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On almost polynomial structures from classical linear connections

ABSTRACT. Let $\mathcal{M}f_m$ be the category of m -dimensional manifolds and local diffeomorphisms and let T be the tangent functor on $\mathcal{M}f_m$. Let \mathcal{V} be the category of real vector spaces and linear maps and let \mathcal{V}_m be the category of m -dimensional real vector spaces and linear isomorphisms. Let w be a polynomial in one variable with real coefficients. We describe all regular covariant functors $F: \mathcal{V}_m \rightarrow \mathcal{V}$ admitting $\mathcal{M}f_m$ -natural operators \tilde{P} transforming classical linear connections ∇ on m -dimensional manifolds M into almost polynomial w -structures $\tilde{P}(\nabla)$ on $F(T)M = \bigcup_{x \in M} F(T_x M)$.

1. Introduction. All manifolds considered in the paper are assumed to be Hausdorff, finite dimensional, second countable, without boundaries and smooth (i.e. of class C^∞). Maps between manifolds are assumed to be of class C^∞ .

The category of m -dimensional manifolds and local diffeomorphisms is denoted by $\mathcal{M}f_m$. The category of vector bundles and vector bundle homomorphisms between them is denoted by \mathcal{VB} . The category of m -dimensional real vector spaces and linear isomorphisms is denoted by \mathcal{V}_m . The category of finite dimensional real vector spaces and linear maps is denoted by \mathcal{V} .

Let w be a polynomial in one variable. A tensor field P of type $(1, 1)$ on a manifold N is called an almost polynomial w -structure on N if $w(P) = 0$ (i.e. $w(P|_x) = 0$ for any $x \in N$).

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In the present paper we solve the following problem.

Problem 1. *Let w be a polynomial in one variable with real coefficients. We characterize all covariant regular functors $F: \mathcal{V}_m \rightarrow \mathcal{V}$ admitting $\mathcal{M}f_m$ -natural operators \tilde{P} transforming classical linear connections ∇ on m -manifolds M into almost polynomial w -structures $\tilde{P}(\nabla)$ on $F(T)M = \bigcup_{x \in M} F(T_x M)$, where $T: \mathcal{M}f_m \rightarrow \mathcal{VB}$ denotes the tangent functor on the category $\mathcal{M}f_m$.*

If $w(t) = t^2 + 1$, then we reobtain the result from [5] on the characterization of covariant regular functors $F: \mathcal{V}_m \rightarrow \mathcal{V}$ admitting $\mathcal{M}f_m$ -natural operators \tilde{J} transforming classical linear connections ∇ on m -manifolds M into almost complex structures $\tilde{J}(\nabla)$ on $F(T)M$.

If $w(t) = t^2 - 1$, then we characterize covariant regular functors $F: \mathcal{V}_m \rightarrow \mathcal{V}$ admitting $\mathcal{M}f_m$ -natural operators \tilde{J} transforming classical linear connections ∇ on m -manifolds M into almost para-complex structures $\tilde{J}(\nabla)$ on $F(T)M$.

2. Basic definitions. The concept of natural bundles and natural operators can be found in the fundamental monograph [3].

Let $F: \mathcal{V}_m \rightarrow \mathcal{V}$ be a covariant regular functor. The regularity of the functor F means that F transforms smoothly parametrized families of isomorphisms into smoothly parametrized families of linear maps. Let $T: \mathcal{M}f_m \rightarrow \mathcal{VB}$ be the tangent functor sending any m -dimensional manifold M into the tangent bundle TM of M and any $\mathcal{M}f_m$ -map $\varphi: M_1 \rightarrow M_2$ into the tangent map $T\varphi: TM_1 \rightarrow TM_2$. Applying F to fibers $T_x M$ of TM , one can define a natural vector bundle $F(T)$ of order 1 over m -manifolds by

$$F(T)M = \bigcup_{x \in M} F(T_x M) \text{ and } F(T)\varphi = \bigcup_{x \in M} F(T_x \varphi): F(T)M_1 \rightarrow F(T)M_2$$

for any m -manifold M and any $\mathcal{M}f_m$ -map $\varphi: M_1 \rightarrow M_2$ between m -manifolds M_1 and M_2 . In particular, if F is the identity functor, then $F(T) = T$.

A classical linear connection on an m -manifold M is an \mathbb{R} -bilinear map $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ such that:

- (1) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$
- (2) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
- (3) $\nabla_X (fY) = Xf \cdot Y + f \cdot \nabla_X Y$,

where $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$ are any vector fields on M and $f, f_1, f_2: M \rightarrow \mathbb{R}$ are any smooth functions on M . Equivalently, a classical linear connection on M is a right invariant decomposition $TLM = H^\nabla \oplus VLM$ of the tangent bundle TLM of LM , where LM is the principal bundle with the structural group $GL(m)$ of linear frames over M and VLM is the vertical bundle of LM , see [2].

Let $w(t) = t^m + a_{m-1}t^{m-1} + \dots + a_1t + a_0$ be the polynomial in one variable with real coefficients a_{m-1}, \dots, a_0 .

A polynomial w -structure on a real vector space W is a linear endomorphism $P: W \rightarrow W$ such that $w(P) = P^m + a_{m-1}P^{m-1} + \dots + a_1P + a_0I = 0$, where P^k denotes the composition $\underbrace{P \circ \dots \circ P}_{k\text{-times}}$ and I denotes the identity

map on W .

An almost polynomial w -structure on manifold N is a tensor field $\tilde{P}: TN \rightarrow TN$ on N of type $(1, 1)$ (affinor) such that $P_x: T_xN \rightarrow T_xN$ is a polynomial w -structure on T_xN for any $x \in N$. In other words, an almost polynomial w -structure is a tensor field P of type $(1, 1)$ on manifold N satisfying a polynomial equation $P^m + a_{m-1}P^{m-1} + \dots + a_1P + a_0I = 0$, where a_{m-1}, \dots, a_0 are real numbers, at every point of N .

The general concept of natural operators can be found in the fundamental monograph [3]. In particular, we have the following definition.

Definition 1. Let $F: \mathcal{V}_m \rightarrow \mathcal{V}$ be a covariant regular functor. An $\mathcal{M}f_m$ -natural operator transforming classical linear connections ∇ on m -manifolds M into almost polynomial w -structures $\tilde{P}(\nabla): TF(T)M \rightarrow TF(T)M$ on $F(T)M$ is an $\mathcal{M}f_m$ -invariant family $\tilde{P}: Q \rightsquigarrow (AwS)F(T)$ of operators

$$\tilde{P}: Q(M) \rightarrow (AwS)(F(T)M)$$

for m -manifolds M , where $Q(M)$ is the set of classical linear connections on M and $(AwS)(F(T)M)$ is the set of almost polynomial w -structures on $F(T)M$. The invariance of \tilde{P} means that if $\nabla_1 \in Q(M_1)$ and $\nabla_2 \in Q(M_2)$ are φ -related by an embedding $\varphi: M_1 \rightarrow M_2$ (i.e. if φ is (∇, ∇_1) -affine embedding), then $\tilde{P}(\nabla_1)$ and $\tilde{P}(\nabla_2)$ are $F(T)\varphi$ -related (i.e. $TF(T)\varphi \circ \tilde{P}(\nabla_1) = \tilde{P}(\nabla_2) \circ TF(T)\varphi$).

Let $F: \mathcal{V}_m \rightarrow \mathcal{V}$ be as above. A \mathcal{V}_m -canonical polynomial w -structure on $V \oplus FV$ is a \mathcal{V}_m -invariant system P of polynomial w -structures

$$P: V \oplus FV \rightarrow V \oplus FV$$

on vector spaces $V \oplus FV$ for m -dimensional real vector spaces V . The invariance of P means that $(\varphi \oplus F\varphi) \circ P = P \circ (\varphi \oplus F\varphi)$ for any linear isomorphism $\varphi: V_1 \rightarrow V_2$ between m -dimensional vector spaces.

3. The main result. The main result of the present note is the following theorem.

Theorem 1. *Let $F: \mathcal{V}_m \rightarrow \mathcal{V}$ be a covariant regular functor and w be a polynomial in one variable with real coefficients. The following conditions are equivalent:*

- (i) *There exists an $\mathcal{M}f_m$ -natural operator $\tilde{P}: Q \rightsquigarrow (AwS)F(T)$.*
- (ii) *There exists a \mathcal{V}_m -canonical polynomial w -structure P on $V \oplus FV$.*

Proof. (i) \Rightarrow (ii). Let $\tilde{P}: Q \rightsquigarrow (AwS)F(T)$ be an $\mathcal{M}f_m$ -natural operator in question. Let V be an m -dimensional vector space from the category \mathcal{V}_m and let ∇^V be the \mathcal{V}_m -canonical torsion free flat classical linear connection on V . Then the almost polynomial w -structure $\tilde{P}(\nabla^V): TF(T)V \rightarrow TF(T)V$ on $F(T)V$ restricts to the polynomial w -structure

$$P := \tilde{P}(\nabla^V)_{0_{0_V}}: T_{0_{0_V}}F(T)V \rightarrow T_{0_{0_V}}F(T)V$$

on the tangent space $T_{0_{0_V}}F(T)V$ of $F(T)(V)$ at $0_{0_V} \in F(T)V$, where 0_V is the zero in V and 0_{0_V} is the zero in $F(T)_{0_V}V$. Since $TV = V \oplus V$, we have $F(T)V = V \oplus FV$. Therefore $T_{0_{0_V}}F(T)V = V \oplus FV$ modulo above identifications. So,

$$P: V \oplus FV \rightarrow V \oplus FV$$

is the polynomial w -structure on $V \oplus FV$ for any \mathcal{V}_m -object V . Because of the canonical character of the construction of P , the structure P is \mathcal{V}_m -canonical.

(ii) \Rightarrow (i). Suppose $P: V \oplus FV \rightarrow V \oplus FV$ is a \mathcal{V}_m -canonical polynomial w -structure. Let $\nabla \in Q(M)$ be a classical linear connection on an m -manifold M . Let $v \in F(T)_xM$, $x \in M$. Since $F(T)$ is of order 1, $F(T)M = LM[F(T)_0\mathbb{R}^m]$ (the associated space). Then ∇ -decomposition $TLM = H^\nabla \oplus VLM$ induces (in obvious way) ∇ -decomposition $TF(T)M = \tilde{H}^\nabla \oplus VF(T)M$. Then we have the identification

$$T_vF(T)M = \tilde{H}_v^\nabla \oplus V_vF(T)M \cong T_xM \oplus F(T)_xM = T_xM \oplus F(T_xM)$$

canonically depending on ∇ , where the equality is the connection decomposition, the identification \cong is the usual one (namely, $\tilde{H}_v^\nabla = T_xM$ modulo the tangent of the projection of $F(T)M$ and $V_vF(T)M = T_v(F(T)_xM) = F(T)_xM$ modulo the standard identification) and the second equality is by the definition of $F(T)M$. We define $\tilde{P}(\nabla)|_v: T_vF(T)M \rightarrow T_vF(T)M$ by

$$\tilde{P}(\nabla)|_v := P: T_xM \oplus F(T_xM) \rightarrow T_xM \oplus F(T_xM)$$

modulo the above identification $T_vF(T)M \cong T_xM \oplus F(T_xM)$. Then $\tilde{P}(\nabla): TF(T)M \rightarrow TF(T)M$ is an almost polynomial w -structure on $F(T)M$. By the canonical character of $\tilde{P}(\nabla)$, the resulting family $\tilde{P}: Q \rightsquigarrow (AwS)F(T)$ is an $\mathcal{M}f_m$ -natural operator. \square

4. An application to para-complex structures. Let $w(t) = t^2 - 1$. Let J be a polynomial w -structure on a vector space W . Then $W = W_+ \oplus W_-$, where $W_\pm = \{v \in W: J(v) = \pm v\}$. If additionally $\dim(W_+) = \dim(W_-)$, then J is called a para-complex structure on W , see [6].

An almost para-complex structure on a manifold N is an affiner $J: TN \rightarrow TN$ on N such that $J_x: T_xN \rightarrow T_xN$ is a para-complex structure on T_xN for any $x \in N$. In other words, an almost para-complex structure is a smooth

(1,1)-tensor field on the manifold N of even dimension m , if the following conditions are satisfied:

- (1) $J^2 = id_{TN}$
- (2) for each point $x \in N$, the eigenspaces T_x^+N and T_x^-N of J_x (the value of J at x) are both $\frac{m}{2}$ -dimensional subspaces of the tangent space T_xN at x , [1], [7].

Corollary 1. *Let $F: \mathcal{V}_m \rightarrow \mathcal{V}$ be a regular covariant functor. The following conditions are equivalent:*

- (a) *There is an $\mathcal{M}f_m$ -natural operator $\tilde{J}: Q \rightsquigarrow (APC)F(T)$ transforming classical linear connections ∇ on m -manifolds M into almost para-complex structures $\tilde{J}(\nabla)$ on $F(T)M$.*
- (b) *There exists a \mathcal{V}_m -canonical para-complex structure J on $V \oplus FV$.*

Proof. This is a simple consequence of Theorem 1. □

Lemma 1. *Let p be a positive integer. Let $F: \mathcal{V}_m \rightarrow \mathcal{V}$ be a covariant regular functor given by $FV = V \times \cdots \times V$ ($(p-1)$ times of V) and $F\varphi = \varphi \times \cdots \times \varphi$ ($(p-1)$ times of φ). If p is even, there is a \mathcal{V}_m -canonical para-complex structure on $V \oplus FV$.*

Proof. If p is even, we have the \mathcal{V}_m -canonical para-complex structure on $V \times \cdots \times V$ (p times of V). Namely, we have the $\frac{p}{2}$ copies of the canonical para-complex structure on $V \times V$ given by $(v, w) \rightarrow (v, -w)$. □

A Weil algebra A is a finite dimensional, commutative, associative and unital algebra of the form $A = \mathbb{R} \times N$, where N is the ideal of all nilpotent elements of A .

Lemma 2 (Lemma 5.1 in [4]). *Let A be a p -dimensional Weil algebra and let T^A be the corresponding Weil functor. For any classical linear connection ∇ on an m -manifold M , we have the base-preserving fibred diffeomorphism $I_{\nabla}^A: T^AM \rightarrow TM \otimes \mathbb{R}^{p-1}$ canonically depending on ∇ .*

We see that $TM \otimes \mathbb{R}^{p-1} = TM \times_M \cdots \times_M TM$ ($(p-1)$ times of TM) = $F(T)M$, where $F: \mathcal{V}_m \rightarrow \mathcal{V}$, $FV = V \times \cdots \times V$ ($(p-1)$ times of V), $F\varphi = \varphi \times \cdots \times \varphi$ ($(p-1)$ times of φ). So, from Corollary 1, Lemma 1 and Lemma 2 we obtain

Proposition 1. *Let A be a Weil algebra. If A is even dimensional, there exists an $\mathcal{M}f_m$ -natural operator $\tilde{J}: Q \rightsquigarrow (APC)T^A$ sending classical linear connections ∇ on m -manifolds M into almost para-complex structures $\tilde{J}(\nabla)$ on T^AM .*

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