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On the Growth of Generalized Powers

Dedicated to Eligiusz Złotkiewicz
on the occasion of his 60th birthday

ABSTRACT. It is shown here that generalized powers $[\lambda(z-z_0)^n]_{\nu,\mu}$, n being a nonzero integer, satisfy an inequality

$$\kappa^{-|n|}|\lambda(z-z_0)^n| \le |[\lambda(z-z_0)^n]_{\nu,\mu}| \le \kappa^{|n|}|\lambda(z-z_0)^n|,$$

where κ is a constant depending only on (certain quantities of) the coefficients ν, μ of the corresponding Cauchy-Riemann system. An application to convergence of generalized power series is given.

I. Generalized powers are special solutions of a Cauchy-Riemann system

(1) $$f_{\bar{z}} = \nu f_z + \mu \overline{f_z}$$

in $\mathbb{C}$ with the topological structure

(2) $$f(z) = (\chi(z) - \chi(z_0))^n\,, \quad n \in \mathbb{Z} \setminus \{0\}\,,$$

where $\chi(z)$ is a quasiconformal mapping of $\mathbb{C}$ onto itself. We denote them by $[a(z-z_0)^n]_{\mu,\nu}$ (for the notation used here and in the following cf. [5], [7]).

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These special solutions possess interesting properties (cf. [4], [6], [7]). In particular, under the conditions on ν, μ from [8], every solution f of (1) in a disk $\{|z-z_0|<R\}$ admits a series expansion

$$f(z)=\sum_{n=0}^{\infty}[a_n(z-z_0)^n]_{\nu,\mu} \text{ in } \{|z-z_0|<\vartheta R\}, \tag{3}$$

where ϑ is a constant from $(0,1]$, which is independent of f but depends on the growth behaviour of generalized powers, namely on bounds for

$$\sup\left\{\left|\frac{\chi(z)}{z-z_0}\right| : z\in\mathbb{C}\setminus\{z_0\}\right\} \tag{4}$$

and

$$\inf\left\{\left|\frac{\chi(z)}{z-z_0}\right| : z\in\mathbb{C}\setminus\{z_0\}\right\} \tag{5}$$

where $(\chi(z))^n=[\lambda(z-z_0)^n]_{\nu,\mu}$ with $\lambda=1$. We want to determine such bounds here. Without loss of generality we may assume that $z_0=0$.

II. Let (1) satisfy the usual conditions

$$\nu\,,\,\mu\in L_\infty\,,\quad \|\,|\nu|+|\mu|\,\|_{L_\infty}=:k<1. \tag{6}$$

Additionally we suppose

$$\nu(z)=\mu(z)\equiv 0 \quad \text{for} \quad |z|>R\geq 1, \tag{7}$$

as well as the validity of the Bojarski condition (cf. [2, p. 499]) at $z_0=0$, i.e.

$$\frac{\nu(z)-\nu(0)}{z}\,,\ \frac{\mu(z)-\mu(0)}{z}\in L_{p^*} \quad \text{with a} \quad p^*>2. \tag{8}$$

Further, we may assume (by diminution of $p^*>2$ if necessary) that

$$kC(p^*)<1, \tag{9}$$

where $C(p)$ means the norm of the complex Hilbert transformation T in L_p. By reasons which become clear later (cf. (48) below) we also choose a p' such that

$$2<p'\leq p^*\,,\ k'C(p')<1 \ \text{with} \ k':=1-\frac{(1-k)^3}{(1+k)^2}\,. \tag{10}$$

Let

$$w(z) = \left(\frac{[\lambda z^n]_{\nu,\mu}}{z^n}\right)^l, \quad n \in \mathbb{Z}\setminus\{0\}, \, l \in \{+1,-1\}, \lambda \in \mathbb{C}\setminus\{0\} \tag{11}$$

and

$$d(\nu,\mu;p) = \left\|\frac{\nu(z)-\nu(0)}{z}\right\|_{L_p} + \left\|\frac{\mu(z)-\mu(0)}{z}\right\|_{L_p}. \tag{12}$$

We want to prove the following

Theorem. *Let ν, μ satisfy (6), (7), (8), and let p' be chosen so that (10) is satisfied. There exists a constant $\kappa \geq 1$ depending only on k, p', R and $d(\nu,\mu;p')$ such that, for every $w(z)$ defined by (11) with $|\lambda| = 1$,*

$$\kappa^{-|n|} \leq |w(z)| \leq \kappa^{|n|} \quad \forall_z \in \mathbb{C}\setminus\{0\}.$$

For a different type of generalized powers (connected with Carleman-Vekua systems), while using the theory of pseudoanalytic functions, a (formally) similar result has been established in [1].

As a consequence of the theorem we obtain (according to [8])

$$\vartheta = 1/\kappa^2 \tag{13}$$

(κ given by (50) below) is one ϑ for (3) to hold. (By the way, this also implies that (3) remains valid even without continuity of ν, μ at z_0 (condition (6) in [8]).)

The proof of the theorem essentially rests on the following

Lemma. *Let ν^*, μ^* satisfy (6), (7), (8), (9) and $\nu^*(0) = \mu^*(0) = 0$. There exists a positive constant $r(k,p^*)$ (explicitly given by (20) below) such that for every $w(z)$ defined by (11) (with ν,μ replaced by ν^*,μ^*) we have*

$$|\lambda|^l e^{-2|n|r(k,p^*)D(\nu^*,\mu^*;p^*)} \leq w(z)| \leq |\lambda|^l e^{2|n|r(k,p^*)D(\nu^*,\mu^*;p^*)} \quad \forall z \in \mathbb{C}.$$

Here

$$D(\nu^*,\mu^*;p^*) = \left\|\frac{\nu^*(z)}{z}\right\|_{L^{p^*,q^*}} + \left\|\frac{\mu^*(z)}{z}\right\|_{L^{p^*,q^*}}, \quad \frac{1}{p^*}+\frac{1}{q^*} = 1, \tag{14}$$

and $||g||_{L^{p,q}} := \max\{||g||_{L_p}, ||g||_{L_q}\}$ for $g \in L_p \cap L_q$.

Proof of Lemma. Let $f(z) := [\lambda z^n]_{\nu^*,\mu^*}$. Then $w(z) = (f(z))^l \cdot z^{-nl}$ satisfies

$$w_{\overline{z}} = \nu^* w_z + \mu^* \frac{\overline{f} w}{f \overline{w}} \overline{w_z} + \nu^* \frac{nl}{z} w + \mu^* \frac{\overline{f}}{f} \frac{nl}{\overline{z}} \overline{w}. \tag{15}$$

At $z = 0$, $f(z)$ possesses the asymptotic expansion

$$f(z) = \lambda z^n + O(|z|^{n+\alpha}) \text{ with } \alpha > 0,$$

and at infinity, $f(z) = \alpha_n(\lambda) \cdot z^n + O(|z|^{n-1})$ with an unknown but well-defined constant $\alpha_n(\lambda) \neq 0$. Hence

$$\lim_{z\to 0} w(z) = \lambda^l =: w(0), \quad \lim_{z\to\infty} w(z) = (\alpha_n(\lambda))^l. \tag{16}$$

Thus, $w(z)$ is a solution of (15) bounded in $\mathbb{C}$. By the Bers-Nirenberg Representation Theorem and Liouville's Theorem for analytic functions,

$$w(z) = \text{const} \cdot e^{s(z)}, \tag{17}$$

where $s(z)$ is Hölder continuous in $\mathbb{C}$ and $\lim_{z\to\infty} s(z) = 0$. Hence

$$w(z) = (\alpha_n(\lambda))^l e^{s(z)}. \tag{18}$$

Moreover, by [5, p. 45], $s(z)$ satisfies the estimate

$$|s(z)| \leq K_{p^*,q^*}(1 - kC(p^*))^{-1} |n| D(\nu^*, \mu^*; p^*) \tag{19}$$

($C(p^*) = C(q^*)$ because of $(1/p^*) + (1/q^*) = 1$, (cf. e.g. [3], [9, p. 33]).

Let

$$r := r(k, p^*) := K_{p^*,q^*}(1 - kC(p^*))^{-1}. \tag{20}$$

Because $w(0) = \lambda^l = (\alpha_n(\lambda))^l e^{s(0)}$ we have

$$e^{-|n| r D(\nu^*,\mu^*;p^*)} \leq \left| \frac{\alpha_n(\lambda)}{\lambda} \right| \leq e^{+|n| r D(\nu^*,\mu^*;p^*)}. \tag{21}$$

Then (18), (19), (21) yield the assertion of the lemma.

III. Proof of Theorem. We have to reduce the ν, μ of the theorem to ν^*, μ^* of the lemma.

To this end we first submit the z-plane to the mapping

$$(22) \qquad t = \begin{cases} z + \mathfrak{b}\overline{z} & \text{if } |z| < R \\ z + \dfrac{\mathfrak{b}R^2}{z} & \text{if } |z| \geq R\,, \end{cases}$$

where

$$(23) \qquad \mathfrak{b} = \frac{2\sigma}{1+\sqrt{1-4|\sigma|^2}}\,, \quad \sigma = \frac{\nu(0)}{1+|\nu(0)|^2-|\mu(0)|^2}$$

Then $\mathfrak{b}$ satisfies $|\mathfrak{b}| \leq k$ (cf. [5, p. 52]). A (ν,μ)-solution $f(z)$ in $\mathbb{C}\setminus\{0\}$ changes into a (ν_1,μ_1)-solution $g(t) := f(z(t))$ in $\mathbb{C}\setminus\{0\}$, where

$$\nu_1(t) = \begin{cases} \dfrac{\overline{\nu}\mathfrak{b}^2 - (1+|\nu|^2-|\mu|^2)\mathfrak{b} + \nu}{N_1} & \text{if } |z(t)| < R \\ 0 & \text{if } |z(t)| \geq R, \end{cases}$$

$$N_1 = |1-\overline{\nu}\mathfrak{b}|^2 - |\mu\mathfrak{b}|^2\,, \quad N_1 \geq (1-k^2)^2\,,$$

$$\mu_1(t) = \begin{cases} \mu\dfrac{1-|\mathfrak{b}|^2}{N_1} & \text{if } |z(t)| < R \\ 0 & \text{if } |z(t)| \geq R, \end{cases}$$

$\nu = \nu(z(t))$, $\mu = \mu(z(t))$, cf. [5, p. 51].

Then, in particular,

$$(24) \qquad \nu_1(t) = \mu_1(t) = 0 \ \text{ if } \ |t| > R(1+k) =: R^*,$$

$$(25) \qquad \| |\nu_1| + |\mu_1| \|_{L_\infty} \leq 1 - \frac{(1-k)^2}{(1+k)} =: k_1 < 1$$

cf. [4, 1.11], and

$$(26) \qquad \nu_1(0) = 0.$$

An elementary calculation gives

$$(27) \qquad |\nu_1(t)| \leq \frac{1}{(1-k)^2}(|\nu - \nu(0)| + |\mu - \mu(0)|)$$

and

$$(28) \qquad |\mu_1(t) - \mu_1(0)| \le \frac{2}{(1-k)^3}(|\nu - \nu(0)| + |\mu - \mu(0)|).$$

For $f(z) = [\lambda z^n]_{\nu,\mu}$ we obtain $f(z(t)) =: g(t) = [\lambda t^n]_{\nu_1,\mu_1}$, i.e. λ remains unchanged. The latter is a consequence of the asymptotic expansion

$$(29) \qquad f(z) = \lambda(z + b\overline{z})^n - b\overline{\lambda}(\overline{z} + \overline{b}z)^n + O(|z|^{n+\alpha})$$

of f at $z = 0$, cf. [5, p. 70].

Next we have to apply appropriate transformations of the g-plane to arrive at a (ν^*, μ^*)–system satisfying the conditions of the lemma. Here we have to distinguish the cases $n \ge 1$ and $n \le -1$. First let $n \ge 1$. Then, with $g = g(t) = [\lambda t^n]_{\nu_1,\mu_1}$, we put

$$(30) \qquad h_1(t) = \begin{cases} g + b\overline{g} & \text{if } |g(t)| < \rho \\ s(g) := g + \dfrac{b\rho^2}{g} & \text{if } |g(t)| \ge \rho, \end{cases}$$

ρ being a positive constant to be specified later and

$$(31) \qquad b = -\mu_1(0).$$

Again $|b| \le k(\le k_1)$ (by the way, it is the same b as in (29)).

Then

$$(32) \qquad h_1(t) = [\lambda^* t^n]_{\nu^*,\mu^*} \text{ with } \lambda^* = (1 - |b|^2)\lambda$$

and

$$\nu^*(t) = \begin{cases} \nu_1(t)\,\dfrac{1-|b|^2}{N_0} & \text{if } |g(t)| < \rho \\ \nu_1(t) & \text{if } |g(t)| \ge \rho, \end{cases}$$

$$\mu^*(t) = \begin{cases} \dfrac{\overline{\mu_1}b^2 + (1 + |\mu_1|^2 - |\nu_1|^2)b + \mu_1}{N_0} & \text{if } |g(t)| < \rho \\ \mu_1(t)\, s'(g(t))/\overline{s'(g(t))} & \text{if } |g(t)| \ge \rho, \end{cases}$$

with

$$N_0 = |1 + \overline{\mu_1}b|^2 - |\nu_1 b|^2, \; N_0 \ge (1 - k_1|b|)^2 \ge (1 - k_1^2)^2.$$

Now

$$(33) \qquad \nu^*(0) = \mu^*(0) = 0.$$

A similar, but even simpler calculation as with (27), (28) yields

$$|\nu^*(t)| \le \frac{1}{1-k_1^2}\,|\nu_1(t)|\,,\quad |\mu^*(t)| \le \frac{1}{(1-k_1)^2}(|\nu_1(t)| + |\mu_1(t) - \mu_1(0)|).$$

From (27), (28), (25) we finally obtain

$$(34)\qquad |\nu^*(t)| \le \frac{(1+k)^2}{(1-k)^6}\left(|\nu(z(t)) - \nu(0)| + |\mu(z(t)) - \mu(0)|\right),$$

$$(35)\qquad |\mu^*(t)| \le 3\frac{(1+k)^2}{(1-k)^7}\left(|\nu(z(t)) - \nu(0)| + |\mu(z(t)) - \mu(0)|\right)$$

if $|g(t)| < \rho$.

As to an appropriate specification of ρ we first observe that

$$g(t) = [\lambda t^n]_{\nu_1,\mu_1} = \alpha_n(\lambda)(X(t))^n$$

where $X(t)$ is the unique homeomorphic solution of $X_{\bar t} = \nu_g X_t\,,\ \nu_g(t) = \nu_1(t) + \mu_1(t)\overline{g_t}/g_t$, satisfying

$$X(0) = 0\,,\quad X(t) = t + \alpha_0 + \frac{a_{-1}}{t} + \cdots \quad \text{for } |t| > R^*.$$

This homeomorphism admits the representation $X(t) = t + Ph(t) - Ph(0)$ with P the Cauchy transformation and h the unique solution of $h = \nu_g + \nu_g Th$ in $L^{p',q'}$ with $p' > 2$, $(1/p') + (1/q') = 1$ (note that $\|\nu_g\|_{L_\infty} \le k_1 < k' < 1$). Hence

$$(36)\qquad \|h\|_{L^{p',q'}} \le \frac{k'}{1 - k'C(p')}\|1_{R^*}\|_{L^{q'}} =: Q,$$

where 1_{R^*} is the characteristic function of $\{|t| < R^*\}$.

Then $X(t)$ satisfies (cf. [5, p.14])

$$(37)\qquad |t| - K \le |X(t)| \le |t| + K \text{ with } K := 2K_{p',q'}\cdot Q.$$

Now we put

$$(38)\qquad \rho = \rho_g = |\alpha_n(\lambda)|(R^* + K)^n\,.$$

Then

$$(39)\qquad \{t : |g(t)| \ge \rho\} \subseteq \{t : |t| \ge R^*\},$$

hence

$$\nu^*(t) = \mu^*(t) = 0 \text{ if } |g(t)| \geq \rho. \tag{40}$$

Further, if $|t| \geq R^* + 2K$ then $|g(t)| \geq \rho$, thus

$$\{t : |g(t)| < \rho\} \subseteq \{t : |t| < R^* + 2K\}. \tag{41}$$

Now we come to the case of $g(t) = [\lambda t^n]_{\nu_1,\mu_1}$, $n \leq -1$. We put, with b from (31),

$$h_2(t) = \begin{cases} (1 - |b|^2)g & \text{if } |g + b\overline{g}| < \rho \\ s_*(g + b\overline{g}) & \text{if } |g + b\overline{g}| \geq \rho, \end{cases}$$

where $s_*(w) := w - (b\rho^2)/w$. Then $h_2(t) = [\lambda^* t^n]_{\nu^*,\mu^*}$, $\lambda^* = (1 - |b|^2)\lambda$, where

$$\nu^*(t) = \begin{cases} \nu_1(t) & \text{if } |g + b\overline{g}| < \rho \\ \nu_1(t)\dfrac{1 - |b|^2}{N_0} & \text{if } |g + b\overline{g}| \geq \rho, \end{cases}$$

$$\mu^*(t) = \begin{cases} \mu_1(t) & \text{if } |g + b\overline{g}| < \rho \\ \dfrac{\overline{\mu_1}b^2 + (1 + |\mu_1|^2 - |\nu_1|^2)b + \mu_1}{N_0} \cdot \dfrac{\overline{s'_*(g + b\overline{g})}}{s'_*(g + b\overline{g})} & \text{if } |g + b\overline{g}| \geq \rho, \end{cases}$$

N_0 as above.

Again (34), (35) hold, first if $|g + b\overline{g}| \geq \rho$, and again $g(t) = \alpha_n(\lambda)(X(t))^n$ with X as above which, in particular, satisfies (37). We now put

$$\rho = \rho_g := (1 - k)|\alpha_n(\lambda)|(R^* + K)^n. \tag{42}$$

Then $\{t : |g + b\overline{g}| < \rho\} \subseteq \{t : |t| > R^*\}$, hence

$$\nu^*(t) = \mu^*(t) = 0 \text{ if } |g + b\overline{g}| < \rho. \tag{43}$$

Further, $|g + b\overline{g}| \geq \rho$ implies $|X(t)|^{|n|} \leq [(1 + k)/(1 - k)](R^* + K)^{|n|}$, and this again implies

$$|t| \leq \frac{1 + k}{1 - k}(R^* + K\,\frac{2}{1 + k}) < \frac{1}{1 - k}(4R + 2K) =: R'. \tag{44}$$

Thus

$$\{t : |g + b\overline{g}| \geq \rho\} \subseteq \{t : |t| < R'\}. \tag{45}$$

Because of (34), (35), (40), (41), (43), (45) we obtain for any $p > 2$ (in both cases $n \geq 1$ and $n \leq -1$)

$$\left\| \frac{\nu^*(t)}{t} \right\|_{L_p} \leq \frac{(1+k)^2}{(1-k)^6} \left[\left(\int_{\{|t|<R'\}} \left| \frac{\nu - \nu(0)}{t} \right|^p d\sigma_t \right)^{1/p} + \left(\int_{\{|t|<R'\}} \left| \frac{\mu - \mu(0)}{t} \right|^p d\sigma_t \right)^{1/p} \right].$$

Further

$$\int_{\{|t|<R'\}} \left| \frac{\nu(z(t)) - \nu(0)}{t} \right|^p d\sigma_t \leq \int_C \left| \frac{\nu(z) - \nu(0)}{z} \right|^p \left| \frac{z}{t} \right|^p \frac{d\sigma_t}{d\sigma_z} \, d\sigma_z$$
$$\leq \int_C \left| \frac{\nu(z) - \nu(0)}{z} \right|^p \frac{(1+k)^2}{(1-k)^p} \, d\sigma_z \, ,$$

and the same inequality holds with ν replaced by μ. Hence

$$\left\| \frac{\nu^*(t)}{t} \right\|_{L_p} \leq \frac{(1+k)^3}{(1-k)^7} \, d(\nu, \mu; p).$$

In the same way we obtain

$$\left\| \frac{\mu^*(t)}{t} \right\|_{L_p} \leq 3 \frac{(1+k)^3}{(1-k)^8} \, d(\nu, \mu; p).$$

By Hölder's inequality, for any $a \in L_p$ vanishing outside $\{|t| \leq R'\}$,

$$||a||_{L_q} \leq ||a||_{L_p} (\pi R'^2)^{1-(2/p)} \quad \text{if} \quad p > 2 \, , \; \frac{1}{p} + \frac{1}{q} = 1.$$

This finally gives the crucial estimate

$$D(\nu^*, \mu^*; p') \leq 4 \, \frac{(1+k)^3}{(1-k)^8} (\pi R'^2)^{1-(2/p')} \cdot d(\nu, \mu; p'). \tag{46}$$

Let now $w(z)$ be defined by (11) above. Then

$$|w(z)| = |h_j(t)/t^n|^l \cdot |t/z|^{nl} \cdot |g(t)/h_j(t)|^l \, , \tag{47}$$

$j = 1$ or $= 2$ if $n \geq 1$ or ≤ -1, respectively.

Since

$$\| |\nu^*| + |\mu^*| \|_{L_\infty} \le k' \tag{48}$$

(cf. [4, 1.11]), we first obtain by the Lemma

$$\begin{aligned} |\lambda^*|^l e^{-2|n|r(k',p')D(\nu^*,\mu^*;p')} &\le \left|\frac{h_j(t)}{t^n}\right|^l \\ &\le |\lambda^*|^l e^{2|n|r(k',p')D(\nu^*,\mu^*;p')}. \end{aligned} \tag{49}$$

Since (for every $\rho > 0$) $1 - k \le |t/z| \le 1/(1-k)$ and

$$(1-k)^2 \le \left|\frac{g(t)}{h_j(t)}\right| \le 1/(1-k)^2,$$

we have

$$\begin{aligned} |\lambda|^l(1-k)^{|n|+3}(1+k)e^{-|n|\delta} &\le |w(z)| \\ &\le |\lambda|^l(1-k)^{-(|n|+3)}(1+k)^{-1}e^{|n|\delta} \end{aligned}$$

in $\mathbb{C} \setminus \{0\}$, where

$$\delta := 8r(k',p')\,\frac{(1+k)^3}{(1-k)^8}(\pi R'^2)^{1-(2/p')} \cdot d(\nu,\mu;p').$$

Thus

$$\kappa = \frac{e^\delta}{(1-k)^4(1+k)} \tag{50}$$

satisfies the assertions of the Theorem.

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