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MISAKO KIKKAWA and WATARU TAKAHASHI

Weak and strong convergence of an implicit iterative process for a countable family of nonexpansive mappings in Banach spaces

Dedicated to W.A. Kirk on the occasion of His Honorary Doctorate of Maria Curie-Skłodowska University

Abstract. In this paper, we introduce an implicit sequence for an infinite family of nonexpansive mappings in a uniformly convex Banach space and prove weak and strong convergence theorems for finding a common fixed point of the mappings.

1. Introduction. Let H be a Hilbert space and let C be a closed convex subset of H. Let $\{T_1, T_2, \dots, T_N\}$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i)$ is nonempty. In 2001, Xu and Ori [15] introduced an implicit iteration process $\{x_n\}$ for a finite family of nonexpansive mappings as follows: $x_0 \in C$ and

$$\begin{aligned} x_1 &= t_1 x_0 + (1-t_1)T_1 x_1, \\ x_2 &= t_2 x_1 + (1-t_2)T_2 x_2, \\ &\vdots \\ x_N &= t_N x_{N-1} + (1-t_N)T_N x_N, \end{aligned}$$

$$x_{N+1} = t_{N+1}x_N + (1 - t_{N+1})T_1 x_{N+1},$$
$$x_{N+2} = t_{N+2}x_{N+1} + (1 - t_{N+2})T_2 x_{N+2},$$
$$\vdots$$

where $\{t_n\}$ is a real sequence in $(0, 1)$ and they proved that this process converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in a Hilbert space setting. Further, Xu and Ori [15] pointed out that it is yet unclear what assumptions on the mappings $\{T_1, T_2, \dots, T_N\}$ and/or the parameters $\{t_n\}$ are sufficient to guarantee the strong convergence of $\{x_n\}$. In 2002, Liu [5] gave an affirmative answer to that question as follows (see also [10]): Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Let $\{T_i : i = 1, 2, \dots, N\}$ be a finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by implicit iteration process. If $\{t_n\}$ and d satisfy $0 < d < 1$ and $0 < t_n \le d < 1$ and there exists some $T \in \{T_i : i = 1, 2, \dots, N\}$ which is semi-compact, then, $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^N F(T_i)$. Further, in 2003, Sun [9] proved that the modified implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the mappings in a uniformly convex Banach space, requiring one member T in the family to be semi-compact.

In this paper, we introduce an implicit sequence for an infinite family of nonexpansive mappings in a uniformly convex Banach space and prove weak and strong convergence theorems for finding a common fixed point of the mappings.

2. Preliminaries and lemmas. Let E be a real Banach space. Let C be a nonempty closed convex subset of E. Then a mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in C$. For a mapping T of C into itself, we denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. We also denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. A Banach space E is called uniformly convex if for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|x_n + y_n\| = 2$, $\lim_{n\to\infty}\|x_n - y_n\| = 0$ holds. E is said to satisfy Opial's condition [6] if for any sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to $z \in E$, $\liminf_{n\to\infty}\|x_n - z\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for all $y \in E$ with $y \ne z$. All Hilbert spaces and l^p $(1 < p < \infty)$ satisfy Opial's condition, while L^p $(1 < p < \infty, p \ne 2)$ do not.

Let $T_1, T_2, \dots$ be an infinite sequence of mappings of C into itself and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, Takahashi [11] (see also [8], [13]) defined a mapping W_n of C

into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned}$$

Such a mapping W_n is called the W-*mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Using [8] and [1], we obtain the following two lemmas.

Lemma 1. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for any $i = 2, 3, \ldots$. Then for every $x \in C$ and $k \in \mathbb{N}$, the $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 1, for $k \in \mathbb{N}$, we define mappings $U_{\infty,k}$ and U of C into itself as follows:

$$U_{\infty,k}x = \lim_{n\to\infty} U_{n,k}x$$

and

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a U is called the W-*mapping* generated by $T_1, T_2, \ldots$, and $\lambda_1, \lambda_2, \ldots$.

Lemma 2. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for any $i = 2, 3, \ldots$. Let W_n $(n = 1, 2, \ldots)$ be the W-*mappings *of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ and let U be the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Then $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ and $F(U) = \bigcap_{i=1}^{\infty} F(T_i)$.*

The following lemma was proved by Xu [14].

Lemma 3. *Let E be a uniformly convex Banach space and let $r > 0$. Then, there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$, where $B_r = \{x \in E : \|x\| \leq r\}$.

We also know the following lemma proved by Schu [7].

Lemma 4. *Let E be a uniformly convex Banach space, let $\{t_n\}$ be a real sequence such that $0 < b \leq t_n \leq c < 1$ for $n \geq 1$ and let $a \geq 0$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following lemma was proved by Browder [2].

Lemma 5. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $Tz = z$.*

3. Weak convergence theorem. In this section, we prove a weak convergence theorem of the implicit iteration process for finding a common fixed point of a countable family of nonexpansive mappings in a Banach space.

Theorem 6. *Let E be a uniformly convex Banach space which satisfies Opial's condition. Let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself with a nonempty common fixed point set $\bigcap_{i=1}^{\infty} F(T_i)$. Let b be a real number with $0 < b < 1$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for every $i = 2, 3, \ldots$. Let W_n $(n = 1, 2, \ldots)$ be W-mappings of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Let U be the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e.,*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$$

for every $x \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n) W_n x_n, \qquad n = 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ and d satisfy $0 < d < 1$ and $0 < \alpha_n \leq d < 1$. Then, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. From Lemma 2, we obtain $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(W_n) = F(U)$. Let $x \in C$. Then for $u \in \bigcap_{n=1}^{\infty} F(T_n)$, we obtain that $D = \{y \in C : \|y-u\| \leq \|x-u\|\}$ is a bounded closed convex subset of C and $x \in D$. Further, for any $y \in D$, we have $T_n y \in C$ and

$$\begin{aligned}\|T_n y - u\| &\leq \|y-u\| \\ &\leq \|x-u\|.\end{aligned}$$

Then D is invariant under T_n for all $n \in \mathbb{N}$. So, without loss of generality, we may assume that C is bounded.

Let $x_0 \in C$ and define S_1 by $S_1 x = \alpha_1 x_0 + (1-\alpha_1) W_1 x$ for all $x \in C$. Then, we have, for all $x, y \in C$,

$$\begin{aligned}\|S_1 x - S_1 y\| &\leq (1-\alpha_1)\|W_1 x - W_1 y\| \\ &\leq (1-\alpha_1)\|x-y\|.\end{aligned}$$

So, we obtain that S_1 is a contraction mapping C into itself. By the Banach contraction principle, there exists a unique point x_1 such that $x_1 = S_1 x_1$. Similarly, for $n \in \mathbb{N}$, we define S_n by $S_n x = \alpha_n x_0 + (1-\alpha_n) W_n x$ for all $x \in C$ and obtain a unique point $x_n \in C$ such that $x_n = S_n x_n$. Let $u \in F(U)$. By the definition of $\{x_n\}$ and Lemma 3, we have

$$\begin{aligned}\|x_n - u\|^2 &= \|\alpha_n(x_{n-1}-u) + (1-\alpha_n)(W_n x_n - u)\|^2 \\ &\leq \alpha_n \|x_{n-1}-u\|^2 + (1-\alpha_n)\|W_n x_n - u\|^2 \\ &\qquad - \alpha_n(1-\alpha_n) g(\|W_n x_n - x_{n-1}\|) \\ &\leq \alpha_n \|x_{n-1}-u\|^2 + (1-\alpha_n)\|x_n - u\|^2 \\ &\qquad - \alpha_n(1-\alpha_n) g(\|W_n x_n - x_{n-1}\|) \\ &\leq \alpha_n \|x_{n-1}-u\|^2 + (1-\alpha_n)\|x_n - u\|^2\end{aligned}$$

for some $g : \mathbb{R}^+ \to \mathbb{R}^+$, which is continuous, strictly increasing, convex and $g(0) = 0$. Therefore, we obtain $\|x_n - u\| \leq \|x_{n-1} - u\|$, and hence the limit of $\{\|x_n - u\|\}$ exists for $u \in F(U)$. Since

$$\alpha_n(1-\alpha_n) g(\|W_n x_n - x_{n-1}\|) \leq \alpha_n(\|x_{n-1}-u\|^2 - \|x_n - u\|^2)$$

for all $n \in \mathbb{N}$ and from $0 < \alpha_n \leq d < 1$, we have

$$(1-d) g(\|W_n x_n - x_{n-1}\|) \leq \|x_{n-1}-u\|^2 - \|x_n - u\|^2$$

and hence $\lim_{n\to\infty} g(\|W_n x_n - x_{n-1}\|) = 0$. This implies

$$\lim_{n\to\infty} \|W_n x_n - x_{n-1}\| = 0. \tag{1}$$

Therefore, from $\|x_n - x_{n-1}\| \leq (1-\alpha_n)\|W_n x_n - x_{n-1}\|$, we have

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{2}$$

Further, from $\|W_n x_n - x_n\| \leq \|W_n x_n - x_{n-1}\| + \|x_{n-1} - x_n\|$, (1) and (2), we obtain

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{3}$$

Since $\{x_n\}$ is bounded, we assume that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to w. Suppose that $w \neq Uw$. From Opial's condition, the definition of U and (3), we have

$$\begin{aligned}
\liminf_{j\to\infty} \|x_{n_j} - w\| &< \liminf_{j\to\infty} \|x_{n_j} - Uw\| \\
&\leq \liminf_{j\to\infty} (\|x_{n_j} - W_{n_j} x_{n_j}\| \\
&\qquad + \|W_{n_j} x_{n_j} - W_{n_j} w\| + \|W_{n_j} w - Uw\|) \\
&\leq \liminf_{j\to\infty} (\|x_{n_j} - W_{n_j} x_{n_j}\| \\
&\qquad + \|x_{n_j} - w\| + \|W_{n_j} w - Uw\|) \\
&= \liminf_{j\to\infty} \|x_{n_j} - w\|.
\end{aligned}$$

This is a contradiction. Hence, we obtain $w \in F(U)$. To complete the proof, we prove that $\{x_n\}$ has at most one weak subsequential limit. We assume that z_1 and z_2 are two distinct weak subsequential limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. From Opial's condition, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_{n_i} - z_1\| < \lim_{i\to\infty} \|x_{n_i} - z_2\| = \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{j\to\infty} \|x_{n_j} - z_2\| < \lim_{j\to\infty} \|x_{n_j} - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is a contradiction. So, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

As a direct consequence of Theorem 6, we obtain the following result.

Corollary 7. *Let X be a Hilbert space. Let C be a nonempty closed convex subset of X. Let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty. Let b be a real number with $0 < b < 1$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for every $i = 2, 3, \ldots$. Let W_n $(n = 1, 2, \ldots)$ be W-mappings of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Let U be the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e.,*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$$

for every $x \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n) W_n x_n, \qquad n = 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ and d satisfy $0 < d < 1$ and $0 < \alpha_n \leq d < 1$. Then, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

4. Strong convergence theorem. In this section, we consider the strong convergence of the implicit iterative process generated by a countable family of nonexpansive mappings in a Banach space. We need the following definition [3].

Definition 1. Let C be a closed subset of a Banach space E. A mapping T from C into itself is said to be semi-compact, if for any sequence $\{x_n\}$ in C such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in C$, where $\to$ denotes the strong convergence.

Theorem 8. *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself with a nonempty common fixed point set $\bigcap_{i=1}^{\infty} F(T_i)$. Let a and b be real numbers with $0 < a \leq b < 1$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < a \leq \lambda_i \leq b < 1$ for every $i = 1, 2, \ldots$. Let W_n $(n = 1, 2, \ldots)$ be W-mappings of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Let U be the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e.,*

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$$

for every $x \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n) W_n x_n, \qquad n = 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ and d satisfy $0 < d < 1$ and $0 < \alpha_n \leq d < 1$. If there exists some $T \in \{T_i : i \in \mathbb{N}\}$ which is semi-compact, then $\{x_n\}$ converges strongly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$,

Proof. Since a uniformly convex Banach space is strictly convex, from Lemma 2, we have $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(W_n) = F(U)$. As in the proof of Theorem 5, we may assume that C is bounded and obtain that the limit of $\{\|x_n - u\|\}$ exists for any $u \in F(U)$. Let $c = \lim_{n\to\infty}\|x_n - u\|$. Fix $k \in \mathbb{N}$. For all $n \in \mathbb{N}$ with $n \geq k$, we have

$$\|U_{n,k} x_n - u\| \leq \|x_n - u\|.$$

So, we obtain $\limsup_{n\to\infty}\|U_{n,k}x_n - u\| \le c$. By the definition of $\{x_n\}$, we have

$$\begin{aligned}
\|x_n - u\| &= \|\alpha_n(x_{n-1} - u) + (1-\alpha_n)(W_n x_n - u)\| \\
&\le \alpha_n\|x_{n-1} - u\| + (1-\alpha_n)\|W_n x_n - u\| \\
&\le \alpha_n\|x_{n-1} - u\| \\
&\qquad + (1-\alpha_n)\{\lambda_1\|T_1 U_{n,2}x_n - u\| + (1-\lambda_1)\|x_n - u\|\} \\
&\le \alpha_n\|x_{n-1} - u\| \\
&\qquad + (1-\alpha_n)\{\lambda_1\|U_{n,2}x_n - u\| + (1-\lambda_1)\|x_n - u\|\} \\
&\le \alpha_n\|x_{n-1} - u\| \\
&\qquad + (1-\alpha_n)\{\lambda_1\lambda_2\|U_{n,3}x_n - u\| + (1-\lambda_1\lambda_2)\|x_n - u\|\} \\
&\vdots \\
&\le \alpha_n\|x_{n-1} - u\| + (1-\alpha_n)\prod_{i=1}^{k-1}\lambda_i\|U_{n,k}x_n - u\| \\
&\qquad\qquad + (1-\alpha_n)(1 - \prod_{i=1}^{k-1}\lambda_i)\|x_n - u\|.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|x_n - u\| &\le \frac{\alpha_n}{(1-\alpha_n)\prod_{i=1}^{k-1}\lambda_i}(\|x_{n-1} - u\| - \|x_n - u\|) + \|U_{n,k}x_n - u\| \\
&\le \frac{d}{(1-d)\prod_{i=1}^{k-1}\lambda_i}(\|x_{n-1} - u\| - \|x_n - u\|) + \|U_{n,k}x_n - u\|.
\end{aligned}$$

Consequently, we have $c \le \liminf_{n\to\infty}\|U_{n,k}x_n - u\|$ and hence

$$\lim_{n\to\infty}\|U_{n,k}x_n - u\| = c$$

for all $k \in \mathbb{N}$. Moreover, since

$$\begin{aligned}
c &= \lim_{n\to\infty}\|U_{n,k}x_n - u\| \\
&= \lim_{n\to\infty}\|\lambda_k(T_k U_{n,k+1}x_n - u) + (1-\lambda_k)(x_n - u)\|
\end{aligned}$$

and

$$\limsup_{n\to\infty}\|T_k U_{n,k+1}x_n - u\| \le \limsup_{n\to\infty}\|U_{n,k+1}x_n - u\| \le c,$$

we obtain $\lim_{n\to\infty}\|T_kU_{n,k+1}x_n - x_n\| = 0$ by Lemma 4. For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\|T_kx_n - x_n\| &\le \|T_kx_n - T_kU_{n,k+1}x_n\| + \|T_kU_{n,k+1}x_n - x_n\|\\ &\le \|x_n - U_{n,k+1}x_n\| + \|T_kU_{n,k+1}x_n - x_n\|\\ &\le \lambda_{k+1}\|T_{k+1}U_{n,k+2}x_n - x_n\| + \|T_kU_{n,k+1}x_n - x_n\|.\end{aligned}$$

Hence we have $\limsup_{n\to\infty}\|T_kx_n - x_n\| \le 0$. This implies

$$\lim_{n\to\infty}\|T_kx_n - x_n\| = 0. \tag{4}$$

for all $k \in \mathbb{N}$. By the assumption, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$ as $i \to \infty$. From (4), we have

$$\|p - T_kp\| = \lim_{i\to\infty}\|x_{n_i} - T_kx_{n_i}\| = 0$$

for all $k \in \mathbb{N}$. This implies $p \in F(T_k)$ for all $k \in \mathbb{N}$. Therefore we have $\liminf_{n\to\infty} d(x_n, F(U)) = 0$. For any $u \in F(U)$, we have

$$\|x_n - u\| \le \|x_{n-1} - u\|$$

and hence

$$d(x_n, F(U)) \le d(x_{n-1}, F(U)).$$

So, we obtain $\lim_{n\to\infty} d(x_n, F(U)) = 0$. Let us prove that $\{x_n\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$, we have

$$\|x_{n+m} - u\| \le \|x_n - u\|$$

for any $u \in F(U)$. Since $\lim_{n\to\infty} d(x_n, F(U)) = 0$, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, F(U)) < \frac{\epsilon}{2}$ for any $n \ge n_0$. Hence there exists $u_1 \in F(U)$ such that $\|x_{n_0} - u_1\| < \frac{\epsilon}{2}$. So, for any $n, m \ge n_0$, we have

$$\begin{aligned}\|x_m - x_n\| &\le \|x_m - u_1\| + \|x_n - u_1\|\\ &\le \|x_{n_0} - u_1\| + \|x_{n_0} - u_1\|\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Then, $\{x_n\}$ is a Cauchy sequence, and hence $\lim_{n\to\infty} x_n$ exists in C. Let $u = \lim_{n\to\infty} x_n$. From (4) and Lemma 5, we have $u \in F(T_k)$ for all $k \in \mathbb{N}$. So, $\{x_n\}$ converges strongly to $u \in F(U)$. □

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Misako Kikkawa
Department of Mathematical
and Computing Sciences
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo, 152-8552
Japan
e-mail: takahata@is.titech.ac.jp

Wataru Takahashi
Department of Mathematical
and Computing Sciences
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo, 152-8552
Japan
e-mail: wataru@is.titech.ac.jp

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