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On Characterization of Chebyshev Optimal Starting and Transformed Approximations by Families Having a Degree

Twierdzenie charakteryzacyjne dla optymalnych aproksymacji startowych i transformowanych elementami rodzin nieliniowych

Теоремы характеризующие стартерные и трансформированные оптимальные апроксимации

1. Introduction.

Let $C[a, b]$ be the space of real valued functions defined and continuous on $[a, b]$ normed by

$$\|f\| = \max\{|f(x)|: x \in [a, b]\}.$$

Denote by G a nonlinear approximating family of functions from $C[a, b]$. The following definitions are given in [7] (see also [3]).

Definition 1. The family G has property Z of degree n at $g \in G$ if for every $h \in G$ the function $(h-g)$ has at most $(n-1)$ zeros on $[a, b]$ or vanishes identically.

Definition 2. G has property A of degree n at $g \in G$ if given

(i) an integer m, $0 \leqslant m < n$

(ii) a set $\{x_1, \ldots, x_m\}$ with $a = x_0 < x_1 < \ldots < x_m < x_{m+1} = b$

(iii) ε with $0 < \varepsilon < \frac{1}{2} \min\{x_{j+1} - x_j: j = 0, \ldots, m\}$, and

(iv) a sign $\sigma \in \{-1, 1\}$,

there exists $h \in G$ with $\|h - g\| < \varepsilon$ and

$$\operatorname{sign}[h(x) - g(x)] = \begin{cases} \sigma, & a \leqslant x \leqslant x_1 - \varepsilon \\ (-1)^i \sigma, & x_i + \varepsilon \leqslant x \leqslant x_{i+1} - \varepsilon \text{ and } i = 1, \ldots, m-1 \\ (-1)^m \sigma, & x_m + \varepsilon \leqslant x \leqslant b. \end{cases}$$

In the case $m = 0$, we require

$$\operatorname{sign}[h(x)-g(x)] = \sigma, \ a \leqslant x \leqslant b.$$

Definition 3. The family G has degree n at $g \in G$ if G has property Z and property A of degree n at g.

Definition 4. A zero x of a continuous function f on $[a, b]$ is called a double zero if x is an interior point of $[a, b]$ and f does not change sign at x. Otherwise, it is called a simple zero.

Definition 5. The points a_i, $a \leqslant a_0 < \ldots < a_n \leqslant b$ are called alternation points of the function f if $f(a_i) = (-1)^i f(a_0) \neq 0$ for $i = 1, \ldots, n$.

In the paper [9] we have introduced the nonlinear family of approximating functions having the weak betweenness property and have presented some of its properties.

Definition 6. A subset G of $C[a, b]$ has the weak betweenness property if for any two distinct elements g and h in G and for every closed subset D of $[a, b]$ such that $h(x) \neq g(x)$ for all $x \in D$ there exists a sequence $\{g_i\}$ of elements of G such that

(i) $\lim\limits_{i\to\infty} \|g - g_i\| = 0$

(ii) numbers $g_i(x)$, where $x \in D$ and $i = 1, 2, \ldots$, lie strictly between $g(x)$ and $h(x)$ (i.e. either

$$g(x) < g_i(x) < h(x) \ \text{or} \ h(x) < g_i(x) < g(x)).$$

Let us assume that the operator Φ: $K \to C[a, b]$ is defined and continuous on the set $K \subset C[a, b]$ and that M is an arbitrary nonempty subset of K.

Definition 7 (see [6]). The element $g \in M$ is said to be an optimal starting approximation in M for $f \in K$ if $\|\Phi(f) - \Phi(g)\| \leqslant \|\Phi(f) - \Phi(h)\|$ for all $h \in M$. The approximation of this type was considered in papers [4, 6, 8, 9].

Definition 8 (see [1, 2]).

The element $g \in M$ is called the optimal transformed approximation in M for $f \in C[a, b]$ if

$$\|f - \Phi(g)\| \leqslant \|f - \Phi(h)\|$$

for all $h \in M$.

The optimal transformed approximation, with respect to M equal to polynomial and rational families and Φ equal to an ordered function, was considered by Dunham in [1, 2]. The main purpose of this paper is to prove the alternation theorems for optimal starting and transformed approximations, with M equal to a nonempty subset of family G having a degree at all $g \in G$. In particular, we shall generalize Theorems 3.2 and 2 characterizing the optimal starting approximation from [4] and [6] respectively.

Additionally, we shall obtain characterization theorems of Kolmogorov type for optimal transformed approximation by families with weak beweenness property. These theorems are similar to characterization thetorems from [9].

2. Optimal starting aproximation

Let us denote by $D(g)$, where $g \in G$, the closed subset of $[a, b]$ defined by

$$D(g) = \{x\colon\ |\Phi(f)(x) - \Phi(g)(x)| = \|\Phi(f) - \Phi(g)\|\}.$$

n this section the following definitions from [6] will be useful.

Definition 9. The operator Φ is called pointwise strictly monotone at $f \in K$ if for each $h, g \in K$ we have

$$|\Phi(f)(z) - \Phi(h)(z)| < |\Phi(f)(z) - \Phi(g)(z)|$$

for each $z \in [a, b]$, where either $g(z) < h(z) \leqslant f(z)$ or $f(z) \leqslant h(z) < g(z)$

Definition 10. The operator Φ is said to be pointwise fixed at $f \in K$, if $h \in K$ with $h(z) = f(z)$ for $z \in [a, b]$ implies $\Phi(h)(z) = \Phi(f)(z)$.

Now we shall prove two lemmas characterizing the family G having a degree.

Lemma 1. *Let the family G have a degree at all $g \in G$. Then G has weak betweenness property.*

Proof. Let g, h be two arbitrary distinct elements of G and let n denote a degree at g. Thus there exists k, $k < n$, simple zeros x_j of $(h-g)$ in (a, b). Let D be any closed subset of $[a, b]$ such that $\delta = \min\{|h(x) - g(x)|\colon x \in D\} > 0$. If $k = 0$ then setting in Definition 2 $\sigma = \operatorname{sign}\{h(x) - g(x)\colon x \in D\}$ we conclude that for every ε, $0 < \varepsilon < \delta$, there exists $p \in M$ such that $\|p - g\| < \varepsilon$ and $p(x)$ lies strictly between $g(x)$ and $h(x)$ for every $x \in D$. From this the thesis of the lemma is obvious, because we may select g_i in Definition 6 which corresponds to $\varepsilon = \dfrac{1}{s+i}$, where an integer s is such that $0 < \dfrac{1}{s} < \delta$. Otherwise, suppose that an integer l is so chosen that sets $\left(x_i - \dfrac{1}{\nu}, x_i + \dfrac{1}{\nu}\right) \cap D$, $i = 1, \ldots, k$, are empty for all $\nu \geqslant 1$.

From Definition 2 for each $\nu \geqslant 1$, $0 < \varepsilon < \min\left\{\frac{1}{\nu}, \delta\right\}$ and $\sigma = \operatorname{sign}\{h(x) - g(x): x \in D \cap [a, x_1)\}$ there exists g_ν which lies strictly between $g(x)$ and $h(x)$ for all $x \in D$ and $\|g_\nu - g\| < \varepsilon$. Hence the family G has the weak betweenness property and the proof is completed.

Lemma 2. *Let g be an arbitrary fixed distinct element of G and let $e \in C[a, b]$. Assume that G has a degree n at g. Let D be a nonempty closed subset of $[a, b]$ such that $e(x) \neq 0$ for all $x \in D$. Then the following three conditions are equivalent:*

(i) *the set D contains at least $(n+1)$ alternation points of the function e.*

(ii) *there does not exist any element $h \in G$ such that the inequality*

$$e(x)[h(x) - g(x)] > 0 \tag{1}$$

is satisfied for all $x \in D$.

(iii) *there does not exist any element $h \in G$ distinct from g such that the inequality*

$$e(x)[h(x) - g(x)] \geqslant 0 \tag{2}$$

is satisfied for all $x \in D$.

Proof. The fact that condition (i) implies (ii) by property Z is obvious. Now we shall prove that (ii) implies (iii). Let us suppose on the contrary that there exists an element $h \in G$ distinct from g such that the inequality (2) is satisfied for all $x \in D$. Let $z_1, \ldots, z_k$, $k < n$, be simple zeros of the function $(h - g)$ in (a, b). If $k = 0$ then the proof follows immediately from Definition 2. Otherwise, renumbering if necessary z_i, we assume that $z_i \in D$ for $i = 1, \ldots, l$, where $l \leqslant k$. From the continuity of all considered functions and inequality (2) it follows that for sufficiently small $\lambda > 0$ there exist the sets $0_\lambda(z_i)$ equal for $i = 1, \ldots, l$ to $(z_i - \lambda, z_i)$ or $(z_i, z_i + \lambda)$ and for $i = l+1, \ldots, k$ to $(z_i - \lambda, z_i + \lambda)$ such that $0_\lambda(z_i) \cap D = \emptyset$. Let x_i be the mid-points of intervals $0_\lambda(z_i)$ for $i = 1, \ldots, k$. Denote $\sigma = \operatorname{sign}\{h(x) - g(x): x \in (a, z_1)$ and $h(x) \neq g(x)\}$. From Definition 2 for every $0 < \varepsilon < \frac{\lambda}{2}$ there exists $p \in G$ such that $[p(x) - g(x)][h(x) - g(x)] \geqslant 0$ and $p(x) \neq g(x)$ for all $x \in [a, b] \setminus \bigcup_{i=1}^{k} 0_\lambda(z_i) \supset D$. Hence setting $h = p$ in (1) we obtain the contradiction of (ii). Finally, we prove that (iii) implies (i). Let us suppose on the contrary that the set D contains exactly k, $k \leqslant n$, alternation points a_i, $i = 0, \ldots, k-1$ of the function e. If $k = 1$ then setting $\sigma = \operatorname{sign}\{e(x): x \in D\}$ we conclude from Definition 2 that there exists an element $h \in G$ such that $\sigma = \operatorname{sign}\{h(x) - g(x): x \in [a, b]\}$. Hence the proof is completed. Otherwise, let x_i denote arbitrary fixed zeros of e in intervals (a_{i-1}, a_i), $i = 1, \ldots, k-1$. Additionally, let $\sigma = \operatorname{sign}\{e(x): x \in [a_0, x_1) \cap D\}$ and let $\varepsilon > 0$ be so chosen that $(x_i - \varepsilon, x_i + \varepsilon) \cap D = \emptyset$ for $i = 1, \ldots, k-1$. For these ε, σ and x_i let $h \in G$ be an element defined by Definition 2. Obviously, inequality (2) with this h is satisfied for all $x \in D$. This gives a contradiction, and the lemma is proven.

From Lemmas 1 and 2 in this paper and Theorems 3 and 4 from [9] we immediately obtain the following theorem which generalizes Theorem 3.2 and 2 from [4] and [6] respectively.

Theorem 1. *Let $\Phi: K \to C[a, b]$ be a continuous operator and let G have a degree at all $h \in G$. Fix an element $g \in G$ and denote by n the degree of G at g. Let $M = K \cap G$ be a nonempty relatively open subset of G and let $e = f - g$, where $f \in K \setminus M$. Finally assume that Φ is pointwise strictly monotone and pointwise fixed at f. Then the following four conditions are equivalent:*

(i) *the element g is an optimal starting approximation to f.*

(ii) *there does not exist any element $h \in G$ such that inequality* (1) *is satisfied for all $x \in D(g)$.*

(iii) *there does not exist any element $h \in G$ distinct from g such that inequality* (2) *is satisfied for all $x \in D(g)$.*

(iv) *the set $D(g)$ contains at least $(n+1)$ alternation points of the function e.*

3. Optimal transformed approximation.

Let us denote by $B(g)$, where $g \in G$, the closed subset of $[a, b]$ defined by

$$B(g) = \{x: |f(x) - \Phi(g)(x)| = \|f - \Phi(g)\|\}.$$

In this section the following definitions will be useful.

Definition 11. The operator $\Phi: K \to C[a, b]$ is said to be pointwise strictly increasing at $g \in M$ if for each $h \in M$ and $x \in [a, b]$ the inequality $g(x) < h(x)$ $(g(x) > h(x))$ implies that

$$\Phi(g)(x) < \Phi(h)(x) \; (\Phi(g)(x) > \Phi(h)(x)).$$

The operator Φ is said to pointwise strictly monotone at $g \in M$ if Φ or $-\Phi$ is pointwise strictly increasing at g. If the operator $\pm\Phi$ is a pointwise increasing at $g \in M$ then we set $\sigma = \pm 1$. The ordered functions [2] and more general transformations considered in [5] are examples of operators pointwise strictly monotone at g, where g and M may be arbitrary chosen. For other examples see [6]. In particular, the operator Φ may be equal to the identity operator.

Theorem 2. *Let $\Phi: K \to C[a,b]$ be a continuous operator. Let G be an arbitrary subset of $C[a,b]$ having weak betweenness property and let $M = K \cap \cap G$ be a nonempty relatively open subset of G. Finally assume that Φ is pointwise strictly monotone at $g \in M$. Then a necessary condition for g to be an optimal transformed approximation, with respect to $f \in C[a,b] \setminus M$ is that there does not exist any element $h \in G$ such that*

$$\sigma[f(x) - \Phi(g)(x)][h(x) - g(x)] > 0 \tag{3}$$

for all $x \in B(g)$.

Proof. Let us suppose on the contrary that there exists $h \in G$ such that inequality (3) is satisfied for all $x \in B(g)$. Then for $x \in B(g)$ we have either

$$f(x) > \Phi(g)(x) \text{ and } \sigma h(x) > \sigma g(x)$$

or

$$f(x) < \Phi(g)(x) \text{ and } \sigma h(x) < \sigma g(x).$$

From the continuity of all considered functions there exists the open set $E \supset B(g)$ such that the last inequalities hold for all $x \in \bar{E}$. Because G has the weak betweenness property and M is open in G then there exists the sequence g_i of elements of M such that $g_i(x)$ lies strictly between $\sigma h(x)$ and $\sigma g(x)$ for all $x \in \bar{E}$ and g_i is convergent uniformly on $[a,b]$ to g. Now, from the pointwise monotonicity of Φ at g and the continuity of the operator Φ it follows that there exists an integer m such that $\Phi(g_i)(x)$ for all $i \geqslant m$ and $x \in \bar{E}$ lies strictly between $f(x)$ and $\Phi(g)(x)$. Hence

$$|f(x) - \Phi(g_i)(x)| < |f(x) - \Phi(g)(x)| = \|f - \Phi(g)\| \tag{4}$$

for all $i \geqslant m$ and $x \in \bar{E}$. If $\bar{E} = [a,b]$ then the proof is completed. Otherwise, let us set $V = X \setminus E$ and

$$\delta = \max\{|f(x) - \Phi(g)(x)|:\ x \in V\}.$$

Obviously V is a compact set. Since $V \cap B(g) = \emptyset$, thus $\|f - \Phi(g)\| > \delta$. From the continuity of Φ and uniform convergence g_i to g it follows that there exists an integer k, $k \geqslant m$, such that $\|\Phi(g) - \Phi(g_i)\| < \|f - \Phi(g)\| - \delta$ for all $i \geqslant k$. Hence for all $x \in V$ and $i \geqslant k$ we obtain

$$\begin{aligned}|f(x) - \Phi(g_i)(x)| &\leqslant |f(x) - \Phi(g)(x)| + |\Phi(g)(x) - \Phi(g_i)(x)| \\ &< \delta + \|f - \Phi(g)\| - \delta = \|f - \Phi(g)\|.\end{aligned}$$

Combining this result with (4) we have

$$\|f - \Phi(g_i)\| < \|f - \Phi(g)\| \quad \text{for all } i \geqslant k.$$

This gives a contradiction.

Theorem 3. *Let M be an arbitrary nonempty subset of K and let the operator Φ be pointwise monotone at g. Then a sufficient condition for $g \in M$ to be an optimal transformed approximation to $f \in C[a, b] \setminus M$ is that there does not exists any element $h \in M$ such that*

$$(5) \qquad \sigma[f(x)-\Phi(g)(x)][h(x)-g(x)] \geqslant 0$$

for all $x \in B(g)$.

Proof. Suppose on the contrary that there exists an $h \in M$ such that $\|f-\Phi(h)\| < \|f-\Phi(g)\|$. Hence for all $x \in B(g)$ we have

$$(6) \qquad |f(x)-\Phi(h)(x)| < |f(x)-\Phi(g)(x)|.$$

Now, we must have for $x \in B(g)$ either $f(x) > \Phi(g)(x)$ and $\sigma h(x) \geqslant \sigma g(x)$ or $f(x) < \Phi(g)(x)$ and $\sigma h(x) \leqslant \sigma g(x)$. Indeed, otherwise from the pointwise monotonicity of Φ at g we obtain that $\Phi(g)(x)$ lies strictly between $f(x)$ and $\Phi(h)(x)$ for all $x \in B(g)$. This gives a contradiction of (6). Combining the above inequalities for functions f, $\Phi(g)$, σg and σh we obtain that the inequality (5) is satisfied for all $x \in B(g)$. This completes the proof.

Theorem 4. *Under the assumptions of Theorem 2 and the additiona assumption that*

$$(7) \qquad h(x) = g(x) \text{ implies } \Phi(h)(x) = \Phi(g)(x) \text{ for all } h \in M$$

a necessary and sufficient condition for $g \in M$ to be a transformed approximation to $f \in C[a, b] \setminus M$ is that there does not exist any element $h \in G$ such that

$$\sigma[f(x)-\Phi(g)(x)][h(x)-g(x)] > 0$$

for all $x \in B(g)$.

Proof. From Theorems 2 and 3 and from the fact that the equality $h(x) = g(x)$ for an $x \in B(g)$ in the proof of Theorem 3 from condition (7) is mpossible we immediately obtain the proof of this theorem.

Note that condition (7) is satisfied if the operator Φ is the identity oqerator, ordered function [2] or transformation from [5]. Finally from Lemma 1 and 2 and Theorems 2 and 3 we have the theorem.

Theorem 5. *Let $\Phi: K \to C[a, b]$ be a continuous operator and let G have a degree at all $h \in G$. Fix an element $g \in G$ and denote by n the degree of G at g. Let $M = K \cap G$ be nonempty relatively open subset of G and let $e = f - \Phi(g)$, where $f \in C[a, b] \setminus M$. Finally assume that Φ is pointwise strictly monotone at g. Then the following four conditions are equivalent:*

(i) *the element g is an optimal transformed approximation to f.*

(ii) *there does not exist any element $h \in G$ such that the inequality $\sigma e(x)\,[h(x)-g(x)] > 0$ is satisfied for all $x \in B(g)$.*

(iii) *there does not exist any element $h \in G$ distinct from g such that the inequality $\sigma e(x)[h(x)-g(x)] \geqslant 0$ is satisfied for all $x \in B(g)$.*

(iv) *the set $B(g)$ contains at least $(n+1)$ alternation points of the function e.*

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STRESZCZENIE

W pracy udowodniono twierdzenia o alternansie, charakteryzujące nieliniową optymalną aproksymację startową i transformowaną. Ponadto, dla optymalnej aproksymacji transformowanej podane zostały twierdzenia charakteryzacyjne typu Kołmogorowa.

РЕЗЮМЕ

В данной работе доказано теоремы о альтернансе, характеризирующие нелинейную стартерную и трансформированную оптимальную апроксимацию. Кроме того, для оптимальной трансформированной апроксимации представлены характеризующие теоремы типа Колмогорова.