the restriction of d to A^{n-1} cannot have a local maximum at z^*. As a consequence, $d|_{A^{n-1}} \cap \Delta^n$ attains its maximum on the "boundary" $A^{n-1} \cap \partial(\Delta^n)$.

Remark. In [3], Miller investigated extremal polynomials $p_{z^*}(\zeta)$ for which the restriction of $d(z)$ to $\{z \in \mathbb{C}^n : z_1 = z_1^*\} \cap \Delta^n$ is maximal at z^*. He proved that $2r + s \geq n - 1$, where s denotes the number of roots of the extremal polynomial on the unit circle and r is the number of critical points on the circle $|w - z_1^*| = d(z^*)$ which is called the critical circle.

From Theorem 1′ we are led to

Problem 2. *Is it true that the restriction $d|_{A^{n-1}}$ of d to an arbitrary $(n--1)$-dimensional plane $A^{n-1} \subset \mathbb{C}^n$ with $A^{n-1} \cap \Delta^n \neq \emptyset$ attains its maximum at points of $A^{n-1} \cap \partial(\Delta^n)$?*

A partial solution of Problem 2 is given by

Theorem 2. *Let $A^2 \subset \mathbb{C}^3$ be an arbitrary complex plane of dimension 2 with $A^2 \cap \Delta^3 \neq \emptyset$. Then*

$$\max_{z \in A^2 \cap \Delta^3} d(z) = \max_{z \in A^2 \cap \partial(\Delta^3)} d(z) .$$

Proof. Let $z^* \in A^2 \cap \Delta^3$ such that $d(z^*) = \max_{z \in A^2 \cap \Delta^3} d(z) =: d^*$. If $z^* \in A^2 \cap \partial(\Delta^3)$, there is nothing to prove. Suppose therefore that $z^* = (z_1^*, z_2^*, z_3^*)$ is an inner point of Δ^3. We may assume that $d^* =$ dist $(z_1^*, C(z^*))$. In the following it will be shown that there exists an at least one-dimensional analytic set $S \subset \mathbb{C}^3$ through z^* with $d|_S(z) \equiv d^*$. As is well known, each such set S must intersect $\partial(\Delta^3)$ (cf. [4], Chapt. 2, Corollary 4), and thus the conclusion of Theorem 2 follows. In order to prove the existence of such an analytic set S we will distinguish two cases, depending on the critical points w_1^*, w_2^* of the extremal polynomial $p_{z^*}(\zeta) = \prod_{k=1}^3 (\zeta - z_k^*)$.

Case 1: $|z_1^* - w_1^*| = |z_1^* - w_2^*|$.

Note that the polynomial $p_z(\zeta) = \prod_{k=1}^3 (\zeta - z_k)$ has the critical points

$$w_{1,2} = \frac{1}{3}(\sigma_1(z) \pm \sqrt{\sigma_1^2(z) - 3\sigma_2(z)}\ ) ,$$

with

$$\sigma_1(z) = z_1 + z_2 + z_3, \qquad \sigma_2(z) = z_1 z_2 + z_1 z_3 + z_2 z_3 .$$

Consider first the particular case $w_1^* \neq w_2^*$. Then $\sigma_1^2(z^*) - 3\sigma_2(z^*) \neq 0$ and thus there is a unique number t such that

$$(z_1^* - \frac{1}{3}\sigma_1(z^*))^2 = -\frac{t^2}{9}(\sigma_1^2(z^*) - 3\sigma_2(z^*)) .$$

It follows from $|z_1^\star - w_1^\star| = |z_1^\star - w_2^\star|$ that the vectors $\pm\sqrt{\sigma_1^2(z^\star) - 3\sigma_2(z^\star)}$ have to be orthogonal to the vector $z_1^\star - \frac{1}{3}\sigma_1(z^\star)$, hence t must be real. With t as above define

$$M = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : (z_1 - \frac{1}{3}\sigma_1(z))^2 = -\frac{t^2}{9}(\sigma_1^2(z) - 3\sigma_2(z))\ \}$$

and let S be an irreducible component of the analytic set $A^2 \cap M$ through $z^\star$. Clearly $z^\star \in S$ and

$$\dim_{z^\star} S \geq \dim_{z^\star} A^2 + \dim_{z^\star} M - 3 \geq 1\ .$$

By construction, we have

$$z_1 - \frac{1}{3}\sigma_1(z) = \pm\frac{it}{3}\sqrt{\sigma_1^2(z) - 3\sigma_2(z)} \quad (z \in S)$$

and therefore

$$|z_1 - w_1(z)| = |z_1 - w_2(z)| = |(\frac{it}{3} - \frac{1}{3})\sqrt{\sigma_1^2(z) - 3\sigma_2(z)}| \quad (z \in S).$$

Since $d|_{A^2}$ has a maximum at $z^\star \in S$, the same is true for the restriction $d|_S$. Because $\sqrt{\sigma_1^2(z) - 3\sigma_2(z)} \neq 0$ has a holomorphic branch near $z^\star$, it follows from the maximum principle for holomorphic functions on analytic sets (cf. [4], Chapt. 4, Theorem 2 G), that $\sigma_1^2(z) - 3\sigma_2(z)$ is a constant on S and therefore

$$d(z) = |z_1 - w_1(z)| = |z_1 - w_2(z)| \equiv d^\star \quad (z \in S)\ .$$

In the particular case $w_1^\star = w_2^\star$ we have $\sigma_1^2(z^\star) - 3\sigma_2(z^\star) = 0$ and one can proceed similarly. Define M in this case by

$$M = \{z \in \mathbb{C}^3 : \sigma_1^2(z) - 3\sigma_2(z) = 0\}$$

and let S again be an irreducible component of $A^2 \cap M$ through $z^\star$. Then clearly $z^\star \in S$ and

$$d(z) = |z_1 - w_1(z)| = |z_1 - w_2(z)| = |z_1 - \frac{1}{3}\sigma_1(z)| \quad (z \in S)\ .$$

From this we conclude $d(z) \equiv d^\star$ on S in view of the maximum principle.

Case 2: $d^\star = |z_1^\star - w_1^\star| < |z_1^\star - w_2^\star|$.
In this case we have

$$d(z) = |z_1 - \frac{1}{3}\sigma_1(z) - \frac{1}{3}\sqrt{\sigma_1^2(z) - 3\sigma_2(z)}\ |$$

locally on A^2 near the point $z^\star$, with an appropriate branch of the root.

Because $d|_{A^2}$ has a local maximum at $z^\star$, the maximum principle yields

$$z_1 - \frac{1}{3}\sigma_1(z) - \frac{1}{3}\sqrt{\sigma_1^2(z) - 3\sigma_2(z)} \equiv \text{ const}$$

locally near $z^\star$ on A^2. It follows that $|z_1 - w_1(z)| \equiv d^\star$ on $A^2 \cap \Delta^3$. Therefore the conclusion of Theorem 2 follows if $d(z) \equiv |z_1 - w_1(z)|$ on $A^2 \cap \Delta^3$. If $d(z) \not\equiv |z_1 - w_1(z)|$ on $A^2 \cap \Delta^3$, there must be a point $\tilde{z} \in A^2 \cap \Delta^3$ such that

$$|\tilde{z}_1 - w_1(\tilde{z})| = |\tilde{z}_1 - w_2(\tilde{z})| = d^\star \ .$$

But then we are done due to the case 1, which completes the proof of Theorem 2.

Problem 3. *Determine the greatest codimension k such that the following is true: If A^{n-k} is an arbitrary complex plane of dimension $n-k$ with $A^{n-k} \cap \Delta^n \neq \emptyset$, then*

$$\max_{z \in A^{n-k} \cap \Delta^n} d(z) = \max_{z \in A^{n-k} \cap \partial(\Delta^n)} d(z) \, .$$

With k as in the problem 3, $k = n-1$ would imply $(\star\star)$ and therefore the Sendov-Conjecture. However, in general $k < n-1$. This can be seen from the figure 2, which deals with degree $n = 8$. Shown there is the graph of $d(z_1, z_2^\star, \ldots, z_8^\star)$ plotted over the unit disk $|z_1| \leq 1$, for $z_2^\star = 1$, $z_{3,4}^\star = e^{\pm i\pi/6}$, $z_{5,6}^\star = e^{\pm i\pi/3}$, $z_{7,8}^\star = \pm i$. According to the figure 2, the restriction of $d(z)$ to the one-dimensional z_1-plane has maxima at inner points but not on the boundary of the unit disk.

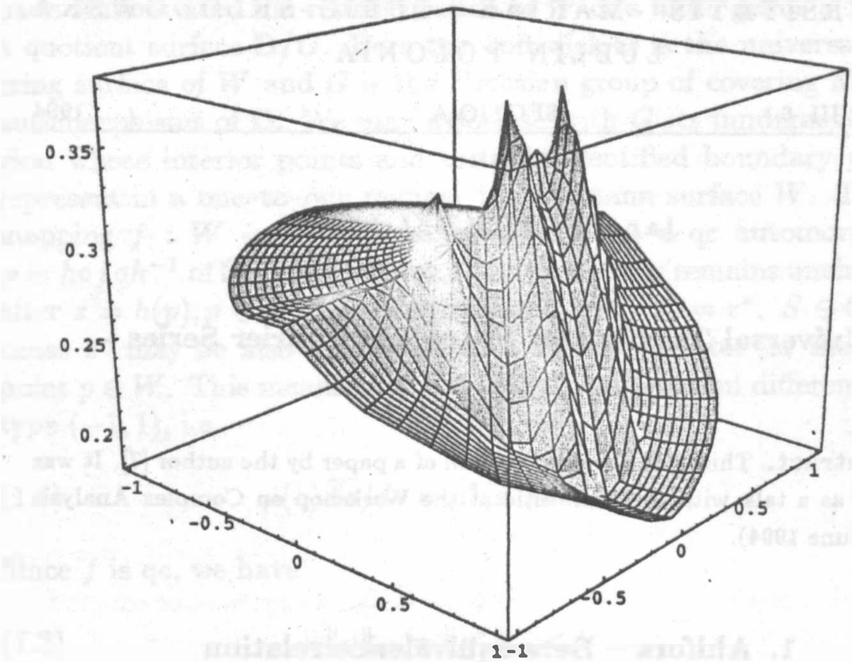

Fig. 2

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