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Multipliers of Cauchy Integrals of Logarithmic Potentials II*

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. Let $\Gamma = \{z : |z| = 1, z \in \mathbb{C}\}$ and $\Delta = \{z : |z| < 1, z \in \mathbb{C}\}$. For each function $f : \Gamma \to \mathbb{C}$ and for each real numbers t and s define

$$D(f;t,s) = f(e^{i(t-s)}) - 2f(e^{it}) + f(e^{i(t-s)}).$$

We prove that if $f \in H^\infty$ and $I(t) = \int_{-\pi}^{\pi} \frac{|D(f;t,s)|}{s^2} \left[\log \frac{2\pi}{|s|}\right] ds$ is integrable on $[-\pi, \pi]$, then f is a multiplier of the class of analytic Cauchy integrals of logarithmic potentials on Δ.

1. Introduction.

Let $\Delta = \{z : |z < 1, z \in \mathbb{C}\}$ and let $\mathbb{T} = \{z : |z| = 1, z \in \mathbb{C}\}$. Let $\mathcal{M}$ denote the set of complex-valued Borel measures on $\mathbb{T}$. Let $\mathcal{F}_0$ denote the family of functions f having the property that there exists a measure μ on $\mathcal{M}$ such that

$$f(z) = f(0) + \int_{\mathbb{T}} \log\left(\frac{1}{1 - \overline{x}z}\right) d\mu(x) \tag{1}$$

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for $|z| < 1$. In (1) and throughout this paper each logarithm means the principal branch. $\mathcal{F}_0$ is a Banach space with respect to the norm defined by $\|f\|_{\mathcal{F}_0} = \inf\{\|\mu\|\} + |f(0)|$ where μ varies over all measures in $\mathcal{M}$ for which (1) holds. A function f is called a multiplier of $\mathcal{F}_0$ if $fg \in \mathcal{F}_0$ for every g in $\mathcal{F}_0$.

Let M_o denote the set of multipliers of $\mathcal{F}_0$. In [2] it was proved that if $f' \in H^p$ for some $p > 1$ then $f \in M_o$ while $f' \in H^1$ is not sufficient for $f \in M_o$. In [3] it was proved that $\int_0^1 \log\frac{1}{1-r}\int_{-\pi}^{\pi}|f''(re^{i\theta})|d\theta dr < +\infty$, for f analytic on Δ, implies $f \in M_o$. Finally in [4] an example was constructed of a function $f \in M_o$ which is not continuous in $\overline{\Delta}$. In this paper we generalize the theorem from [3] mentioned above. For each function $f : \mathbb{T} \to \mathbb{C}$ and for each real numbers t and s define $D(f;t,s) = f(e^{i(t+s)}) - 2f(e^{it}) + f(e^{i(t-s)})$.

The following theorem is the main result of this paper.

Theorem 1. *Suppose $f \in H^\infty$ and $I(t) = \int_{-\pi}^{\pi}\frac{|D(f;t,s)|}{s^2}\left[\log\frac{2\pi}{|s|}\right]ds$. If $\int_{-\pi}^{\pi} I(t)dt < +\infty$, then $f \in M_o$.*

2. Preliminary lemmas.

Lemma 1. *Suppose $0 < t \le \pi$ and $x \ge 2$. Then there exists a constant C_1 such that*

$$\log\frac{\pi x}{t} \le C_1\frac{\log x}{t}. \tag{2}$$

Proof. Note that for $0 < t \le \pi$ and $x \ge 2$ we have

$$\frac{t\log\frac{\pi x}{t}}{\log x} = \frac{t\log\pi}{\log x} + t - \frac{t\log t}{\log x} \le \frac{\pi\log\pi}{\log 2} + \pi + \frac{|t\log t|}{\log 2}. \tag{3}$$

It is easily verified that $|t\log t| \le \pi\log\pi$ on $0 < t \le \pi$ and so we infer from (3) that (2) holds with $C_1 = (2\pi\log\pi)/(\log 2) + \pi$.

Lemma 2. *Let*

$$I(\beta,\gamma,t) = \int_0^1 \frac{(1-r)^\beta \log\frac{1}{1-r}}{|1-re^{it}|^{\gamma+1}}dr$$

and suppose $\beta > -1$ and $\gamma \ge \beta + 1$. Then there exists a constant C_2 such that

$$I(\beta,\gamma,t) = I(\beta,\gamma) \le C_2\frac{[\log 2\pi/|t|]}{|t|^{\gamma-\beta}} \quad \text{for} \quad 0 < |t| \le \pi. \tag{4}$$

Proof. Since $I(\beta,\gamma,t) = I(\beta,\gamma,-t)$ for $0 < |t| \le \pi$ we may assume that $0 < t \le \pi$. Then $|1 - re^{it}|^2 = 1 - 2r\cos t + r^2 = (1-r)^2 + 4r\sin^2\frac{t}{2} \ge (1-r)^2 + 4r^2t^2/\pi^2$ for $0 \le r < 1$. Hence

$$I(\beta,\gamma) \le \int_0^1 \frac{(1-r)^\beta \log\frac{1}{1-r}}{\left[(1-r)^2 + \frac{4r^2t^2}{\pi^2}\right]^{(\gamma+1)/2}} dr \equiv J(\beta,\gamma). \tag{5}$$

The change of variables $x = (2t)/\pi \cdot r/(1-r)$ yields $1/(1-r) = 1 + \pi x/2t$ and $dr = (\pi/2t)(1-r)^2 dx$ and so

$$J(\beta,\gamma) = \frac{\pi}{2t}\int_0^\infty \frac{\left(1+\frac{\pi x}{2t}\right)^\delta \log\left(1+\frac{\pi x}{2t}\right)}{(1+x^2)^{\frac{\gamma+1}{2}}} dx \tag{6}$$

where $\delta = \gamma - \beta - 1 > 0$.
For $\gamma \ge 1$ we have $1 + \gamma \le 2\gamma$ and so $1 + (\pi/2)(x/t) \le 1 + \pi/t \le 2\pi/t$ for $0 \le x \le 2$. Likewise for $x \ge 2$ we have $1 + (\pi/2)(x/t) \le \pi x/t$. Hence

$$\begin{aligned} J(\beta,\gamma) &\le \frac{\pi}{2t}\int_0^2 \left(\frac{2\pi}{t}\right)^\delta \frac{\log\frac{2\pi}{t}}{(1+x^2)^{\frac{\gamma+1}{2}}} dx \\ &\quad + \frac{\pi}{2t}\int_2^\infty \left(\frac{\pi x}{t}\right)^\delta \frac{\log\frac{\pi x}{t}}{(1+x^2)^{\frac{\gamma+1}{2}}} dx \\ &= \frac{\pi(2\pi)^\delta}{2t^{\delta+1}} \log\left(\frac{2\pi}{t}\right) \int_0^2 \frac{1}{(1+x^2)^{\frac{\gamma+1}{2}}} dx \\ &\quad + \frac{\pi^{\delta+1}}{2t^\delta}\int_2^\infty \frac{x^\delta \log\frac{\pi x}{t}}{(1+x^2)^{\frac{\gamma+1}{2}}} dx. \end{aligned} \tag{7}$$

Using (4) we infer from (7) that

$$\begin{aligned} J(\beta,\gamma) &\le \frac{\pi(2\pi)^\delta}{2t^{\delta+1}} \log\left(\frac{2\pi}{t}\right) \int_0^2 \frac{1}{(1+x^2)^{\frac{\gamma+1}{2}}} dx \\ &\quad + \frac{\pi^{\delta+1}C_1}{2t^{\delta+1}}\int_2^\infty \frac{x^\delta \log x}{(1+x^2)^{\frac{\gamma+1}{2}}} dx. \end{aligned} \tag{8}$$

Now $1 \le 2\log[2\pi/t]$ for $0 < t \le \pi$ and so (8) yields

$$\begin{aligned} J(\beta,\gamma) &\le \frac{\pi(2\pi)^\delta}{2t^{\delta+1}} \log\left(\frac{2\pi}{t}\right) \int_0^2 \frac{1}{(1+x^2)^{\frac{\gamma+1}{2}}} dx \\ &\quad + \frac{\pi^{\delta+1}}{t^{\delta+1}} C_1 \log\left(\frac{2\pi}{t}\right)\int_2^\infty \frac{x^\delta \log x}{(1+x^2)^{\frac{\gamma+1}{2}}} dx. \end{aligned} \tag{9}$$

Since $\gamma+1-\delta=\beta+2>1$ the last integral in (9) is finite.

Note also that $\delta+1=\gamma-\beta$. Define

$$C_2=\frac{\pi(2\pi)^{\delta}}{2}\int_0^2\frac{1}{(1+x^2)^{\frac{\gamma+1}{2}}}dx+\pi^{\delta+1}C_1\int_2^{\infty}\frac{x^{\delta}\log x}{(1+x^2)^{\frac{\gamma+1}{2}}}dx.$$

Now (5) and (9) imply (4) for $0<t\le\pi$ which gives (4) for $0<|t|\le\pi$.

Lemma 3. *If $f\in H^{\infty}$ then there exists a constant C_3 such that*

$$\begin{aligned}(10)\qquad &\int_{-\pi}^{\pi}\left(\int_0^1\log\frac{1}{1-r}|f'(re^{it})|dr\right)dt\\ &\le C_3\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{|\log\frac{2\pi}{|s|}|}{|s|^2}|D(f;t,s)|ds\right)dt.\end{aligned}$$

Proof. It was shown in [5] that if $f\in H^{\infty}$ then

$$(11)\qquad |f'(re^{it})|\le\frac{1}{\pi}\int_0^{\pi}\left\{\frac{(1-r)^2+s^2}{|1-re^{is}|^4}\right\}|D(f;t,s)|ds.$$

We infer from (11) that

$$\begin{aligned}(12)\qquad &\int_0^1\log\frac{1}{1-r}|f'(re^{it})|dr\\ &\le\frac{1}{\pi}\int_0^{\pi}\left[\int_0^1\frac{(1-r)^2\log\frac{1}{1-r}}{|1-re^{is}|^4}dr\right]|D(f;t,s)|ds\\ &+\frac{1}{\pi}\int_0^{\pi}s^2\left[\int_0^1\frac{\log\frac{1}{1-r}}{|1-re^{is}|^4}dr\right]|D(f;t,s)|ds.\end{aligned}$$

Note that (4) and (12) yield constants A_2 and A_3 such that

$$\begin{aligned}(13)\qquad &\int_{-\pi}^{\pi}\left(\int_0^1\log\frac{1}{1-r}|f'(re^{it})|dr\right)dt\\ &\le\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\int_0^{\pi}\left(\int_0^1\frac{(1-r)^2\log\frac{1}{1-r}}{|1-re^{is}|^4}dr\right)|D(f;t,s)|ds\right]dt\end{aligned}$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\int_0^{\pi}s^2\left(\int_0^1\frac{\log\frac{1}{1-r}}{|1-re^{is}|^4}dr\right)|D(f;t,s)|ds\right]dt$$

$$\leq\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\int_0^{\pi}A_2\frac{\log\frac{2\pi}{|s|}}{|s|}|D(f;t,s)|ds\right]dt$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\int_0^{\pi}A_3s^2\frac{\log\frac{2\pi}{|s|}}{|s|^3}|D(f;t,s)|ds\right]dt.$$

Since $1/s\leq\pi/s^2$ for $0<s\leq\pi$ and $\frac{\log 2\pi/|s|}{|s|^2}|D(f;t,s)|$ is an even function of s, (13) implies (10) with $C_3=(A_2+A_3)/\pi$.

Lemma 4. *Let $f\in H^\infty$ and set $z=re^{it}$. Then there exists a constant C_4 such that*

$$\int_{-\pi}^{\pi}\left(\int_0^1\log\frac{1}{1-r}|f_{tt}(re^{it})|dr\right)dt \tag{14}$$
$$\leq C_4\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}\frac{\log\frac{2\pi}{|s|}}{|s|^2}|D(f;t,s)|ds\right]dt.$$

Proof. Let $P(r,s)$ denote the Poisson kernel. We have [1, p.77] $f_{tt}(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_{ss}(r,s)D(f;t,s)ds$. Hence

$$|f_{tt}(z)|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_{ss}(r,s)||D(f;t,s)|ds \tag{15}$$

where $P_{ss}(r,s)=(1-r^2)\left[\frac{8r^2\sin^2 s}{(1-2r\cos s+r^2)^3}-\frac{2r\cos s}{(1-2r\cos s+r^2)^2}\right]$.

Since $|P_{ss}(r,s)|\leq\frac{16s^2(1-r)}{|1-re^{is}|^6}+\frac{4(1-r)}{|1-re^{is}|^4}$, we have

$$\int_0^1\log\frac{1}{1-r}|P_{ss}(r,s)|dr\leq 16s^2\int_0^1\frac{(1-r)\log\frac{1}{1-r}}{|1-re^{is}|^6}dr \tag{16}$$
$$+4\int_0^1\frac{(1-r)\log\frac{1}{1-r}}{|1-re^{is}|^4}dr.$$

Now (16) and two applications of (4) give constants, say A_4 and A_5, such that

$$\int_0^1\log\frac{1}{1-r}|P_{ss}(r,s)|dr\leq 16s^2A_4\frac{\log\frac{2\pi}{|s|}}{|s|^4}+4A_5\frac{\log\frac{2\pi}{|s|}}{|s|^2}\leq A_6\frac{\log\frac{2\pi}{|s|}}{|s|^2} \tag{17}$$

where $A_6 = 16A_4 + 4A_5$. It follows from (15) and (17) that

$$\begin{aligned}
&\int_0^1 \log\frac{1}{1-r}|f_{tt}(re^{it})|dr \\
(18)\qquad &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^1 \log\frac{1}{1-r}|P_{ss}(r,s)|dr\right)|D(f;t,s)|ds \\
&\le \frac{A_6}{2\pi}\int_{-\pi}^{\pi}\frac{\log\frac{2\pi}{|s|}}{|s|^2}|D(f;t,s)|ds.
\end{aligned}$$

If we let $C_4 = A_6/2\pi$ then (18) implies (14).

Proof of Theorem 1. Suppose $f \in H^\infty$ and $z = re^{it}$, then we have $f''(z) = \frac{1}{z^2}\{if_t(z) - f_{tt}(z)\}$. Fix r_0 in (0,1). Then if $r_0 \le r < 1$, since $|f_t(z)| \le |f'(z)|$ we have

$$(19)\qquad |f''(z)| \le \frac{1}{r_0^2}\{|f_t(z)| + |f_{tt}(z)|\} \le \frac{1}{r_0^2}\{|f'(z)| + |f_{tt}(z)|\}.$$

It follows from (10), (14) and (19) that

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left(\int_{r_0}^1 \log\frac{1}{1-r}|f''(re^{it})|dr\right)dt \\
&\le \frac{1}{r_0^2}\int_{-\pi}^{\pi}\left(\int_{r_0}^1 \log\frac{1}{1-r}|f'(re^{it})|dr\right)dt \\
&\quad + \frac{1}{r_0^2}\int_{-\pi}^{\pi}\left(\int_{r_0}^1 \log\frac{1}{1-r}|f_{tt}(re^{it})|dr\right)dt \\
(20)\qquad &\le \frac{C_3}{r_0^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{\log\frac{2\pi}{|s|}}{|s|^2}|D(f;t,s)|ds\right)dt \\
&\quad + \frac{C_4}{r_0^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{\log\frac{2\pi}{|s|}}{|s|^2}|D(f;t,s)|ds\right)dt \\
&= \frac{C_3 + C_4}{r_0^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{\log\frac{2\pi}{|s|}}{|s|^2}|D(f;t,s)|ds\right)dt.
\end{aligned}$$

Recalling that $I(t) = \int_{-\pi}^{\pi}\frac{|D(f;t,s)|}{s^2}\left[\log\frac{2\pi}{|s|}\right]ds$ we see that (20) and our assumption that $I(t)$ is integrable implies that

$$(21)\qquad \int_{-\pi}^{\pi}\left(\int_0^1 \log\frac{1}{1-r}|f''(re^{it})|dr\right)dt < +\infty.$$

It follows from Theorem 1 in [3] that $f \in M_0$.

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