ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XXXI, 15 SECTIO A 1977

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin
Zakład Zastosowań Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

JAN STANKIEWICZ, ZOFIA STANKIEWICZ

Some Remarks on Subordination and Majorization of Functions

Pewne uwagi o podporządkowaniu i majorazacji funkcji

Некоторые заметки о подчинению и мажорации функций

Let $f(z)$, $F(z)$ be two regular functions in K_r, $K_r = \{z: |z| < r\}$. The function $f(z)$ is said to be subordinate to $F(z)$ in K_r if there exists a regular function $\omega(z)$, bounded in K_r, $|\omega(z)| \leqslant |z| < r$ for which

$$f(z) = F(\omega(z)) \text{ in } K_r.$$

In this case we write

$$f(z) \prec_r F(z).$$

If for every $z \in K_r$ we have

$$|f(z)| \leqslant |F(z)|$$

then we say, that $f(z)$ is majorized by $F(z)$ in K_r. In this case we write

$$f(z) \ll_r F(z).$$

In 1935 M. Biernacki initiated the study of relationship between the subordination in K_1 and majorization in some smaller disk K_r of two functions f, F. Next this problem was investigated by many other authors. A natural problem was to find the largest number $r \in (0, 1\rangle$ such that the implication

$$f(z) \prec_1 F(z) \Rightarrow f(z) \ll_r F(z) \tag{1}$$

holds, where $f(z)$ and $F(z)$ range over some fixed classes of regular functions respectively.

In 1971 Z. Bogucki and J. Waniurski [3] modified M. Biernacki's problem in the following way.

Assuming that $f(z) \dashv_1 F(z)$ they tried to determine the function

$$\mathscr{R}(r, n, S_0) = \sup_{|z|=r}\left|\frac{f(z)}{F(z)}\right|$$

where $f(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots, n = 1, 2, \ldots, a_n \geqslant 0$ is arbitrary regular function in K_1 and $F(z)$ ranges over some fixed subclass $S_0 \subset S$ (S is the class of univalent, normalized function in K_1). They gave a general method to determine the function $\mathscr{R}(r, n, S_0)$ and using this method they determined $\mathscr{R}(r, n, S_0)$ for $S_0 = S^c$ — the class of univalent convex normalized functions in K_1 and $S_0 = S_a^*$ ($a = 0$, $a = 1/2$) where S_a^* is the class of normalized, univalent functions starlike of order a in K_1.

In this paper we give another method to determine the function the $\mathscr{R}(r, n, S_0)$ in the case $n \geqslant 2$. Using this method we can determine $\mathscr{R}(r, n, S_0)$ for different classes of functions $F(z)$. This method works also in some cases when S_0 is not a subclass of the class S of univalent functions.

Let be given two functions $m(r)$, $M(r)$ determined, monotonous and continuous in $\langle 0, 1)$ and such that $m(0) = M(0) = 0$, $m'(0) = M'(0) = 1$, $0 \leqslant m(r) \leqslant M(r)$.

Definition 1. Denote by $S(m, M)$ the class of all regular and normalized functions ($F(0) = F'(0) - 1 = 0$) in K_1 and such, that for $|z| = r < 1$ the inequality

$$m(r) \leqslant |F(z)| \leqslant M(r)$$

holds.

Remark 1. If we put $m(r) = r/(1+r)^2$, $M(r) = r/(1-r)^2$ the class $S(m, M)$ is not empty because the class S of normalized univalent functions in K_1 is contained in $S(m, M)$.

Definition 2. Denote by H_n, $n = 1, 2, \ldots$, the class of all functions which are regular in K_1 and have the expansion of the form

$$f(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots$$

Theorem 1. *If the function $f(z) \in H_n$ is subordinate to $F(z)$ in K_1, $F(z) \in S(m, M)$ then for $n \geqslant 2$ and $|z| = r < 1$ we have*

$$|f(z)| \leqslant T(r, n, S(m, M)) \cdot |F(z)| \tag{2}$$

where

$$T(r, n, S(m, M)) = \frac{M(r^n)}{m(r)}.$$

Proof. If $f(z) \in H_n$ and $f(z) \dashv_1 F(z)$ then $f(z) = F(\omega(z))$ where $|\omega(z)| \leqslant |z|^n$. Therefore

(3) $$|f(z)| = |F(\omega(z))| \leqslant \max_{|\zeta| \leqslant r^n} |F(\zeta)| \leqslant M(r^n).$$

Furthermore for $|z| = r$

$$|F(z)| \geqslant m(r)$$

or in an equivalent form

(4) $$\frac{|F(z)|}{m(r)} \geqslant 1.$$

From (3) and (4) it follows

$$|f(z)| \leqslant M(r^n) \leqslant M(r^n)\frac{|F(z)|}{m(r)}$$

which is equivalent to (2).

Remark 2. If there exists an extremal function $F_e(z) \in S(m, M)$ such that

$$|F_e(r)| = M(r) \text{ and } |F_e(-r)| = m(r)$$

then the result of Theorem 1 is the best possible. It is the best possible in this sense that we cannot replace the function $T(r, n, S(m, M))$, $n \geqslant 2$, by any smaller function of r. In this case we have

$$T(r, n, S(m, M)) = \mathscr{R}(r, n, S(m, M)).$$

Proof. Let us put

$$F(z) = e^{-i\theta}F_e(e^{i\theta}z), \quad \omega(z) = z^n.$$

Then $F(z) \in S(m, M)$, $f(z) = F(z^n) \in H_n$ and $f(z) \dashv F(z)$ in K_1. We can choose z and θ such that

$$e^{i\theta}z^n = r^n$$

$$e^{i\theta}z = -r.$$

In particular we can put

$$\theta = \frac{n-2}{n-1}\pi$$

$$z = re^{\frac{i\pi}{n-1}}.$$

Thus

(5) $$|f(z)| = |F(z^n)| = |e^{-i\theta}F_e(r^n)| = M(r^n)$$

and

$$(6) \quad T(r, m, S(m, M))|F(z)| = T(r, n, S(m, M))|e^{-i\theta}F_\varepsilon(e^{i\theta}z)| = = \frac{M(r^n)}{m(r)}m(r) = M(r^n).$$

From (5) and (6) we obtain that for such z we have equality in (2).

Remark 3. Under the substitution of Remark 2, the additional condition $a_n \geqslant 0$ in Theorem 1 does not improve the result of this Theorem.

Proof. This follows from the fact that the functions $f(z)$, $F(z)$ in the proof of Remark 2 which give equality in (2) have $a_n = 1 > 0$.

From Theorem 1 we can obtain the solution of Biernacki's problem. Namely, we can determine the number $r(n, S(m, M))$ for which the implication (1) holds.

Immediately from Theorem 1 we have the following theorem.

Theorem 2. *Let* $f(z) \in H_n$, $n \geqslant 2$, *and* $F(z) \in S(m, M)$. *If* $f(z) \dashv_1 F(z)$ *then*

$$f(z) \leqslant_{r(n, S(m, M))} F(z),$$

where

$$r(n, S(m, M)) = \inf\{r\colon r \in (0, 1), T(r, n, S(m, M)) > 1\}.$$

Remark 4. In many cases the number $r(n, S(m, M))$ is the smallest positive root of the equation

$$m(r) = M(r^n).$$

It is true in this case when the smallest root of this equation is isolated from the other roots of the equation.

Remark 5. For many classes S_0 of regular functions the estimations on $|F(z)|$ are known and therefore we know the class $S(m, M)$ which contains this given class S_0. Thus for this class we have

$$\mathscr{R}(r, n, S_0) \leqslant T(r, n, S(m, M)).$$

If the extremal function mentioned in Remark 2 belongs to the class S_0, then

$$\mathscr{R}(r, n, S_0) = T(r, n, S(m, M)).$$

Let us put $S_0 = S_a^*$ where S_a^* denotes the class of functions starlike of order a, $F(0) = F'(0)-1 = 0$, $\operatorname{Re}\{zF'(z)/F(z)\} > a$ for $z \in K_1$, $0 \leqslant a < 1$. Then for $|z| = r < 1$ we have (see for example [2])

$$(7) \quad m(r) \leqslant |F(z)| \leqslant M(r)$$

where

$$m(r) = \frac{r}{(1+r)^{2(1-a)}},$$

(8)

$$M(r) = \frac{r}{(1-r)^{2(1-a)}}.$$

Thus we have the following

Theorem 3.

$$\mathscr{R}(r, n, S_a^*) = T\big(r, n, S(m, M)\big) = r^{n-1}\left(\frac{1+r}{1-r}\right)^{2(1-a)}$$

where $m(r)$, $M(r)$ *are given by* (8).

Proof. From (7) and (8) we have that $S_a^* \subset S(m, M)$. The function

$$F_e(z) = \frac{z}{(1-z)^{2(1-a)}} \in S_a^*$$

and

$$|F_e(-r)| = m(r), \; |F_e(r)| = M(r)$$

where $m(r)$, $M(r)$ are given by (8). Now Theorem 3 follows from the Remark 5.

In the special cases, $a = 0$ and $a = 1/2$ we obtain the functions $\mathscr{R}(r, n, S_0)$ and $\mathscr{R}(r, n, S_{1/2}^*)$ respectively if $n \geqslant 2$.

In the mentioned cases the results of paper [3] and our paper are the same although in the paper [3] the authors gave an additional condition $a_n \geqslant 0$.

The method used in [3] based on the domain of variability of $F(z)/F(\xi)$ when $F(z)$ ranges over the class S_a^*. Therefore the authors could not determine $\mathscr{R}(r, n, S_a^*)$ for other a because this domain of variability is not known.

Because in our method we need only $\sup\limits_{|z|=r^n, |\xi|=r} \left|\frac{F(z)}{F(\xi)}\right|$ or a corresponding infimum thus we can determine $\mathscr{R}(r, n, S_a^*)$ for all $a \in \langle 0, 1)$ and for some other classes which were not investigated in [3].

Corollary 1. *For* $n \geqslant 2$ *we have*

$$\mathscr{R}(r, n, S) = \mathscr{R}(r, n, S_0^*) = r^{n-1}\left(\frac{1+r}{1-r}\right)^2$$

Proof. It is easy to see that

$$S_0^* \subset S \subset S(m, M)$$

where $m(r)$, $M(r)$ are given as in the Remark 1. Thus

$$\mathscr{R}(r, n, S_0^*) \leqslant \mathscr{R}(r, n, S) \leqslant T(r, n, S(m, M)) = \mathscr{R}(r, n, S^*)$$

and Corollary 1 is proved.

It is known that if $F(z) \in S^c = \left\{f(z)\colon f(0) = f'(0) - 1 = 0,\ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ in } K_1\right\}$ and $|z| = r < 1$ then

$$m(r) \leqslant |F(z)| \leqslant M(r)$$

where

$$m(r) = \frac{r}{1+r}, \quad M(r) = \frac{r}{1-r}. \tag{9}$$

The function $F_e = \frac{z}{1-z} \in S^c \subset S(m, M)$ is extremal in the class $S(m, M)$ where $m(r)$, $M(r)$ are given by (9).

From the above and Remark 5 we have

Corollary 2. *If $n \geqslant 2$ then*

$$\mathscr{R}(r, n, S^c) = r^{n-1}\left(\frac{1+r}{1-r}\right).$$

By Remark 5 we can extend the Theorem 2 on the classes $S_0 \subset S(m, M)$. Using the Theorem 3, by Remarks 4 and 5 we obtain immediately

Corollary 3. *If $f(z) \in H_n$, $n \geqslant 2$, $F(z) \in S_\alpha^*$, $0 \leqslant \alpha < 1$ and $f(z) \dashv_1 F(z)$ then*

$$f(z) \ll_{r(n, S_\alpha^*)} F(z)$$

where $r(n, S_\alpha^*)$ is the smallest positive root of the equation

$$\left(\frac{1-r^n}{1+r}\right)^{2(1-\alpha)} = r^{n-1}.$$

For $S_0 = S^c$ by Theorem 2 using the equalities (9) we obtain

Corollary 4. *If $f(z) \in H_n$, $n \geqslant 2$, $F(z) \in S^c$ and $f(z) \dashv_1 F(z)$ then*

$$f(z) \ll_{r(n, S^c)} F(z)$$

where $r(n, S^c)$ *is the smallest positive root of the equation*

$$2r^n + r^{n-1} - 1 = 0.$$

In the case $S_0 = S$ we can use the Corollary 1 and then by Theorem 2 we have

Corollary 5. *If* $f(z) \in H_n$, $n \geqslant 2$, $F(z) \in S$ *and* $f(z) \prec_1 F(z)$ *then*

$$f(z) \ll_{r(n,S)} F(z)$$

where $r(n, S) = r(n, S_0)$ *is the smallest positive root of the equation*

$$r^{2n} - r^{n+1} - 4r^n - r^{n-1} + 1 = 0.$$

The Corollaries 3, 4 and 5 coincitde with corresponding results of [2].

REFERENCES

[1] Biernacki M. *Sur quelques majorantes de la théorie des fonctions univalentes*, C. R. Acad. Sci. Paris Ser. A-B 201 (1935), 256-258.

[2] Bogowski, F. Jabłoński F. F., Stankiewicz J., *Subordination en domaine et inégalités des modules pour certaines classes de fonctions holomorphes dans le cercle unité*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 20 (1966), 23-28.

[3] Bogucki Z., Waniurski, J., *The relative growth of subordinate functions*, Michigan Math. J. 18 (1971), 357-363.

STRESZCZENIE

Niech $f(z), F(z)$ będą funkcjami regularnymi w K_1, $K_r = \{z: |z| < r\}$. Jeżeli istnieje funkcja regularna $\omega(z)$, $|\omega(z)| \leqslant |z|$, $z \in K_r$, taka, że $f(z) = F(\omega(z))$ dla $z \in K_r$ to mówimy, że $f(z)$ jest podporządkowana $F(z)$ w K_r i piszemy $f(z) \prec_r F(z)$. Jeżeli $|f(z)| \leqslant |F(z)|$ dla $z \in K_r$ to mówimy, że $f(z)$ jest zmajoryzowana przez $F(z)$ w K_r i piszemy $f(z) \ll_r F(z)$.

Problem znalezienia możliwie najlepszej liczby r takiej, żeby zachodziła implikacja

$$f(z) \prec_1 F(z) \Rightarrow f(z) \ll_r F(z)$$

gdy $F(z)$ przebiega ustaloną klasę S_0 został zapoczątkowany w 1935 r przez M. Biernackiego [1]. Z. Bogucki i J. Waniurski [3] badali nieco ogólniejszy problem, chodziło o wyznaczenie funkcji $T(r, n, S_0)$ takiej, aby prawdziwa była implikacja

$$f(z) \prec_1 F(z) \Rightarrow |f(z)| \leqslant T(|z|, n, S_0)|F(z)| \text{ dla } z \in K_1.$$

Podali oni pewną metodę ogólną i stosując ją wyznaczyli funkcję $T(r, n, S_0$ dla pewnych podklas funkcji jednolistnych.

W tej pracy podana jest inna metoda ogólna pozwalająca na wyznaczenie $T(r, n, S_0)$ przy $n \geqslant 2$ dla wielu klas S_0 dla których nie można było tej funkcji wyznaczyć metodą z pracy [3]. Wydaje się, że metoda podana w tej pracy pozwala na wyznaczenie możliwie najlepszej takiej funkcji dla bardzo wielu klas funkcji regularnych nawet niekoniecznie jednolistnych. Uzyskane rezultaty w szczególnych przypadkach dają niektóre wyniki uzyskane w pracy [2] dla pierwszego ze wspomnianych problemów i w pracy [3] dla drugiego problemu.

РЕЗЮМЕ

Пусть $f(z)$, $F(z)$ будут регулярными функциями в K_r, $K_r = \{z\colon |z| < r\}$. Если существует регулярная функция $\omega(z)$, $|\omega(z)| \leqslant |z|$, $z \in K_r$ такая, что $f(z) = F(\omega(z))$ для $z \in K_r$ тогда говорим, что $f(z)$ является подчиненной $F(z)$ в K_r и пишем $f(z) \prec_r F(z)$. Если $|t(z)| \leqslant F(z)|$ для $z \in K_r$ тогда говорим, что $f(z)$ мажорна $F(z)$ в K_r и пишем $f(z) \leqslant_r F(z)$.

Вопрос отыскания возможно самого лучшего числа r, такого чтобы наступила импликация

$$f(z) \prec_1 F(z) \Rightarrow f(z) \ll_r F(z)$$

когда $F(z)$ проходит определенный класс S_0, был поднят М. Бернацким в 1935 году. [1]. З. Богутски и Ю. Ванюрски [3] исследовали немного общую проблему, хотели определить такую функцию $T(r, n, S_0)$, для которой была бы исполнена импликация

$$f(z) \prec_1 F(z) \Rightarrow |f(z)| \leqslant T(|z|, n, S_0)|F(z)| \text{ для } z \in K_1.$$

Представили они общий метод и используя его определили функцию $T(r, n, S_0)$ для некоторых подклассов однолистных функций.

В данной работе представлен другой общий метод, дающий возможность определить $T(r, n, S_0)$ при $n \geqslant 2$ для многих классов S_0 для которых не было возможно определить эту функцию методом взятым из данной работы [3]. Кажется, что метод представлен в этойн работе дает возможность найлучше определить такую функцию для многих классов регулярных функций и не всегда однолистных. Полученные результаты в особых случаях совпадаются с результатами полученными в работе [2] для первой из вспоминаемых проблем и в работе [3] для другой проблемы.