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Quasisubordination and Quasimajorization of Analytic Functions

Quasipodporządkowanie i quasimajoryzacja funkcji analitycznych

Квазиподчинение и квазимажорация аналитических функций

Let B be the class of analytic functions in K_R, $K_R = \{z: |z| < R\}$ and bounded $|\varphi(z)| \leqslant 1$ for $z \in K_R$.

Let Ω denote the class of analytic functions in K_R, such that $|\omega(z)| \leqslant |z|$ for $z \in K_R$.

Let $f(z)$, $F(z)$ be two functions which are single-valued and analytic in the disc K_R.

Definition 1. The function $f(z)$ is said to be subordinate to $F(z)$ in K_R if there exists a function $\omega(z) \in \Omega$, for which

$$f(z) = F(\omega(z)), \ z \in K_R.$$

In this case we write

$$f(z) \dashv F(z) \text{ in } K_R.$$

Definition 2. The function $f(z)$ is said to be majorized by $F(z)$ in K_R, if there exists a function $\varphi(z) \in B$, such that

$$f(z) = \varphi(z)F(z) \text{ for } z \in K_R.$$

We write then

$$f(z) \ll F(z) \text{ in } K_R.$$

Note that $f(z) \ll F(z)$ in K_R, if and only if, for every $z \in K_R$ we have $|f(z)| \leqslant |F(z)|$.

Several theorems exist in the literature that relate $f(z)$ and $F(z)$ when $f(z) \dashv F(z)$ which have their counterparts that relate $g(z)$ and $G(z)$ when $g(z) \ll G(z)$. In order to establish some semblance of unification for parallel results for subordination and majorization, M. S. Robertson [6] introduced the concept of quasisubordination.

Definition 3. [6]. Let $f(z)$ and $F(z)$ be analytic in K_R. Let $\varphi(z)$ be analytic and bounded for $z \in K_R$, $|\varphi(z)| \leqslant 1$, such that $f(z)/\varphi(z)$ is regular and subordinate to $F(z)$ in K_R. Then $f(z)$ is said to be quasisubordinate to $F(z)$ w.r.t. to $\varphi(z)$ in K_R.

If $f(z)$ is quasisubordinate to $F(z)$ w.r.t. to $\varphi(z)$ we shall often say simply that $f(z)$ is quasisubordinate to $F(z)$ in K_R and write

$$f(z) \dashv_q F(z) \text{ in } K_R.$$

From this definition we have that $f(z) \dashv_q F(z)$ in K_R, if and only if there exists the function $\varphi(z) \in B$ and $\omega(z) \in \Omega$, such that

$$f(z) = \varphi(z) F(\omega(z)) \text{ for } z \in K_R.$$

The concept of quasisubordination can be also defined as follows:

Definition 4. Let $f(z)$, $F(z)$ be analytic in K_R. If there exists an analytic function $g(z)$, such that $f(z) \ll g(z)$ in K_R and $g(z) \dashv F(z)$ in K_R, then $f(z)$ is said to be quasisubordinate to $F(z)$ relative to $g(z)$ in K_R and we shall often say simply that $f(z)$ is quasisubordinate to $F(z)$ in K_R and write $f(z) \dashv_q F(z)$ in K_R.

These two definitions 3 and 4 are equivalent. If $f(z)$ is quasisubordinate to $F(z)$ relative to $\varphi(z)$, then in Definition 4 we can put $g(z) = f(z)/ /\varphi(z)$. Now if $f(z)$ is quasisubordinate to $F(z)$ relative to $g(z)$, then we can put $\varphi(z) = f(z)/g(z)$ in Definition 3.

Thus we have

$$f(z) \dashv_q F(z) \text{ in } K_R \Leftrightarrow \underset{g(z)}{\exists} \big(f(z) \ll g(z) \text{ in } K_R\big) \wedge \big(g(z) \dashv F(z) \text{ in } K_R\big).$$

Using the Definition 4 we can obtain immediately the generalizations on quasisubordination of all these theorems, which have the same conclusions and the assumption subordination $f(z) \dashv F(z)$ is replaced by that of majorization $f(z) \ll F(z)$. In particular we have

Theorem 1, [6]. *If $f(z) \dashv_q F(z)$ in K_R, then for every $\lambda > 0$ and $r \in (0, 1)$ we have*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leqslant \int_0^{2\pi} |F(re^{i\theta})|^\lambda d\theta.$$

Theorem 2, [6]. *If $f(z) \dashv_q F(z)$ in K_R and*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad F(z) = \sum_{k=0}^{\infty} A_k z^k$$

for $z \in K_R$ then for $n = 0, 1, 2, \ldots$ and $r \in (0, R)$

$$\sum_{k=0}^{n} |a_k|^2 r^{2k} \leqslant \sum_{k=0}^{n} |A_k|^2 r^{2k}.$$

If the series $\sum_{k=0}^{\infty} |A_k|^2 r^{2k}$ *is convergent for* $r \leqslant R$ *then*

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \leqslant \sum_{k=0}^{\infty} |A_k|^2 r^{2k}, \; 0 < r \leqslant R.$$

Theorem 3. *If* $f(z) \dashv_q F(z)$ *in* K_1 *then*

$$\sum_{k=0}^{n} |a_k|^2 \leqslant \sum_{k=0}^{n} |A_k|^2, \; n = 0, 1, 2 \ldots,$$

and if the series $\sum_{k=0}^{\infty} |A_k|^2$ *is convergent then*

$$\sum_{k=0}^{\infty} |a_k|^2 \leqslant \sum_{k=0}^{\infty} |A_k|^2.$$

The following theorem is a generalization of Theorem 3 in the paper [3] p. 211.

Theorem 4. *Let* λ_k, $k = 1, 2, \ldots$, *be a sequence of nonnegative real numbers, such that* $\lambda_k \geqslant \lambda_{k+1} \geqslant 0$ *for* $k = 1, 2, \ldots$.

If $f(z) \dashv_q F(z)$ *in* K_1 *then*

$$\sum_{k=1}^{n} \lambda_k |a_k|^2 \leqslant \sum_{k=1}^{n} \lambda_k |A_k|^2, \; n = 1, 2, \ldots$$

and if the series $\sum_{k=1}^{\infty} \lambda_k |A_k|^2$ *is convergent, then*

$$\sum_{k=1}^{\infty} \lambda_k |a_k|^2 \leqslant \sum_{k=1}^{\infty} \lambda_k |A_k|^2.$$

Let A be a class of functions analytic in the unit disc K_1 and normalized by the conditions

$$f(0) = 0, \; f'(0) \geqslant 0.$$

Now we can introduce the concept of so-called quasisubordination in a normalized way.

Definition 5. We say that a function $f(z) \in A$ is quasisubordinate in a normalized way to $F(z) \in A$, if there exists a function $g(z) \in A$ such that $f(z) \ll g(z)$ in K_1 and $g(z) \dashv F(z)$ in K_1. We write then

$$f(z) \dashv_{qu} F(z).$$

For this kind of subordination we can prove the following theorems.

Theorem 5. *Let S_a^* denote ths clase of a-starlike normalized functions in K_1 that is $F(0) = F'(0)-1 = 0$ and $\operatorname{Re}\{zF'(z)/F(z)\} > a$ for $z \in K_1$. If $F(z) \in S_{1/2}^*$ and $f(z) \dashv_{qu} F(z)$ then for $|z| = r < 1$ we have*

$$|f(z)| \leqslant T(r, S_{1/2}^*)\,|F(z)|$$

where

$$T(r, S_{1/2}^*) = \max\left\{1, \frac{r}{1-r}\right\}.$$

Theorem 6. *Let S^c denote the class of convex normalized functions in K_1 that is $F(0) = 0 = F'(0)-1$ and $\operatorname{Re}\{1+zF''(z)/F'(z)\} > 0$ in K_1. If $F(z) \in S^c$ and $f(z) \dashv_{qu} F(z)$ then for $|z| = r < 1$ we have*

$$|f(z)| \leqslant T(r, S^c)\,|F(z)|$$

where

$$T(r, S^c) = T(r, S_{1/2}^*) = \max\left\{1, \frac{r}{1-r}\right\}.$$

Theorem 7. *If $F(z) \in S_0^*$ and $f(z) \dashv_{qu} F(z)$, then for $|z| = r < 1$, we have*

$$|f(z)| \leqslant T(r, S_0^*)\,|F(z)|$$

where

$$T(r, S_0^*) = \max\left\{1, \frac{r}{(1-r)^2}\right\}.$$

The results of the theorems, 5, 6 and 7 are the best possible in this sense that we could not replace the functions $T(r, S_{1/2}^*) = T(r, S^c)$ and $T(r, S_0^*)$ by any smaller functions of r respectively.

The Theorems 5, 6 and 7 follow immediately from one general theorem. In order to formulate it we must introduce first some notations.

Let H_z denote so-called the Rogosinski's domain bounded by an arc of circumference $|\zeta| = |z|^2$ and two arcs of circumferences going through the point $\zeta = z$ and tangent to the circumference $|\zeta| = |z|^2$ at the points $\zeta_1 = iz|z|$, $\zeta_2 = -iz|z|$ respectively.

Let U be an arbitrary fixed subclass of the class S of normalized ($f(0) = 0$, $f'(0) = 1$) and univalent functions in K_1, with the following property: $f(z) \in U \Rightarrow e^{-i\theta}f(e^{i\theta}z) \in U$ for all real θ.

Let us put

$$Q(z, U) = \left\{w\colon w = \frac{F(\zeta)}{F(z)}, \zeta \in H_z, F(z) \in U\right\}.$$

Theorem 8. *If $f(z) \prec_{qu} F(z)$ and $F(z) \in U$ then for $r \in (0, 1)$ we have*

$$\sup_{|z|=r} \left|\frac{f(z)}{F(z)}\right| \leqslant T(r, U),$$

where

$$T(r, U) = \sup\{|w|\colon w \in Q(r, U)\}.$$

Proof of Theorem 8. $f(z) \prec_{qu} F(z)$ implies that there exists a function $g(z)$ such that $f(z) \ll g(z)$, that is $|f(z)/g(z)| < 1$ for $z \in K_1$ and $g(z) \prec F(z)$ in K_1. Now by the results of paper [2] (Theorem 1 and Corollary 1) we have that $|g(z)/F(z)| \leqslant T(r, U)$ for $|z| = r < 1$. Therefore

$$\left|\frac{f(z)}{F(z)}\right| = \left|\frac{f(z)}{g(z)}\right| \cdot \left|\frac{g(z)}{F(z)}\right| \leqslant T(r, U).$$

Because the above mentioned results of paper [2] are best possible and subordination $f(z) \prec F(z)$ in K_1 implies quasisubordination $f(z) \prec_q F(z)$ in K_1, therefore the result given in Theorem 8 is the best possible, too.

Proof of Theorems 5, 6 and 7. It is enough to use the Theorem 8 and the functions $T(r, S^*_{1/2})$, $T(r, S^c)$ and $T(r, S^*_0)$ determined in [2].

If in the definitions 4 and 5 we change the role of subordination and majorization then we obtain one new concept.

Definition 6. Let $f(z)$, $F(z)$ be analytic in K_R. If there exists an analytic function $h(z)$ such that $f(z) \prec h(z)$ in K_R and $h(z) \ll F(z)$ in K_R, then $f(z)$ is said to be quasimajorized by $F(z)$ relative to $h(z)$ in K_R and we shall often say simply that $f(z)$ is quasimajorized by $F(z)$ in K_R and write $f(z) \ll_q F(z)$ in K_R.

It is easy to see that $f(z) \ll_q F(z)$ in K_R if and only if there exist the functions $\varphi_1(z) \in B$ and $\omega_1(z) \in \Omega$ such that

$$f(z) = \varphi_1(\omega_1(z)) \cdot F(\omega_1(z)), \text{ for } z \in K_R.$$

Definition 7. We shall say that a function $f(z) \in A$ is quasimajorized in a normalized way by a function $F(z) \in A$, if there exists a function $h(z) \in A$ such that $f(z) \prec h(z)$ in K_1 and $h(z) \ll F(z)$ in K_1. We shall write then

$$f(z) \ll_{qu} F(z).$$

Quasimajorization and quasisubordination are connected by the following relations.

Lemma 1. *If* $f(z) \ll_q F(z)$ *in* K_R *then* $f(z) \dashv_q F(z)$ *in* K_R.

Lemma 2. *If* $f(z) \ll_{qu} F(z)$ *then* $f(z) \dashv_{qu} F(z)$.

Proof. In order to prove it it is enough to put $\varphi(z) = \varphi_1(\omega_1(z))$ and $\omega(z) = \omega_1(z)$. Furthermore if f, h, F belong to A then $\varphi_1(0) \geqslant 0$, $\omega_1'(0) \geqslant 0$ and therefore $\varphi(0) \geqslant 0$, $\omega'(0) \geqslant 0$. This implies that $h(z) = f(z)/\varphi(z) \in A$ and the both lemmas are proved.

Remark. By the lemmas 1 and 2 it follows that the theorems 1-8 of this paper are valid if we replace in them quasisubordination by quasi-majorization.

Problem. Lemma 1 says that $f(z) \ll_q F(z) \Rightarrow f(z) \dashv_q F(z)$. We can change the direction of this implication that is, the concepts of quasi-subordination and quasimajorization are equivalent.

Remark. If $F(z) = z$ is identity function then $f(z) \ll_{qu} F(z)$ if and only if $f(z) \dashv_{qu} F(z)$.

Proof of the remark. Necessity follows by Lemma 2. Suffiency. In this case $f(z) \dashv_{qu} z$ and we have

$$f(z) = \varphi(z)\omega(z) = \varphi_1(\omega_1(z))\omega_1(z)$$

where $\omega_1(z) = \varphi(z)\omega(z)$, $\varphi_1(z) \equiv 1$. Therefore $f(z) \ll_{qu} F(z)$.

Thus we see that for identity these two concepts are equivalent. In a general case the problem is open.

We can generalize also some such theorems on quasimajorization of which we could not generalize on quasisubordination.

Theorem 9. *If* $f(z) \ll_{qu} F(z)$ *and* $F(z) \in S_0^*$ *then for every* $R \in (0, 1\rangle$ *we have*

$$f(K_{r(R)}) \subset F(K_R).$$

where

$$r(R) = r(R, S_0^*) = \min\left\{R, \frac{\sqrt{R}}{1+\sqrt{R+R}}\right\}.$$

Theorem 10. *If* $f(z) \ll_{qu} F(z)$ *and* $F(z) \in S_{1/2}^*$ *then for every* $R \in (0, 1\rangle$ *we have*

$$f(K_{r(R)}) \subset F(K_R),$$

where

$$r(R) = r(R, S_{1/2}^*) = \min\left\{R, \frac{\sqrt{5R^2+4R}-R}{2(1+R)}\right\}.$$

Corollary. *Theorem 10 is valid if the hypothesis $F(z) \in S^*_{1/}$ is replaced by $F(z) \in S^c$ since $S^*_{1/2} \supset S^c$.*

The theorems 9 and 10 follow from some general theorem which is a generalization of Theorem 2.2 of paper [4]. Now we formulate

Theorem 11. *If $f(z) \ll_{qu} F(z)$ and $F(z) \in U$ then for every $R \in (0,1)$ we have*

$$f(K_{r(R)}) \subset F(K_R)$$

where

$$r(R) = r(R, U) = \sup_{r \leqslant R}\{r\colon O_r \cap D(R, r, U) = \emptyset\}$$

and

$$O_r = \{w\colon w = \varphi(z), |z| \leqslant r, \varphi(z) \in B, \varphi(0) \geqslant 0\},$$

$$D(R, r, U) = \left\{w\colon w = \frac{F(z)}{F(\zeta)},\ |z| = R,\ |\zeta| = r,\ F \in U\right\}.$$

Proof of Theorem 11. By the hypothesis $f(z) \ll_{qu} F(z)$ we have

1° $f(z) \dashv h(z)$ in K_1, $h(z) \in A$,

2° $h(z) \ll F(z)$ in K_1.

The functions $h(z), F(z)$ satisfy the hypotheses of Theorem 2.2 [4] and therefore

$$h(K_{r(R)}) \subset F(K_R).$$

From 1° we have that

$$f(K_\varrho) \subset h(K_\varrho)$$

for every $\varrho \in (0, 1\rangle$.

Thus we obtain

$$f(K_{r(R)}) \subset F(K_R)$$

and the proof is complete. In an analogous way: Theorem 9 follows from Theorem 2 of paper [5] p. 924 and Theorem 10 follows from Theorem 2 of paper [1] p. 7.

Let us put in the Theorems 11, 10 and 9 $R \to 1$. Then we have the following corollaries.

Corollaries.

Suppose $f(z) \ll_{qu} F(z)$.

1. *If $F(z) \in U$, then*

$$f(K_{r_1}) \subset F(K_1)$$

where

$$r_1 = \lim_{R \to 1} r(R).$$

2. *If* $F(z) \in S^*_{1/2}$ *then* $f(K_{1/2}) \subset F(K_1)$.

3. *If* $F(z) \in S^*_0$ *then* $f(K_{1/3}) \subset F(K_1)$.

The Theorems 1-11 are not all which we can obtain. There are many other theorems which can be extended on quasisubordination and quasi-majorization. Some of these generalizations will be studied in the next paper.

REFERENCES

[1] Bogowski F., Stankiewicz Z., *Sur la majoration modulaire des fonctions et l'inclusion des domaines dans la classe* $S^*_{1/2}$, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, 25 (1971), 5-14.

[2] Bogucki Z., Waniurski J., *The relative growth of subordinate functions*, Michigan Math. J. 18 (1971), 357-363.

[3] Golusin G. M., *On majorants of subordinate analytic functions* I, Mat. Sb. N. S. 29 (1951), 209-224.

[4] Lewandowski Z., Stankiewicz J., *Majorante modulaire des fonctions et inclusion des domaines*, Bull. Acad. Polon. Sci., 10 (1971), 917-922.

[5] —, —, *Les majorantes modulaires étoilees et l'inclusion des domaines*, Bull. Acad. Polon. Sci., 10 (1971), 932-929.

[6] Robertson M. S., *Quasi-subordinate functions*, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, Athens, Ohio 1970, 311-330.

STRESZCZENIE

Od dawna znane są pojęcia podporządkowania $f \dashv F$ i majoryzacji (modułowej) $f \ll F$. M. S. Robertson w 1970 r. uogólnił te dwa pojęcia wprowadzając nowe pojęcie quasipodporządkowania $f \dashv_q F$, które w szczególnych przypadkach daje podporządkowania lub majoryzację. W tej pracy podana jest definicja quasipodporządkowania w postaci

$$f \dashv_q F \Leftrightarrow \bigvee_g \{(f \ll g) \wedge (g \dashv F)\}.$$

Definicja ta pozwala na otrzymanie niemal bez dowodów wielu twierdzeń dotyczących funkcji quasipodporządkowanych.

Wprowadzone jest tu również pojęcie quasimajoryzacji $f \ll_q F$ poprzez zamianą roli majoryzacji i podporządkowania w definicji quasipodporządkowania.

$$f \ll_q F \Leftrightarrow \bigvee_g \{(f \dashv h) \wedge (h \ll F)\}.$$

Okazuje się, że $f \ll_q F \Rightarrow f \dashv_q F$, problem odwrotny pozostaje otwarty. Ponadto podanych jest kilka przykładowych twierdzeń wiążących quasipodporządkowanie ze wzajemnym wzrostem funkcji f i F oraz quasimajoryzację też ze wzrostem funckji i dodatkowo z zawieraniem się obszarów $f(K_r)$, $F(K_R)$ gdzie $K_r = \{z: |z| < r\}$.

РЕЗЮМЕ

Уже давно известно понятие подчинения $f \prec F$ и мажорации (модульной) $f \ll F$. Робертсон М. С. в 1970 г. обобщил эти два понятия, вводя новое понятие квазиподчинения $f \dashv_q F$, которое в особенных случаях создает подчинение или мажорации. В данной работе представлена дефиниция квазиподчинения в виде:

$$f \dashv_q F \Leftrightarrow \bigvee_g \{(f \ll g) \wedge (g \dashv F)\}$$

Эта дефиниция дает возможность получить почти без доказательств много теорем, относящихся к квазиподчиненным функциям.

Введено здесь понятие квазимажорации $f \ll_q F$, заменяя роль мажорации и подчинения в дефиницию квазиподчинения.

$$f \prec_q F \Leftrightarrow \bigvee_h \{(f \prec h) \wedge (h \ll F)\}$$

Оказывается, что $f \ll_q F \Rightarrow f \dashv_q F$ оборотная проблема остаётся открытой. Также было представлено несколько примерных теорем, связывающих квазиподчинение с обоюдным ростом функций f и F, а также квазимажорации с ростом функции и дополнительно с возмещающимися областями $f(K_r)$, $F(K_R)$, где $K_r = \{z\colon |z| < r\}$.

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