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Instytut Matematyki
Uniwersytet Marii Curie-Sklodowskiej

W. Cieślak, J. Zając

Polar Decomposition of Involutive Groups

Rozklad biegunowy grup involutywnych

Полярное разложение инволютивных групп

Let G denote a group. A mapping $*: G \to G$ satisfying the conditions:

$$(a^*)^* = a \quad \text{for all} \quad a \in G, \tag{1}$$

$$(ab)^* = b^*a^* \quad \text{for all} \quad a, b \in G, \tag{2}$$

$$\text{there exists} \quad g \in G \quad \text{such that} \quad g^* \neq g^{-1}, \tag{3}$$

will be called an involutive mapping of G. The pair $(G, *)$ will be called an involutive group.

Let $(G, *)$ be an involutive group and let e denote the unit element of G. We introduce the following notations:

$$S = \{s \in G : s^* = s\} \tag{4}$$

$$T = \{gg^* : g \in G\} \tag{5}$$

$$H = \{h \in G : h^*h = e\}. \tag{6}$$

It is easy to see that H is a subgroup of G.

Definition 1. The subset $A \subset G$ will be called a square set if for arbitrary $a \in A$ there exists exactly one element $b \in A$ such that $a = b^2$. In particular the group G will be called a square group if G is a square set.

Let us consider the family:

$$\mathcal{A} = \{A \subset S : A \quad \text{is a square set}\}$$

partly ordered by the inclusion. We note that $\{e\} \in \mathcal{A}$ and if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. Let $\mathcal{L}$ be a chain in $\mathcal{A}$ and $L_0 = \bigcup_{L \in \mathcal{L}} L$. We have $L_0 \in \mathcal{A}$ and $L \subset L_0$ for all $L \in \mathcal{L}$. The Kuratowski-Zorn Lemma implies that there exists the maximal element S^+ in $\mathcal{A}$. The maximal element is exactly one. Indeed, if S_1^+ and S_2^+ are two maximal elements, then we have $S_1^+, S_2^+ \subset S_1^+ \cup S_2^+ \in \mathcal{A}$ and $S_1^+ = S_2^+ = S_1^+ \cup S_2^+$. It implies the equality $S_1^+ = S_2^+$.

Definition 2. We say that an involutive group $(G, *)$ has the polar decomposition if $G = S^+H$ and the decomposition $g = sh$ of an element $g \in G$ ($s \in S^+, h \in H$) is unique.

Theorem 1. *An involutive group $(G,*)$ has the polar decomposition iff:*

$$T \subset S^+ . \tag{7}$$

Proof. Let $T \subset S^+$ and $a \in G$. Then $aa^* \in T \subset S^+$ and $aa^* = b^2$ for some $b \in S^+$. It is sufficient to show that $h = b^{-1}a \in H$. We have $h^*h = a^*b^{-2}a = a^*(aa^*)^{-1}a = e$. Thus $G = S^+H$. Uniqueness of the decomposition needs to be proved.

For $a = s_1h_1$ and $a = s_2h_2$ we have $aa^* = s_1^2 = s_2^2$. Thus uniqueness of decomposition results from the uniqueness of the root.

We assume that an involutive group $(G,*)$ has the polar decomposition. Let $a \in G$ and $a = sh$ for some $s \in S^+$ and $h \in H$. Hence $a^* = h^{-1}s$ and $aa^* = s^2$. It implies the inclusion (7).

Considering the group $GL(n, R)$ with transposition the classical polar decomposition of the linear group is obtained [1]. Similarly, considering the multiplicative group of complex numbers with conjugation the trigonometric form of complex numbers is obtained.

Let K, A denote a group and an abelian square group respectively. Let us consider the semi-direct products of the groups K and A if it exists. A group multiplication in $K \times A$ is of the form:

$$(k,a)(l,b) = (kl, r_k b + a) ,$$

r denotes here the action of K on A. It is easy to see that:

$$(k,a)^* = (k^{-1}, r_{k^{-1}}a) \tag{8}$$

is an involutive mapping.

Theorem 2. *The involutive group $(K \times A, *)$ has the polar decomposition.*

Proof. We note that:

$$H = \{(k,0) : k \in K\} \quad \text{is isomorphic with} \quad K ,$$

$$L = \{(1,s) : s \in A\} \quad \text{is isomorphic with} \quad A .$$

Further, we have $(k,a) = (1,a)(k,0)$ and this decomposition is unique. Thus $K \times A = LH$. Since A is a square group then L is a square group too, and $L = S^+$. Thus the proof is complete.

Let us assume an involutive group $(G, *)$ to have the polar decomposition. We determine the action of the group G on S^+ as follows:

$$r_g s = g^* s g \quad \text{for} g \in G, \quad s \in S^+ .$$

G acts transitively on S^+. Really, let $s, t \in S^+$. By s', t' we denote such elements of the set S^+ that $s = (s')^2$, $t = (t')^2$. Then for $g = (s')^{-1} t'$ we have $r_g s = t$. It follows from (9) that H is an isotropy group of the point e. Hence $f : G/H \to S^+$ given by the formula $f(gH) = r_g e = g^* g$ is a bijective mapping [1].

REFERENCES

[1] Želobenko, D.F., *Compact Lie Groups and their Representations*, (Russian), Moscow 1970

STRESZCZENIE

W pracy wprowadzono pojęcie grupy inwolutywnej i rozkładu biegunowego takich grup. Twierdzenie 1 zawiera warunek konieczny i wystarczający istnienia rozkładu biegunowego grupy inwolutywnej. Dalej pokazano, że iloczyn półprosty dowolnej grupy i kwadratowej grupy abelowej posiada rozkład biegunowy. Praca uogólnia rozkład biegunowy grupy liniowej na pewną klasę grup inwolutywnych.

РЕЗЮМЕ

В данной работе осуществлено понятие инволютивной группы и полярного разложения таких групп. Теорема 1 содержит конечное и достаточное условие существования полярного разложения инволютивной группы. Дальше показано, что полупрямое произведение произвольной группы и квадратной абелевой группы имеет полярное разложение. Работа обобщает полярное разложение линейной группы на некоторый класс инволютивных групп.