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On the Krzyż Conjecture and Related Problems II

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. The coefficient problem for holomorphic bounded and nonvanishing functions in the unit disk related to the Krzyż conjecture is discussed.

1. This note may be considered as a continuation of our work [2], where corresponding references can be found. Let $H(\mathbb{D})$ denote the set of holomorphic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In the sequel we consider the following families of functions:

$$(1) \qquad \mathcal{B} := \{f \in H(\mathbb{D}) : f(z) = a_0 + a_1 z + \cdots, |f(z)| < 1, z \in \mathbb{D}\},$$

$$(2) \qquad \mathcal{B}_0 := \{f \in \mathcal{B} : f(z) \neq 0, z \in \mathbb{D}\},$$

$$(3) \qquad \mathcal{P} := \{p \in H(\mathbb{D}) : p(z) = 1 + p_1 z + \cdots, \operatorname{Re} p(z) > 0, z \in \mathbb{D}\}.$$

With no loss of generality we may assume for $f \in \mathcal{B}_0$ the normalization

$$(4) \qquad a_0 = e^{-t}, \quad t > 0.$$

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The Krzyż conjecture [1] asserts that for $f(z) = e^{-t} + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{B}_0$:

$$\max_{f \in \mathcal{B}_0} |a_n| = \frac{2}{e} = 0.73575\ldots, \quad n = 1, 2, \ldots, \tag{5}$$

with the equality (up to the rotation) for the function $F_n(z) = F(z^n)$, $n = 1, 2, \ldots$, where

$$F(z) = \exp\left(\frac{z+1}{z-1}\right) = \frac{1}{e} - \frac{2}{e} z + \cdots, \quad z \in \mathbb{D}. \tag{6}$$

The above conjecture has been proved for $n = 1, 2, 3, 4$ only, and in general, it is known that for every n, $|a_n| < 0.99918\ldots$.

The following lemmas will play a crucial role in our considerations [2].

Lemma 1. *A function $f(z) = e^{-t} + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{B}_0$ if and only if it has the representation*

$$f(z) = \exp\{-tp(z)\}, \quad z \in \mathbb{D}, \ t > 0, \tag{7}$$

where $p \in \mathcal{P}$.

For the coefficients a_n of a function $f \in \mathcal{B}_0$ we have from (3) and (7) the following

Lemma 2. *If $f(z) = e^{-t} + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{B}_0$, then*

$$a_n = (-t) \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) a_j p_{n-j}, \quad a_0 = e^{-t}. \tag{8}$$

The function $p_0(z) = \frac{1+z^n}{1-z^n} \in \mathcal{P}$ which corresponds to the extremal case in the Krzyż conjecture has the property: $p_1 = p_2 = \ldots = p_{n-1} = 0$.

Because of this fact it is natural to consider the subcalsses of $\mathcal{B}_0$ "generated" via formula (7) by such functions $p \in \mathcal{P}$ which are in some sense "close" to the function $p_0(z)$.

For a given integer $n = 3, 4, \ldots$ and $k \in \{1, 2, \ldots, n-1\}$, let $\mathcal{B}_0(n, k) \subset \mathcal{B}_0$ denote the class of functions generated via (7) by the class $\mathcal{P}(n, k) \subset \mathcal{P}$ consisting of functions $p(z) = 1 + p_k z + p_n z^n + \cdots$ with the property: $p_1 = p_2 = \ldots = p_{k-1} = p_{k+1} = \ldots p_{n-1} = 0$.

In [2] we have proved that $|a_n| \leq 2/e$ in the class $\mathcal{B}_0(n, 1)$. Because of the formula (8) we easily see that if $p_1 = p_2 = \ldots = p_k = 0$ and $k \geq \left[\frac{n}{2}\right]$, then $|a_n| = te^{-t}|p_n| \leq 2/e$, and therefore an interesting question arises what would happen if one of the coefficients $p_k \neq 0$ for $k \leq \left[\frac{n}{2}\right]$.

2. We have

Theorem 1. *Let* $n = 3, 4, \dots$ *be a fixed integer and* $s = 2, 3, 4, \dots$ *. For* $f \in \mathcal{B}_0(n, k)$ *we have:*

$$a_n = (-ta_0)p_n \quad \text{if} \quad n = ks + l,\ l = 1, \dots, k-1 \tag{9}$$

and

$$a_n = (-ta_0)\left[p_n + \frac{(-t)^{s-1}}{s!}p_k^s\right] \quad \text{if} \quad n = ks\ . \tag{10}$$

Corollary. *If* $f \in \mathcal{B}(n, k)$ *and* k *is not a divisor of* n *then*

$$|a_n| \le 2te^{-t} \le 2/e.$$

Therefore we have to consider only such $k \in \{1, 2, \dots, n-1\}$ that $n = ks, s = 2, 3, \dots$. Of course, the above remark implies that it is enough to consider only $k \le \left[\frac{n}{2}\right]$.

Proof of Theorem 1. From (8) we obtain that for $f \in \mathcal{B}_0(n, k)$:

$$a_n = (-t)\left[a_0 p_n + \frac{k}{n}a_{n-k}p_k\right].$$

Again by (8) we get

$$a_{n-k} = (-t)\left[a_0 p_{n-k} + \left(1 - \frac{1}{n-k}\right)a_1 p_{n-k-1} + \dots + \left(1 - \frac{n-k-1}{n-k}\right)a_{n-k-1}p_1\right].$$

If $k > n - k$, i.e. $k > n/2$, then $a_{n-k} = 0$ and $a_n = -ta_0 p_n$.
If $k = n/2$ (therefore n is even), then $a_{n-k} = a_{n/2} = -ta_0 p_{n/2}$ and

$$a_n = (-ta_0)\left[p_n - \frac{1}{2}tp_{n/2}\right].$$

If $1 \le k < n - k$, then there exists an integer s, such that $p_{n-k-s} = p_k$. Therefore $s = n - 2k$ and we have

$$a_{n-k} = (-t)\left[\frac{k}{n-k}a_{n-2k}p_k\right]$$

which implies

$$a_n = (-t)\left[a_0 p_n + (-t)\frac{k}{n-k}\cdot\frac{k}{n}a_{n-2k}p_k^2\right].$$

Continuing in this way, we conclude that if $k < n/s, s = 2,3,\dots$, then

$$a_n = (-t)\left[a_0 p_n + \frac{(-t)^{s-1}k^s}{n(n-k)\dots[n-(s-1)k]}a_{n-sk}p_k^s\right], \tag{11}$$

and if $k \neq n/s$, then

$$a_n = (-ta_0)p_n.$$

Using again (8) and (11) we obtain (10).

In order to estimate $|a_n|$ for $f \in \mathcal{B}_0(n,k)$ we will apply the special form of Carathéodory inequalities [2] for the class $\mathcal{P}(n,k)$. These inequalities are equivalent to the nonnegativity of all principal minors of the following determinant:

$$\Delta_n'' = \begin{vmatrix} 2 & 0 & \dots & p_k & \dots & \dots & 0 \\ 0 & 2 & \ddots & 0 & p_k & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & p_k \\ \overline{p_k} & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \overline{p_k} & \dots & \dots & \ddots & 2 \end{vmatrix}_{n\times n} \tag{12}$$

and nonnegativity of the determinant

$$\Delta_{n+1}'' = \begin{vmatrix} 2 & 0 & \dots & p_k & \dots & \dots & p_n \\ 0 & 2 & \ddots & 0 & p_k & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \overline{p_k} & \ddots & \ddots & \ddots & \ddots & \ddots & p_k \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \overline{p_n} & \dots & 0 & \overline{p_k} & \dots & \ddots & 2 \end{vmatrix}_{(n+1)\times(n+1)} \tag{13}$$

Therefore, we will need explicit formulas for some special determinants.

Lemma 3. *Let r_1 and r_2 denote the roots of the equation: $r^2 - cr + ab = 0$, where a, b, c are arbitrary complex numbers. For given integers $n = 3, 4, \ldots$ and $k = 1, 2, \ldots, n-1$ we consider the following $n \times n$ determinants:*

$$(14)\qquad W_n^{k-1} = W_n^{k-1}(a,b,c) = \begin{vmatrix} c & 0 & \ldots & a & 0 & \ldots & 0 \\ 0 & c & \ddots & 0 & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ b & \ddots & \ddots & \ddots & \ddots & \ddots & a \\ 0 & b & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & b & \ldots & 0 & c \end{vmatrix}_{n\times n}$$

(15)

$$V_n^{k-1} = V_n^{k-1}(a,b,c) = \begin{vmatrix} 0 & 0 & \ldots & 0 & a & 0 & \ldots & \ldots & 0 \\ c & 0 & \ddots & \ddots & \ddots & a & \ddots & \ddots & 0 \\ 0 & c & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a \\ b & 0 & \ddots & \ddots & c & \ddots & \ddots & \ddots & 0 \\ 0 & b & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & b & 0 & \ldots & 0 & c & 0 \end{vmatrix}_{n\times n}$$

(The determinant W_n^{k-1} has only three non-zero diagonals consisting of the same elements: c's on the main diagonal and a's and b's which start at $(k+1)$-th column and $(k+1)$-th row respectively.

The determinant V_n^{k-1} has only three non-zero diagonals consisting of the same elements: c's which starts at the second row, a's which starts at the k-th column, and b's which starts at the $(k+2)$-th row).

We have the following formulae ($n = ks + l$, $s = 0, 1, 2, \ldots$, $l = 0, 1, \ldots, k-1$):

$$(16)\qquad W_n^{k-1} = \left(\frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1}\right)^{k-l} \left(\frac{r_2^{s+2} - r_1^{s+2}}{r_2 - r_1}\right)^{l}$$

$$(17)\qquad \begin{aligned} V_n^{k-1} &= (-1)^{(n+1)s} a^s \left(\frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1}\right)^{k-1} && \text{if } l = 0 \\ V_n^{k-1} &= 0 && \text{if } l = 1, 2, \ldots, k-1. \end{aligned}$$

Proof. Two cases $n = ks$ and $n = ks + l$, $l = 1, \dots, k-1$ have to be considered separately. After some elementary manipulations with rows and columns in a similar way as it was done in [3] we obtain that each of the determinants W_n^{k-1} and V_n^{k-1} is the product of some blocks which have the form W_s^0 or W_{s+1}^0. But it is well known that $W_s^0(a,b,c) = \frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1}$ [2], which ends the proof.

Now we can prove our main result.

Theorem 2. *Let $n = 3, 4, \dots$ be a given integer and assume that $f(z) = e^{-t} + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{B}_0(n,k)$. We have the following sharp bounds:*

(a) *If $|p_k| < 1$ and $\kappa = \sqrt{1 - |p_k|^2}$, then we have:*

$$|a_n| \le te^{-t} \cdot \begin{cases} \frac{(1+\kappa)^{s+1} - (1-\kappa)^{s+1} + 2\kappa(1-\kappa^2)^{s/2}}{(1+\kappa)^s - (1-\kappa)^s} - (1-\kappa^2)^{s/2} \frac{t^{s-1}}{s!} & \text{for} \quad t < t_1(\kappa) \\ \frac{(1+\kappa)^{s+1} - (1-\kappa)^{s+1}}{(1+\kappa)^s - (1-\kappa)^s} & \text{for} \quad t = t_1(\kappa) \\ \frac{(1+\kappa)^{s+1} - (1-\kappa)^{s+1} - 2\kappa(1-\kappa^2)^{s/2}}{(1+\kappa)^s - (1-\kappa)^s} + (1-\kappa^2)^{s/2} \frac{t^{s-1}}{s!} & \text{for} \quad t > t_1(\kappa), \end{cases} \tag{18}$$

where

$$t_1(\kappa) = \left[\frac{2\kappa s!}{(1+\kappa)^s - (1-\kappa)^s} \right]^{1/(s-1)}, \quad \kappa \in [0,1]. \tag{19}$$

(b) *If $|p_k| = 1$, then we have*

$$|a_s| \le te^{-t} \cdot \begin{cases} 1 + \frac{2}{s} - \frac{t^{s-1}}{s!} & \text{for } t < \sqrt[s-1]{(s-1)!} \\ 1 + \frac{1}{s} & \text{for } t = \sqrt[s-1]{(s-1)!} \\ 1 + \frac{t^{s-1}}{s!} & \text{for } t > \sqrt[s-1]{(s-1)!}. \end{cases} \tag{20}$$

(c) *If $1 < |p_k| \le \left[\cos \frac{\pi}{s+1}\right]^{-1}$ and $\cos\varphi = \frac{1}{|p_k|}$, $\sin\varphi = \frac{1}{|p_k|}\sqrt{|p_k|^2 - 1}$, then we have*

$$|a_n| \le te^{-t} \cdot \begin{cases} \frac{\sin(s+1)\varphi + \sin\varphi}{\cos\varphi \sin s\varphi} - \frac{1}{\cos^s \varphi} \cdot \frac{t^{s-1}}{s!} & \text{for } t < t_2(\varphi) \\ \frac{\sin(s+1)\varphi}{\cos\varphi \sin s\varphi} & \text{for } t = t_2(\varphi) \\ \frac{\sin(s+1)\varphi - \sin\varphi}{\cos\varphi \sin s\varphi} + \frac{1}{\cos^s\varphi} \cdot \frac{t^{s-1}}{s!} & \text{for } t > t_2(\varphi), \end{cases} \tag{21}$$

where

$$(22)\qquad t_2(\varphi) = \left[s!\frac{\cos^{s-1}\varphi\sin\varphi}{\sin s\varphi}\right]^{1/(s-1)}, \; 0 \le \varphi \le \frac{\pi}{s+1}.$$

All above estimates are sharp.

Proof. If $f \in \mathcal{B}_0(n,k)$, then $f(z) = \exp\{-tp(z)\}$ where $p \in \mathcal{P}(n,k)$ and by (10) we have

$$a_n = (-ta_0)\left[p_n + \frac{(-t)^{s-1}}{s!}p_k^s\right], \; n = ks.$$

By (12) and (13) the Carathéodory inequalities have the form:

$$\begin{aligned}
&\Delta_1'' = 2 > 0, \quad \Delta_2'' = 2^2, \dots, \quad \Delta_k'' = 2^k > 0,\\
&\Delta_q'' = W_q^{k-1}(p_k, \bar{p}_k, 2) \ge 0, \quad q = k+1, k+2, \dots, \quad n = ks,\\
&\Delta_{n+1}'' \ge 0.
\end{aligned}$$

By the formula (16) we have

$$(23)\qquad \Delta_q'' = \left(\frac{r_2^{m+1} - r_1^{m+1}}{r_2 - r_1}\right)^{k-l}\left(\frac{r_2^{m+2} - r_1^{m+2}}{r_2 - r_1}\right)^{l},$$

$q = km + l$, $m = 1, 2, \dots, s-l$, $l = 0, 1, \dots, k-1$ where r_1 and r_2 are the roots of the equation $r^2 - 2r + |p_k|^2 = 0$.
If $|p_k| < 1$, then $r_2 = 1 + \sqrt{1 - |p_k|^2}$, $r_1 = 1 - \sqrt{1 - |p_k|^2}$ and $\Delta_q'' > 0$, $q = k+1, \dots, n$.
If $|p_k| = 1$, then $r_2 = r_1 = 1$ and $\Delta_q'' = (m+1)^{k-l}(m+2)^l > 0$, $m = 0, \dots, s$, $l = 0, 1, \dots, k-1$.
If $1 < |p_k| \le 2$, then denoting by $\cos\varphi = \frac{1}{|p_k|}$, $\sin\varphi = \frac{1}{|p_k|}\sqrt{|p_k|^2 - 1}$ we have $r_2 = |p_k|e^{i\varphi}$, $r_1 = |p_k|e^{-i\varphi}$. In this case we have

$$(24)\qquad \Delta_q'' = |p_k|^{m(k-l)}\left\{\frac{\sin(m+1)\varphi}{\sin\varphi}\right\}^{k-l}|p_k|^{l(m+1)}\left\{\frac{\sin(m+2)\varphi}{\sin\varphi}\right\}^{l},$$
$$m = 0, 1, \dots, s-1.$$

Therefore all Δ_q'' are nonnegative if $\sin(s+1)\varphi \ge 0$ which holds if $0 \le \varphi \le \frac{\pi}{s+1}$ which is equivalent to the inequality $1 < |p_k| \le \frac{1}{\cos\frac{\pi}{s+1}}$, $s = 2, \dots,$ which further has to be assumed.

The inequality $\Delta''_{n+1} \geq 0$ gives the precise region of variability of p_n in terms of p_k and we have [see [2], formula (20) and (21)]

$$\Delta''_{n+1} \geq 0 \Leftrightarrow |p_n - w''_n| \leq R''_n \tag{25}$$

where

$$w''_n = (-1)^n \frac{C''_n}{\Delta''_{n-1}}, \quad R''_n = \frac{\Delta''_n}{\Delta''_{n-1}}, \quad n = ks, \quad s = 2, 3, \ldots . \tag{26}$$

By Lemma 3 and formula (16) and (17) we obtain:

$$\Delta''_n = W_{ks}^{k-1}(p_k, \bar{p}_k, 2) = \left(\frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1} \right)^k$$

$$\begin{cases} = \left[\dfrac{\left(1+\sqrt{1-|p_k|^2}\right)^{s+1} - \left(1-\sqrt{1-|p_k|^2}\right)^{s+1}}{2\sqrt{1-|p_k|^2}} \right]^k & \text{if } |p_k| < 1 \\ (s+1)^k & \text{if } |p_k| = 1 \\ |p_k|^{ks} \left(\dfrac{\sin(s+1)\varphi}{\sin\varphi} \right)^k & \text{if } 1 < |p_k| \end{cases} \tag{27}$$

$$\Delta''_{n-1} = W_{ks-1}^{k-1}(p_k, \bar{p}_k, 2) = \left(\frac{r_2^s - r_1^s}{r_2 - r_1} \right) \left(\frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1} \right)^{k-1}$$

$$\begin{cases} \dfrac{\left[\left(1+\sqrt{1-|p_k|^2}\right)^s - \left(1-\sqrt{1-|p_k|^2}\right)^s\right]\left[\left(1+\sqrt{1-|p_k|^2}\right)^{s+1} - \left(1-\sqrt{1-|p_k|^2}\right)^{s+1}\right]^{k-1}}{\left(2\sqrt{1-|p_k|}\right)^k} & \text{if } \quad |p_k| < 1 \\ s(s+1)^{k-1} & \text{if } \quad |p_k| = 1 \\ |p_k| \dfrac{\sin s\varphi}{\sin\varphi} \cdot \left(\dfrac{\sin(s+1)\varphi}{\sin\varphi} \right)^{k-1} & \text{if } \quad 1 < |p_k| \end{cases} \tag{28}$$

$$C''_n = V_{ks}^{k-1}(p_k, \bar{p}_k, 2) = (-1)^{(n+1)s} p_k^s \left(\frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1} \right)^{k-1} \tag{29}$$

where r_1 and r_2 are as above depending on whether $|p_k| < 1$ or $|p_k| = 1$ or $|p_k| > 1$.

Now we can write by (10) and (25) ($n = ks,\ k = 1, 2, \ldots, \left[\frac{n}{2}\right]$)

$$|a_n| = te^{-t}\left|p_n + \frac{(-t)^{s-1}}{s!}p_k^s\right| \le te^{-t}\left|w_n'' + R_n''e^{i\psi} + \frac{(-t)^{s-1}}{s!}p_k^s\right|$$

$$= te^{-t}\left|(-1)^n(-1)^{(n+1)s}p_k^s\frac{\Delta_n''}{\Delta_{n-1}''}\cdot\frac{r_2-r_1}{r_2^{s+1}-r_1^{s+1}}\right.$$

$$\left.+\frac{\Delta_n''}{\Delta_{n-1}''}e^{i\psi} + \frac{(-t)^{s-1}}{s!}p_k^s\right|_{p_k=|p_k|e^{i\theta}}$$

$$= te^{-t}\left|\frac{\Delta_n''}{\Delta_{n-1}''}e^{i\psi} + (-1)^s e^{i\theta s}|p_k|^s\left[(-1)^{ks+ks^2}\frac{r_2-r_1}{r_2^{s+1}-r_1^{s+1}} - \frac{t^{s-1}}{s!}\right]\right|$$

$$\le te^{-t}\left[\frac{\Delta_n''}{\Delta_{n-1}''} + |p_k|^s\left|\frac{r_2-r_1}{r_2^{s+1}-r_1^{s+1}} - \frac{t^{s-1}}{s!}\right|\right]$$

where ψ and θ can be chosen in such a manner that the sign of equality is always possible.

Applying formulas (27) and (28) we conclude the proof.

Remark 1. We may observe that the right hand side of the inequalities (18), (20) and (21) is exactly the same as in the inequalities (23), (25) and (26) in [2], where for n we have to substitute s and in the place of p_1 we have to put p_k.

Therefore the rest of the conclusions is exactly the same as in [2] and we omit them. The special case $p = 2$ (not covered by the results in [2]) follows from the same formulas.

Finally we have

Corollary. *Let $n = 3, 4, \ldots$ be a given fixed integer and assume that $f(z) = e^{-t} + \sum_{m=1}^{\infty} a_m z^m \in \mathcal{B}_0(n,k)$. Then we have $|a_n| \le 2/e$.*

Remark 2. One should emphasize the importance of inequality (25) which can be formulated as follows.

If $p(z) = 1 + p_k z^k + p_n z^n + \ldots \in \mathcal{P}(n,k)$, then the region of variability of $\{p_n\}$ is the closed disk

$$\left|p_n - (-1)^s p_k^s \frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1}\right| \le \frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1} \quad \text{if} \quad n = ks$$

and

$$|p_n| \le \frac{r_2^{s+1} - r_1^{s+1}}{r_2 - r_1} \text{ if } n = ks + l,\ l = 1, \dots, k-1 \tag{30}$$

where r_1 and r_2 are the roots of equation $r^2 - 2r + |p_k|^2 = 0$.

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