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Local Ramsey numbers for linear forests

Abstract. Let L be a disjoint union of nontrivial paths. Such a graph we call a linear forest. We study the relation between the 2-local Ramsey number $R_{2\text{-}loc}(L)$ and the Ramsey number $R(L)$, where L is a linear forest.

L will be called an (n, j)-linear forest if L has n vertices and j maximal paths having an odd number of vertices. If L is an (n, j)-linear forest, then $R_{2\text{-}loc}(L) = (3n - j)/2 + \lceil j/2 \rceil - 1$.

Introduction. Let G, H be simple graphs with at least two vertices. The Ramsey number $R(G, H)$ is the smallest integer n such that in arbitrary two-colouring (say red and blue) of edges of the complete graph K_n a red copy of G or a blue copy of H is contained (as subgraphs). If G and H are isomorphic we write $R(G)$ instead of $R(G, G)$. For a graph G and positive integer n by nG we denote the graph consisting of n disjoint copies of G. Moreover, $K_{1,n}$ denotes a star with n edges, and P_n denotes a path with n vertices.

A *local k-colouring* of a graph F is a colouring of the edges of F in such a way that the edges incident to each vertex of F are coloured with at most k different colours. *The k-local Ramsey number* $R_{k\text{-}loc}(G)$ of a graph G is defined as the smallest integer n such that K_n contains a monochromatic

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subgraph G for every local k-colouring of K_n. The existence of $R_{k\text{-}loc}(G)$ is proved by Gyárfás, Lehel, Schelp and Tuza in [8]. Most of the results for local k-colourings can be found in [1], [2], [5], [8], [9], [11]–[14].

Let L be a disjoint union of nontrivial paths. Such a graph we call *a linear forest*. L will be called *a* (n, j)*-linear forest* if L has n vertices and j maximal paths having an odd number of vertices.

We study the relation between the 2-local Ramsey number $R_{2\text{-}loc}(L)$ and the Ramsey number $R(L)$, where L is an (n, j)-linear forest.

If K_n is locally 2-coloured and $m,\ m \geq 2$, is the number of colours, we can define a partition $\mathcal{P}(K_n)$ into nonempty sets on the vertices of K_n as follows. Let A_{ij} denote the set of vertices in K_n incident to edges of colour i and colour j, where $i \neq j$ (we mean that $A_{ij} = A_{ji}$). The vertices incident to edges of only one colour, say i, can be distributed arbitrarily in the sets A_{ij}, where $1 \leq j \leq m,\ j \neq i$. So every partition set A_{ij} induces a 2-coloured complete subgraph in K_n.

The following result is a key tool in studying of the 2-local Ramsey number of graphs.

Proposition 1 (Gyárfás, et al. [8])**.** *Let K_n be locally 2-coloured with colours $1, 2, \ldots, m$, where $m \geq 2$. Then either $m = 3$ and*

$$\mathcal{P}(K_n) = \{A_{12}, A_{13}, A_{23}\}$$

or there exists a colour, say 1, such that

$$\mathcal{P}(K_n) = \{A_{12}, A_{13}, \ldots, A_{1m}\}.$$

The following interesting results are useful here.

Proposition 2 (Gyárfás, et al. [8])**.** *Let P_n denote the path on n vertices. Then*

$$R_{2\text{-}loc}(P_{2k}) = 3k - 1 \text{ if } k \geq 1,$$

$$R_{2\text{-}loc}(P_{2k+1}) = 3k + 1 \text{ if } k \geq 1.$$

For disconnected acyclic graphs G the following results are known.

Proposition 3. (Gyárfás, et al. [8]) $R_{2\text{-}loc}(nK_2) = 3n - 1$;
(Cockayne, et al. [6]) $R(nK_2) = 3n - 1$ *if* $n \geq 2$.
Moreover, if $a \geq b \geq 1$ *then*
(Grossman [7]) $R(K_{1,a} \cup K_{1,b}) = \max\{a + 2b, 2a + 1, a + b + 3\}$;
(Bielak [2]) $R_{2\text{-}loc}(K_{1,a} \cup K_{1,b}) = 2a + b + 2$.

For unions of cycles the following relations between the Ramsey and the 2-local Ramsey numbers are known.

Theorem 4. (Burr et al. [3]) $R(nC_3) = 5n,\ n \geq 2$;
(Gyárfás et al. [8]) $R_{2\text{-}loc}(nC_3) = 7n - 2,\ n \geq 2$.

Theorem 5 (Bielak [2])**.** *$R_{2\text{-}loc}(nC_4) = 6n-1 = R(nC_4)$ for $n \geq 2$;*
$R_{2\text{-}loc}(nC_{2k+1}) = n(4k+3)-2 > R(nC_{2k+1})$ for $k \geq 2$ or $n \geq 2$;
$R_{2\text{-}loc}(k(C_3 \cup C_4)) = 15k-2 > R(k(C_3 \cup C_4))$ for $k \geq 1$.

Mizuno and Sato [10] proved that $R(k(C_3 \cup C_4)) = 11k-1$.

There is a question for which disconnected graphs the 2-local Ramsey number $R_{2\text{-}loc}(G)$ is equal to the Ramsey number $R(G)$. In this paper we study this problem for linear forests.

Investigation of linear forests. Let the complement of a graph G be denoted by $\overline{G}$. Burr and Roberts proved the following lemma and theorem.

Lemma 6 (Burr, et al. [4])**.** *Let $m \geq 2k-2 > 0$ and let G be a graph of order $m+k$ containing a path $P_m = u_1 \ldots u_m$ of order m but no path of order $m+1$. Then $\overline{G}$ contains two disjoint paths, each of the form $v^{(1)}u^{(1)}v^{(2)} \ldots u^{(s-1)}v^{(s)}$, where each $u^{(i)}$ is a u_j in P_m with $2 \leq j \leq 2k-3$, each of $v^{(i)}$ is a vertex not in P_m, and the two paths have a total of $2k-2$ vertices.*

Theorem 7 (Burr, et al. [4])**.** *If L is an (n,j)-linear forest, then $R(L) = (3n-j)/2-1$.*

The Ramsey number for an (n,j)-linear forest depends on the number of vertices n and the number of odd components j. A natural question is: what is the 2-local Ramsey number for an (n,j)-linear forest? The answer to this question is the principal result of this paper and is presented in the following theorem.

Theorem 8. *If L is an (n,j)-linear forest, then $R_{2\text{-}loc}(L) = (3n-j)/2 + \lceil j/2 \rceil - 1$.*

Proof. Let $t = (3n-j)/2 + \lceil j/2 \rceil - 1$. First let us consider the colour-partition $\mathcal{P}(K_{t-1}) = \{A_{12}, A_{13}, A_{23}\}$ such that $|A_{12}| = (n-j)/2 + \lceil j/2 \rceil - 1 = |A_{13}|$, $|A_{23}| = (n-j)/2 + \lfloor j/2 \rfloor$. Note that $|A_{12} \cup A_{13}| \leq |A_{12} \cup A_{23}| = |A_{13} \cup A_{23}| < n$. So, there exists no monochromatic L in this local 2-colouring of K_{t-1}.

Thus $R_{2\text{-}loc}(L) \geq t$. We should prove that $R_{2\text{-}loc}(L) \leq t$. Let us consider a local 2-colouring of the edges of K_t with m colours. We can assume that $m \geq 2$, else there exists monochromatic L in this local 2-colouring of K_t.

Let P_{2s} and P_q be any paths of L. Let L' be formed from L by replacing these two paths with a path P_{2s+q}. Note that L is a subgraph of L' and the parameter j is the same for L and L'. So $R_{2\text{-}loc}(L') \geq R_{2\text{-}loc}(L)$, and the inequality to be proved remains the same.

Therefore, it suffices to consider only the cases in which L consists of a single path of even order or in which L contains only paths of odd order.

The first case is covered by Proposition 2. Let us consider the second case. The inequality $R_{2\text{-}loc}(L) \leq t$ can be proved by induction on j. Again, the case $j = 1$ is covered by Proposition 2.

Assume the result to be true for any linear forest with $j-1$ paths of odd order, $j \geq 2$. Let L consist of j paths of odd order and have n vertices and let P_l be a shortest path in L.

Note that

$$l \leq \lfloor n/j \rfloor \leq \lfloor n/2 \rfloor. \tag{1}$$

Case 1. $\mathcal{P}(K_t) = \{A_{12}, A_{13}, A_{23}\}$. Without loss of generality we assume that $|A_{12}| \geq |A_{13}| \geq 1$ and $|A_{12}| \geq |A_{23}| \geq 1$. Then $|A_{12}| \geq (n-j)/2+\lceil j/2 \rceil$. Evidently if $|A_{13}| \geq (n-j)/2+\lfloor j/2 \rfloor$ then we can easily find L in the colour 1 in the subgraph $\langle A_{12} \cup A_{13} \rangle$. So let $|A_{13}| \leq (n-j)/2+\lfloor j/2 \rfloor -1$. Similarly we can assume that $|A_{23}| \leq (n-j)/2+\lfloor j/2 \rfloor -1$. Note that $|A_{13} \cup A_{23}| \leq n-2$.

Suppose that $|A_{13}|$, $|A_{23}| \geq (l-1)/2$. Since $|A_{12}| \geq (l-1)/2+1$, we define X as a $(3(l-1)/2+1)$-element subset consisting of $(l-1)/2$ vertices of A_{13} and of A_{23}, and $(l-1)/2+1$ vertices of A_{12}. Evidently $\langle X \rangle$ contains P_l of colour 1 and of colour 2 in the colouring. Note that $|A_{13} \cup A_{23} - X| \leq n-l-1$. Hence $K_t - X$ does not contain $L - P_l$ of colour 3 in the colouring. Since $t - |X| = (3(n-l)-(j-1))/2 + \lceil j/2 \rceil - 1$, by inductive hypothesis there exists a linear forest $L - P_l$ in colour 1 or 2 in the colouring. Thus we get the result.

Assume that without loss of generality $|A_{23}| = a \leq (l-1)/2-1$. Suppose that $|A_{13}| = b \geq (l-1)/2$ and define X as follows: $|X \cap A_{13}| = (l-1)/2$, $|X \cap A_{12}| = l-a$, $|X \cap A_{23}| = a$. Moreover, let $\langle X \rangle$ contain all vertices of a P_{l-2a} in colour 2 from $< A_{12} >$ (if it exists). Thus $\langle X \rangle$ contains P_l in colour 1 and in colour 2 (if it is available). Since $|A_{13} \cup A_{23} - X| \leq (n-j)/2 + \lfloor j/2 \rfloor - 1 + a - (l-1)/2 - a < \lfloor n/2 \rfloor$, $K_t - X$ does not contain $L - P_l$ of colour 3 in the colouring. Thus, by inductive hypothesis, $L - P_l$ is of colour 1 or of colour 2 in the colouring of $K_t - X$ and we get the result as above.

Therefore, we can assume that $|A_{13}| = b \leq (l-1)/2-1$ and $b \geq a$. Then $|A_{13} \cup A_{23}| \leq l-3$ and $\langle A_{13} \cup A_{23} \rangle$ does not contain any $L - L_l$ in the colour 3. Moreover,

$$|A_{12}| \geq \lfloor 3n/2 \rfloor - 1 - (a+b) = \lfloor 3(n-2(a+b)/3)/2 \rfloor - 1. \tag{2}$$

Hence, in view of Theorem 7, there exists a monochromatic path $P = P_{n-\lceil 2(a+b)/3 \rceil}$ in $\langle A_{12} \rangle$.

Let $S = A_{12} - V(P)$ and $|S| = s$. Note that

$$s \geq \max\{\lfloor (n - \lceil 2(a+b)/3 \rceil /2 \rfloor - 1, b\}$$

and

$$\lceil \lceil 2(a+b)/3 \rceil /2 \rceil \leq b.$$

Therefore, if P is in colour 1 then it can be extended to P_n of the same colour by using vertices of A_{13} and vertices of S.

Let us assume that P is in colour 2. We can assume that $\lceil 2(a+b)/3\rceil \geq 2a+1$, in the opposite case P can be extended to P_n of colour 2 by using vertices of A_{23} and vertices of S.

Then $a \leq \lceil b/2\rceil - 1$ and $a+b < 3(l-2)/4$. Let $P_m = u_1u_2\ldots u_m$ be a longest path of colour 2 in $\langle A_{12}\rangle$.

Set $k = (l-1)/2$. Evidently by (1)

$$m \geq n - \lceil 2(a+b)/3\rceil \geq 2l - \lceil (l-2)/2\rceil \geq l+2 > l-3 = 2k-2.$$

Set $S' = A_{12} - V(P_m)$. We can assume that $m \leq n-2a-1$, else since $|A_{12} - (n-2a)| \geq a$ we can find P_n in colour 2.

Then, by (1) and (2), we get

$$\begin{aligned} |S'| &\geq \lfloor 3n/2\rfloor - 1 - (a+b) - (n-2a-1) \\ &= \lfloor n/2\rfloor + a - b \geq \lfloor n/2\rfloor - (l-1)/2 + 2 \\ &\geq (l+1)/2 + 2 > k. \end{aligned}$$

Suppose for a while that $k \geq 2$. Let us consider a subgraph G of $\langle A_{12}\rangle$ containing all vertices of the path P_m and k vertices of S'. Since P_m is in colour 2, in view of Lemma 6 there are two disjoint paths in colour 1 having a total $2k-2$ vertices, each path beginning and ending outside the set $V(P_m)$ and not using the vertices $u_1, u_{l-3}, u_{l-2}, \ldots, u_m$. By maximality of m we have that the edges between u_1 and end vertices of these paths are in colour 1. Therefore we get a path of order $2k-1$ in colour 1 covering k vertices u_i, where $i \leq 2k-3 = l-4$. Using a vertex of A_{13} and the vertex $u_{l-3} = u_{2k-2}$ we can easily extend this path to a path P' of order $2k+1 = l$ in colour 1.

Let

$$X = V(P') \cup \bigcup_{i=1}^{2k-3} \{u_i\}.$$

Suppose that $k=1$. Then let $X = V(P_3) \cup \{v\}$, where P_3 is a path in $\langle A_{12}\rangle$ of the colour 2 and $v \in A_{13}$.

Note that in the both cases, $\langle X\rangle$ contains paths of order l in colour 1 and 2. Since $|X| = (3l-1)/2$ and $t - |X| = (3(n-l)-(j-1))/2 + \lceil j/2\rceil - 1$, by inductive hypothesis we get $L - P_l$ of colour 1 or 2 in the graph $K_t - X$ in the colouring. The result is proved.

Case 2. $\mathcal{P}(K_t) = \{A_{12}, A_{13}, \ldots,\ A_{1m}\}$. Without loss of generality we can assume that $|A_{12}| \geq |A_{13}| \geq \cdots \geq |A_{1m}|$. Let $M = \max\{q:\ P_q \in L\}$. If $|A_{12}| < M$ then we can change each colour i, for $3 \leq i \leq m$, to colour 2. Since there exists no P_M in colour 2 then in view of Theorem 7 we get L in colour 1. Therefore we can assume that $|A_{12}| \geq M$. Similarly without loss of generality we can assume that $|A_{1i}| \geq l,\ i = 2, \ldots, m$. Moreover, $m \geq 3$, else we have a global 2-colouring and this case is covered by Theorem 7.

If $|A_{13}| \geq \lceil n/2 \rceil$, then we have a P_n of colour 1 in the subgraph $\langle A_{12} \cup A_{13} \rangle$. So L of this colour can be easily created as well.

Thus let $|A_{13}| \leq \lceil n/2 \rceil - 1$. Since $n - l \geq n - \lfloor n/j \rfloor \geq n - \lfloor n/2 \rfloor \geq \lceil n/2 \rceil$, the subgraph $\langle A_{1i} \rangle$ does not contain $L - P_l$ in colour i for $i \geq 3$. Let us define X as a $(l + (l-1)/2)$-element subset of $V(K_t)$ containing $(l-1)/2$ vertices from A_{13} and l vertices of a P_l in colour 2 if it exists (else take l vertices from A_{12} arbitrarily). The graph $K_t - X$ consists of $(3(n-l)-(j-1))/2 + \lceil j/2 \rceil - 1$ vertices so by inductive hypothesis it contains $L - P_l$ of colour 1 or of colour 2 in the colouring. Since $\langle X \rangle$ contains P_l in colour 1 and in colour 2 (if it is available), we get the result. □

Immediately by Theorems 7, 8 we get the following result.

Corollary 9. *If L is an (n, j)-linear forest, then*

$$R_{2\text{-}loc}(L) = R(L), \ \textit{for } j = 0$$

and

$$R_{2\text{-}loc}(L) > R(L), \ \textit{for } j > 0.$$

Final remark. The respective general methods for the study of the local k-colouring for $k > 2$ have not been discovered.

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