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**Some new inequalities  
of Hermite–Hadamard type  
for  $GA$ -convex functions**

ABSTRACT. Some new inequalities of Hermite–Hadamard type for  $GA$ -convex functions defined on positive intervals are given. Refinements and weighted version of known inequalities are provided. Some applications for special means are also obtained.

**1. Introduction.** Let  $J \subset (0, \infty)$  be an interval; a real-valued function  $f : J \rightarrow \mathbb{R}$  is said to be  $GA$ -convex (concave) on  $J$  (see for instance [2]), if

$$(1.1) \quad f\left(x^{1-\lambda}y^\lambda\right) \leq (\geq) (1-\lambda)f(x) + \lambda f(y)$$

for all  $x, y \in J$  and  $\lambda \in [0, 1]$ .

Since the condition (1.1) can be written as

$$(1.2) \quad \begin{aligned} & f \circ \exp((1-\lambda)\ln x + \lambda\ln y) \\ & \leq (\geq) (1-\lambda)f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y), \end{aligned}$$

then we observe that  $f : J \rightarrow \mathbb{R}$  is  $GA$ -convex (concave) on  $J$  if and only if  $f \circ \exp$  is convex (concave) on  $\ln J := \{\ln z, z \in J\}$ . If  $J = [a, b]$ , then  $\ln J = [\ln a, \ln b]$ .

It is known that the function  $f(x) = \ln(1+x)$  is  $GA$ -convex on  $(0, \infty)$  [2].

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For real and positive values of  $x$ , the *Euler gamma* function  $\Gamma$  and its *logarithmic derivative*  $\psi$ , the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [28] that the function  $f : (0, \infty) \rightarrow \mathbb{R}$  defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is *GA-concave* on  $(0, \infty)$  while the function  $g : (0, \infty) \rightarrow \mathbb{R}$  defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is *GA-convex* on  $(0, \infty)$ .

**Example 1.** If  $[a, b] \subset (0, \infty)$  and the function  $g : [\ln a, \ln b] \rightarrow \mathbb{R}$  is convex (concave) on  $[\ln a, \ln b]$ , then the function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f(t) = g(\ln t)$  is *GA-convex* (*concave*) on  $[a, b]$ .

We recall that the classical Hermite–Hadamard inequality states that

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}$$

for any convex function  $f : [a, b] \rightarrow \mathbb{R}$ .

For related results, see [1]–[6], [8]–[11], [12]–[18] and [20]–[27].

The *identric mean*  $I(a, b)$  is defined by

$$I(a, b) := \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a, b) := \frac{b-a}{\ln b - \ln a}$$

where  $0 < a < b$ .

In [28] the authors obtained the following Hermite–Hadamard type inequality.

**Theorem 1.** *If  $b > a > 0$  and  $f : [a, b] \rightarrow \mathbb{R}$  is a differentiable *GA-convex* (*concave*) function on  $[a, b]$ , then*

$$(1.4) \quad \begin{aligned} f(I(a, b)) &\leq (\geq) \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq (\geq) \frac{b-L(a, b)}{b-a} f(b) + \frac{L(a, b)-a}{b-a} f(a). \end{aligned}$$

The differentiability of the function is not necessary in Theorem 1 for the first inequality (1.4) to hold, as shown in [10].

If we take  $\lambda = \frac{1}{2}$  in the definition (1.1) of  $GA$ -convex (concave) function on  $[a, b]$ , then we have

$$(1.5) \quad f(\sqrt{ab}) \leq (\geq) \frac{f(a) + f(b)}{2}.$$

The following refinement of (1.5), which is an inequality of Hermite–Hadamard type, holds (see [25] for an extension for  $GA$   $h$ -convex functions):

**Theorem 2.** *Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a  $GA$ -convex (concave) function on  $[a, b]$ . Then we have*

$$(1.6) \quad f(\sqrt{ab}) \leq (\geq) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq (\geq) \frac{f(a) + f(b)}{2}.$$

Motivated by the above results we provide in the following a refinement of (1.6) for a division of the interval  $[a, b]$ . We also establish a weighted version that generalizes the inequalities (1.4) and (1.6) and provides upper and lower bounds for the moments

$$\frac{1}{b-a} \int_a^b t^p f(t) dt, \text{ for } p \in \mathbb{R} \text{ with } p \neq 0, -1$$

of the  $GA$ -convex (concave) function on  $[a, b]$ .

**2. Some refinements.** In 1994, [5] (see also [17, p. 22]) we proved the following refinement of Hermite–Hadamard inequality. For the sake of completeness we give here a direct proof that is different from the one in [5].

**Lemma 1.** *Let  $g : [c, d] \rightarrow \mathbb{R}$  be a convex function on  $[c, d]$ . Then for any division  $c = x_0 < x_1 < \dots < x_{n-1} < x_n = d$  with  $n \geq 1$  we have the inequalities*

$$(2.1) \quad \begin{aligned} g\left(\frac{c+d}{2}\right) &\leq \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) \\ &\leq \frac{1}{d-c} \int_c^d g(x) dx \\ &\leq \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \\ &\leq \frac{1}{2} [g(c) + g(d)]. \end{aligned}$$

**Proof.** Using the Hermite–Hadamard inequality on the interval  $[x_i, x_{i+1}]$ ,  $i \in \{0, \dots, n-1\}$ , we have

$$(2.2) \quad \begin{aligned} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) &\leq \int_{x_i}^{x_{i+1}} g(x) dx \\ &\leq (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \end{aligned}$$

for any  $i \in \{0, \dots, n-1\}$ .

Summing in (2.2) over  $i \in \{0, \dots, n-1\}$  and dividing by  $d - c$ , we get the second and the third inequalities in (2.1).

Since for  $p_i := \frac{x_{i+1} - x_i}{d - c} \geq 0$  we have  $\sum_{i=0}^{n-1} p_i = 1$ , then by Jensen's inequality we have

$$\begin{aligned} \frac{1}{d - c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) &\geq g\left(\frac{1}{d - c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left(\frac{x_{i+1} + x_i}{2}\right)\right) \\ &= g\left(\frac{1}{d - c} \sum_{i=0}^{n-1} \left(\frac{x_{i+1}^2 - x_i^2}{2}\right)\right) \\ &= g\left(\frac{d^2 - c^2}{2(d - c)}\right) = g\left(\frac{c + d}{2}\right), \end{aligned}$$

which proves the first inequality in (2.1).

For a convex function  $g : [c, d] \rightarrow \mathbb{R}$  we have

$$g(x) = g\left(\frac{(d - x)c + (x - c)d}{d - c}\right) \leq \frac{(d - x)g(c) + (x - c)g(d)}{d - c}$$

for any  $x \in [c, d]$ .

This implies that

$$g(x_i) \leq \frac{1}{d - c} [(d - x_i)g(c) + (x_i - c)g(d)]$$

and

$$g(x_{i+1}) \leq \frac{1}{d - c} [(d - x_{i+1})g(c) + (x_{i+1} - c)g(d)]$$

for any  $i \in \{0, \dots, n-1\}$ .

Therefore,

$$\begin{aligned} &\sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \\ &\leq \frac{1}{2(d - c)} \sum_{i=0}^{n-1} (x_{i+1} - x_i) [(d - x_i)g(c) \\ &\quad + (x_i - c)g(d) + (d - x_{i+1})g(c) + (x_{i+1} - c)g(d)] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d-c} \left[ \left( d(d-c) - \frac{1}{2}(d^2 - c^2) \right) g(c) + \left( \frac{1}{2}(d^2 - c^2) - c(d-c) \right) g(d) \right] \\
&= (d-c) \frac{g(c) + g(d)}{2},
\end{aligned}$$

and the last part of (2.1) is proved.  $\square$

**Theorem 3.** Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a GA-convex function on  $[a, b]$ . Then for any division  $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$  with  $n \geq 1$  we have the inequalities

$$\begin{aligned}
(2.3) \quad f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) f(\sqrt{t_i t_{i+1}}) \\
&\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
&\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{f(t_i) + f(t_{i+1})}{2} \\
&\leq \frac{1}{2} [f(a) + f(b)].
\end{aligned}$$

**Proof.** If  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  is a GA-convex function on  $[a, b]$ , then  $f \circ \exp$  is convex on  $[\ln a, \ln b]$ . Let  $c = \ln a$ ,  $d = \ln b$  and the division  $x_i = \ln t_i$ ,  $i \in \{0, \dots, n-1\}$  of the interval  $[c, d]$ .

If we write the inequality (2.1) for  $g = f \circ \exp$  on the interval  $[c, d]$  and for the division

$$\ln a = \ln t_0 < \ln t_1 < \dots < \ln t_{n-1} < \ln t_n = \ln b,$$

we have

$$\begin{aligned}
(2.4) \quad &(f \circ \exp) \left( \frac{\ln a + \ln b}{2} \right) \\
&\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) (f \circ \exp) \left( \frac{\ln t_{i+1} + \ln t_i}{2} \right) \\
&\leq \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} (f \circ \exp)(x) dx \\
&\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{(f \circ \exp)(\ln t_i) + (f \circ \exp)(\ln t_{i+1})}{2} \\
&\leq \frac{1}{2} [(f \circ \exp)(\ln a) + (f \circ \exp)(\ln b)],
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
 f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) f(\sqrt{t_i t_{i+1}}) \\
 (2.5) \quad &\leq \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} (f \circ \exp)(x) dx \\
 &\leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{f(t_i) + f(t_{i+1})}{2} \\
 &\leq \frac{1}{2} [f(a) + f(b)].
 \end{aligned}$$

By using the change of variable  $\exp(x) = t$ , we have  $x = \ln t$ ,  $dx = \frac{dt}{t}$  and

$$\int_{\ln a}^{\ln b} (f \circ \exp)(x) dx = \int_a^b \frac{f(t)}{t} dt$$

and by (2.5) we get the desired result (2.3).  $\square$

**Corollary 1.** *Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a GA-convex function on  $[a, b]$ . If  $a \leq t \leq b$ , then we have*

$$\begin{aligned}
 f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} [(\ln t - \ln a) f(\sqrt{at}) + (\ln b - \ln t) f(\sqrt{tb})] \\
 (2.6) \quad &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\
 &\leq \frac{1}{2} \left[ f(t) + \frac{(\ln t - \ln a) f(a) + (\ln b - \ln t) f(b)}{\ln b - \ln a} \right] \\
 &\leq \frac{1}{2} [f(a) + f(b)].
 \end{aligned}$$

**Remark 1.** Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a GA-convex function on  $[a, b]$ . For a division  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$  with  $n \geq 1$  of the interval  $[0, 1]$ , consider the division

$$(2.7) \quad t_i := a \left( \frac{b}{a} \right)^{\lambda_i} = a^{1-\lambda_i} b^{\lambda_i}, \quad i \in \{0, \dots, n-1\}$$

of the interval  $[a, b]$ .

Observe that

$$\begin{aligned}
 \ln t_{i+1} - \ln t_i &= \ln \left[ a \left( \frac{b}{a} \right)^{\lambda_{i+1}} \right] - \ln \left[ a \left( \frac{b}{a} \right)^{\lambda_i} \right] \\
 &= \ln a + \lambda_{i+1} \ln \left( \frac{b}{a} \right) - \ln a - \lambda_i \ln \left( \frac{b}{a} \right) \\
 &= (\ln b - \ln a) (\lambda_{i+1} - \lambda_i)
 \end{aligned}$$

and

$$\sqrt{t_i t_{i+1}} = \sqrt{a^{1-\lambda_i} b^{\lambda_i} a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}} = a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}$$

for any  $i \in \{0, \dots, n-1\}$ .

If we write the inequality (2.3) for the division (2.7) we get

$$\begin{aligned} f(\sqrt{ab}) &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right) \\ (2.8) \quad &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{f(a^{1-\lambda_i} b^{\lambda_i}) + f(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}})}{2} \\ &\leq \frac{1}{2} [f(a) + f(b)], \end{aligned}$$

for any division  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$  with  $n \geq 1$  of the interval  $[0, 1]$ .

If we write the inequality (2.8) for  $0 < \lambda < 1$ , then we get

$$\begin{aligned} f(\sqrt{ab}) &\leq (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ (2.9) \quad &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{1}{2} \left[ f\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a) \right] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

The inequality (2.9) was obtained in [10].

In the following section we establish some weighted Hermite–Hadamard type inequalities for  $GA$ -convex functions.

**3. Weighted inequalities.** We have the following weighted inequality:

**Theorem 4.** Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a  $GA$ -convex function on  $[a, b]$  and  $w : [a, b] \rightarrow [0, \infty)$  an integrable function on  $[a, b]$  with  $\int_a^b w(t) dt > 0$ , then

$$\begin{aligned} (3.1) \quad f\left(\exp\left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt}\right)\right) &\leq \frac{\int_a^b w(t) f(t) dt}{\int_a^b w(t) dt} \\ &\leq \frac{\left(\ln b - \frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt}\right) f(a) + \left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned}$$

**Proof.** Observe that for  $t \in [a, b]$  we have

$$\ln t = \frac{(\ln b - \ln t) \ln a + (\ln t - \ln a) \ln b}{\ln b - \ln a}.$$

By the convexity of  $f \circ \exp$  we have

$$\begin{aligned} f(t) &= (f \circ \exp)(\ln t) \\ (3.2) \quad &= (f \circ \exp) \left( \frac{(\ln b - \ln t) \ln a + (\ln t - \ln a) \ln b}{\ln b - \ln a} \right) \\ &\leq \frac{\ln b - \ln t}{\ln b - \ln a} f(a) + \frac{\ln t - \ln a}{\ln b - \ln a} f(b) \end{aligned}$$

for any  $t \in [a, b]$ .

If we multiply (3.2) by  $w(t) \geq 0$ ,  $t \in [a, b]$  and integrate over  $t$  on  $[a, b]$ , then we get

$$\begin{aligned} &\int_a^b f(t) w(t) dt \\ &\leq \frac{\ln b \int_a^b w(t) dt - \int_a^b w(t) \ln t dt}{\ln b - \ln a} f(a) + \frac{\int_a^b w(t) \ln t dt - \ln a \int_a^b w(t) dt}{\ln b - \ln a} f(b) \end{aligned}$$

and the second inequality in (3.1) is proved.

By Jensen's inequality we have

$$\frac{\int_a^b w(t) f(t) dt}{\int_a^b w(t) dt} = \frac{\int_a^b w(t) (f \circ \exp)(\ln t) dt}{\int_a^b w(t) dt} \geq (f \circ \exp) \left( \frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt} \right)$$

and the first part of (3.1) is proved.  $\square$

**Corollary 2** (see Theorem 2). *Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a GA-convex function on  $[a, b]$ , then*

$$(3.3) \quad f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq \frac{f(a) + f(b)}{2}.$$

**Proof.** If we take  $w(t) = \frac{1}{t}$  in (3.1), then we have

$$\begin{aligned} (3.4) \quad &f \left( \exp \left( \frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} \right) \right) \leq \frac{\int_a^b \frac{f(t)}{t} dt}{\int_a^b \frac{1}{t} dt} \\ &\leq \frac{\left( \ln b - \frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} \right) f(a) + \left( \frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} - \ln a \right) f(b)}{\ln b - \ln a}. \end{aligned}$$

Since

$$\int_a^b \frac{\ln t}{t} dt = \frac{1}{2} [(\ln b)^2 - (\ln a)^2], \quad \int_a^b \frac{1}{t} dt = \ln b - \ln a,$$



then we get from (3.4)

$$\begin{aligned} f\left(\exp\left(\frac{\ln b + \ln a}{2}\right)\right) &\leq \frac{\int_a^b \frac{f(t)}{t} dt}{\ln b - \ln a} \\ &\leq \frac{(\ln b - \frac{\ln b + \ln a}{2}) f(a) + (\frac{\ln b + \ln a}{2} - \ln a) f(b)}{\ln b - \ln a} \end{aligned}$$

and the inequality (3.3) is proved.  $\square$

Define the *p*-logarithmic mean as

$$L_p(a, b) := \left(\frac{1}{p+1} \frac{b^{p+1} - a^{p+1}}{b - a}\right)^{1/p}, \quad p \neq -1, 0$$

for  $0 < a < b$ .

**Corollary 3.** Let  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$  be a GA-convex function on  $[a, b]$ , then for any  $p \in \mathbb{R}$  with  $p \neq 0, -1$  we have

$$\begin{aligned} (3.5) \quad &L_p^p(a, b) f\left([I(a^{p+1}, b^{p+1})]^{1/(p+1)}\right) \\ &\leq \frac{1}{b-a} \int_a^b t^p f(t) dt \\ &\leq \frac{(L(a^{p+1}, b^{p+1}) - a^{p+1}) f(a) + (b^{p+1} - L(a^{p+1}, b^{p+1})) f(b)}{(p+1)(b-a)}. \end{aligned}$$

If  $p = 0$ , then we have

$$\begin{aligned} (3.6) \quad &f(I(a, b)) \leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{(L(a, b) - a) f(a) + (b - L(a, b)) f(b)}{b-a}. \end{aligned}$$

**Proof.** If we take  $w(t) = t^p$ , with  $p \neq 0, -1$  in (3.1), then we have

$$\begin{aligned} (3.7) \quad &f\left(\exp\left(\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt}\right)\right) \\ &\leq \frac{\int_a^b t^p f(t) dt}{\int_a^b t^p dt} \\ &\leq \frac{\left(\ln b - \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt}\right) f(a) + \left(\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned}$$

We have

$$\int_a^b t^p dt = \frac{1}{p+1} (b^{p+1} - a^{p+1})$$

and

$$\begin{aligned} \int_a^b t^p \ln t dt &= \frac{1}{p+1} \int_a^b \ln t dt^{p+1} \\ &= \frac{1}{p+1} \left[ t^{p+1} \ln t \Big|_a^b - \int_a^b t^p dt \right] \\ &= \frac{1}{p+1} \left( b^{p+1} \ln b - a^{p+1} \ln a - \frac{1}{p+1} (b^{p+1} - a^{p+1}) \right). \end{aligned}$$

Then

$$\begin{aligned} \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} &= \frac{\frac{1}{p+1} \left( b^{p+1} \ln b - a^{p+1} \ln a - \frac{1}{p+1} (b^{p+1} - a^{p+1}) \right)}{\frac{1}{p+1} (b^{p+1} - a^{p+1})} \\ &= \frac{1}{p+1} \ln I(a^{p+1}, b^{p+1}) = \ln [I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}}, \end{aligned}$$

$$\begin{aligned} \ln b - \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} &= \ln b - \frac{1}{p+1} \left[ \frac{b^{p+1} \ln b^{p+1} - a^{p+1} \ln a^{p+1}}{b^{p+1} - a^{p+1}} - 1 \right] \\ &= \frac{1}{p+1} \left( 1 - \frac{\ln b^{p+1} - \ln a^{p+1}}{b^{p+1} - a^{p+1}} a^{p+1} \right) = \frac{1}{p+1} \left( 1 - \frac{a^{p+1}}{L(a^{p+1}, b^{p+1})} \right) \\ &= \frac{1}{p+1} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} \end{aligned}$$

and, similarly,

$$\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} - \ln a = \frac{1}{p+1} \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})}.$$

From (3.7) we then have

$$\begin{aligned} &f \left( \exp \left( \ln [I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}} \right) \right) \\ &\leq \frac{\frac{1}{b-a} \int_a^b t^p f(t) dt}{\frac{1}{b-a} \int_a^b t^p dt} \\ &\leq \frac{\left( \frac{1}{p+1} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} \right) f(a) + \left( \frac{1}{p+1} \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} \right) f(b)}{\ln b - \ln a}, \end{aligned}$$

i.e.

$$\begin{aligned}
 & f \left( [I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}} \right) \\
 & \leq \frac{\frac{1}{b-a} \int_a^b t^p f(t) dt}{L_p^p(a, b)} \\
 & \leq \frac{\left( \frac{1}{p+1} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} \right) f(a) + \left( \frac{1}{p+1} \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} \right) f(b)}{\ln b - \ln a} \\
 & = \frac{\frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} f(a) + \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} f(b)}{\ln b^{p+1} - \ln a^{p+1}}.
 \end{aligned}$$

By multiplying this inequality with  $L_p^p(a, b)$ , we get

$$\begin{aligned}
 & L_p^p(a, b) f \left( [I(a^{p+1}, b^{p+1})]^{\frac{1}{p+1}} \right) \\
 & \leq \frac{1}{b-a} \int_a^b t^p f(t) dt \\
 & \leq \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \frac{L(a^{p+1}, b^{p+1}) - a^{p+1}}{L(a^{p+1}, b^{p+1})} f(a) + \frac{b^{p+1} - L(a^{p+1}, b^{p+1})}{L(a^{p+1}, b^{p+1})} f(b)}{\ln b^{p+1} - \ln a^{p+1}} \\
 & = \frac{(L(a^{p+1}, b^{p+1}) - a^{p+1}) f(a) + (b^{p+1} - L(a^{p+1}, b^{p+1})) f(b)}{(p+1)(b-a)}.
 \end{aligned}$$

If we perform the calculations in the above inequalities for  $p = 0$ , we get the desired inequality (3.6). We omit the details.  $\square$

**Remark 2.** If we take  $p = 1$  in (3.5), then we get

$$\begin{aligned}
 (3.8) \quad & f \left( \sqrt{I(a^2, b^2)} \right) \leq \frac{1}{b-a} \int_a^b t f(t) dt \\
 & \leq \frac{(A(a, b) L(a, b) - a^2) f(a) + (b^2 - A(a, b) L(a, b)) f(b)}{2(b-a)}.
 \end{aligned}$$

**4. Applications.** Let  $q \neq 0$  and consider the convex function  $g(t) = \exp(qt)$ ,  $t \in \mathbb{R}$ . Then the function  $f_q : (0, \infty) \rightarrow \mathbb{R}$ ,  $f_q(t) = g(\ln t) = \exp(q \ln t) = t^q$  is a  $GA$ -convex function on  $(0, \infty)$ . We observe that for  $0 < a < b$  we have

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b t^q dt & = \begin{cases} \frac{1}{q+1} \frac{b^{q+1} - a^{q+1}}{b-a}, & q \neq -1 \\ \frac{\ln b - \ln a}{b-a}, & q = -1 \end{cases} \\
 & = \begin{cases} L_q^q(a, b), & q \neq -1 \\ L^{-1}(a, b), & q = -1, \end{cases}
 \end{aligned}$$

where  $L_q(a, b)$  ( $q \neq -1$ ) is the  $q$ -Logarithmic mean and  $L$  is the logarithmic mean defined in the introduction.

If we write the inequality (2.8) for the  $GA$ -convex function on  $(0, \infty)$ ,  $f_q : (0, \infty) \rightarrow \mathbb{R}$ ,  $f_q(t) = t^q$ ,  $q \neq 0$ , then we get the following result:

**Proposition 1.** For any division  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$  with  $n \geq 1$  of the interval  $[0, 1]$ ,

$$\begin{aligned}
 (4.1) \quad G^q(a, b) &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) G^q(a^{2-\lambda_i-\lambda_{i+1}}, b^{\lambda_i+\lambda_{i+1}}) \\
 &\leq L(a, b) L_{q-1}^{q-1}(a, b) \\
 &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{G^q(a^{2(1-\lambda_i)}, b^{2\lambda_i}) + G^q(a^{2(1-\lambda_{i+1})}, b^{2\lambda_{i+1}})}{2} \\
 &\leq A(a^q, b^q).
 \end{aligned}$$

If we write the inequality (3.5) for the  $GA$ -convex function on  $(0, \infty)$ ,  $f_q : (0, \infty) \rightarrow \mathbb{R}$ ,  $f_q(t) = t^q$ ,  $q \neq 0$ , then:

**Proposition 2.** If  $0 < a < b$  and  $p, q \in \mathbb{R}$  with  $p \neq 0, -1$ ,  $q \neq 0$ , then

$$\begin{aligned}
 (4.2) \quad L_p^p(a, b) [I(a^{p+1}, b^{p+1})]^{\frac{q}{p+1}} \\
 &\leq L_{p+q}^{p+q}(a, b) \\
 &\leq \frac{(L(a^{p+1}, b^{p+1}) - a^{p+1}) a^q + (b^{p+1} - L(a^{p+1}, b^{p+1})) b^q}{(p+1)(b-a)}.
 \end{aligned}$$

From (3.6) and (3.8) we get:

**Proposition 3.** If  $0 < a < b$  and  $q \in \mathbb{R}$  with  $q \neq 0$ , then

$$(4.3) \quad I^q(a, b) \leq L_q^q(a, b) \leq \frac{(L(a, b) - a) a^q + (b - L(a, b)) b^q}{b - a},$$

and

$$\begin{aligned}
 (4.4) \quad I^{q/2}(a^2, b^2) &\leq L_{q+1}^{q+1}(a, b) \\
 &\leq \frac{(A(a, b) L(a, b) - a^2) a^q + (b^2 - A(a, b) L(a, b)) b^q}{2(b-a)}.
 \end{aligned}$$

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