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Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Lublin

PIOTR BORÓWKO

Some Remarks on the Stability of Ordinary Differential Equations under Persistent Random Disturbances

Pewne uwagi o stabilności równań różniczkowych zwyczajnych względem stale działających zakłóceń losowych

Некоторые примечания об устойчивости обыкновенных дифференциальных уравнений относительно постоянно действующих случайных возмущений

The aim of this paper is to show some results on the stability of ordinary differential equations under persistent random disturbances. H. Bunke [1] discussed the similar problems and gave some conditions for mean square stability as well as for W- and P-stability. The interesting thing is that the mean square stability was investigated without using the typical, for this kind of problems, methods based on Lyapunov function.

In this paper we show similar result for W-stability. Moreover, we define the notion of „equal sample stability under persistent random disturbances" for which analogous results hold and which is related to W-stability.

Let $f(x, t)$ be a continuous function

$$f\colon R^n \times T \to R^n,\ T = \langle 0, \infty)$$

such that $f(0, t) = 0$, $t \in T$. Suppose the differential equation

$$\frac{dx}{dt} = f(x, t) \tag{1}$$

has for each initial point $(x_0, t_0) \in R^n \times T$ an unique solution on T. Suppose, moreover, that the origin is an isolated singular point of (1).

Now assume that $(\Omega, \mathfrak{A}, P)$ is a probability space and denote by M the class of all n-dimensional stochastic processes $g(x, t, \omega)$ defined on $R^n \times T \times \Omega$ with the following properties:

— for almost all $\omega \in \Omega$, g is a continuous function on $R^n \times T$,

— for each initial condition $(x_0, t_0) \in R^n \times T$, there exists an unique sample solution X_t, $t \in T$ of the stochastic differential equation

$$\frac{dX_t}{dt} = f(X_t, t) + g(X_t, t, \omega). \tag{2}$$

Corresponding definitions and existence theorems can be found e.g. in [2].

Let $T_0 = \langle t_0, \infty)$.

Definition 1: The trivial solution of the differential equation (1) is called to be equally sample stable under persistent random disturbances from M, if

$$\bigwedge_{t_0 \geqslant 0} \bigwedge_{\varepsilon > 0} \bigvee_{\delta > 0} \bigvee_{\eta > 0} \bigvee_{\substack{\Omega^* \in \mathfrak{A} \\ P(\Omega^*) = 1}} \bigwedge_{\omega \in \Omega^*} \Big[\big(\|x_0\| < \delta,\ g \in M,\ \sup_{(x,t) \in R^n \times T_0} \|g(x, t, \omega)\| < \eta \big) \Rightarrow \big(\|X_t(\omega)\| < \varepsilon,\ t \geqslant t_0 \big) \Big].$$

Definition 2: The trivial solution of the differential equation (1) is called to be W-stable under persistent random disturbances from M, if

$$\bigwedge_{t_0 \geqslant 0} \bigwedge_{\varepsilon > 0} \bigwedge_{p \in (0,1)} \bigvee_{\delta > 0} \bigvee_{\eta > 0} \Big[\big(\|x_0\| < \delta,\ g \in M,\ E\big\{ \sup_{(x,t) \in R^n \times T_0} \|g(x, t, \omega)\| \big\} < \eta \big) \Rightarrow P\{ \|X_t(\omega)\| < \varepsilon,\ t \geqslant t_0 \} > p \Big].$$

Theorem 1: *Suppose there exists a positive L such that*

$$\bigwedge_{t \in T} \bigwedge_{x, \bar{x} \in R^n} \|f(x, t) - f(\bar{x}, t)\| \leqslant L \|x - \bar{x}\|.$$

If the trivial solution of (1) is uniformly asymptotically stable then it is equally sample stable under persistent random disturbances from M.

Proof: Recall first some results on stability of ordinary differential equation

$$\frac{dx}{dt} = f(x, t),\ f\colon B \times T \to B \tag{3}$$

where B is a Banach space. If f is lipschitzean with respect to x and continuous, a solution of a differential equation

$$\frac{dy}{dt} = f(y, t) + g(y, t),\ g\colon\ B \times T \to B \tag{4}$$

exist, and the trivial solution of the equation (3) is uniformly asyptotically stable i.e.

$$\begin{gathered} \bigwedge_{\varepsilon > 0} \bigvee_{\delta > 0} \bigwedge_{t_0 \geqslant 0} \big[\|x(t_0)\| < \delta \Rightarrow \big(\|x(t)\| < \varepsilon,\ t \geqslant t_0 \big) \big] \\ \bigvee_{\tilde{\varepsilon} > 0} \bigwedge_{\tilde{\delta} > 0} \bigvee_{\tau > 0} \bigwedge_{t_0 \geqslant 0} \big(\|x(t_0)\| < \tilde{\varepsilon} \Rightarrow \|x(t_0 + \tau)\| < \tilde{\delta} \big) \end{gathered} \tag{5}$$

then it is also stable under persistent disturbances i.e. the equation (4) satisfies

$$(6)\quad \bigwedge_{t_0\geqslant 0}\bigwedge_{\varepsilon>0}\bigvee_{\delta>0}\bigvee_{\eta>0}\Big[\big(\|y(t_0)\|<\delta,\quad \sup_{(y,t)\in B\times T_0}\|g(y,t)\|<\eta\big)\Rightarrow\big(\|y(t)\|<\varepsilon,\ t\geqslant t_0\big)\Big].$$

The proof of it can be found e.g. in [3].

For any choosen $t_0 \geqslant 0$ and $\varepsilon > 0$, find δ and η which fulfills (6). Let Ω^* be the set of all $\omega \in \Omega$ for which the sample functions of sample solution X_t of the equation (2) with the initial condition (x_0, t_0) are solutions of the ordinary differential equations obtained from (2), by fixing of ω. It follows from (6) that for all $\omega \in \Omega^*$ if

$$\|x_0\| < \delta,\ g \in M,\quad \sup_{(x,t)\in R^n\times T_0}\|g(x,t,\omega)\| < \eta$$

then

$$\|X_t(\omega)\| < \varepsilon,\ t \geqslant t_0,$$

q.e.d.

Lemma: *Assume that the trivial solution of* (1) *is equally sample stable under persistent random disturbances from M. Then it is also W-stable under the same disturbances.*

Proof: In view of Definition 1 we have

$$(7)\quad \bigwedge_{t_0\geqslant 0}\bigwedge_{\varepsilon>0}\bigvee_{\delta>0}\bigvee_{\bar\eta>0}\bigvee_{\substack{\Omega^*\in\mathfrak{A}\\ P(\Omega^*)=1}}\bigwedge_{\omega\in\Omega^*}\Big[\big(\|x_0\|<\delta,\ g\in M,\ \sup_{(x,t)\in R^n\times T_0}\|g(x,t,\omega)\|<\bar\eta\big)\Rightarrow\big(\|X_t(\omega)\|<\varepsilon,\ t\geqslant t_0\big)\Big].$$

Choose arbitrary $t_0 \geqslant 0$, $\varepsilon > 0$ and $p \in (0, 1)$. Now find $\delta > 0$ and $\bar\eta > 0$ fulfiling (7). Next take η such that $0 < \eta < (1-p)\bar\eta$, and assume

$$(8)\quad \|x_0\| < \delta,\ g \in M,\ E\{\sup_{(x,t)\in R^n\times T_0}\|g(x,t,\omega)\|\} < \eta.$$

Then in view of Chebyshev inequality, we get $\|x_0\| < \delta$, $g \in M$ and

$$P\{\sup_{(x,t)\in R^n\times T_0}\|g(x,t,\omega)\| < \bar\eta\} \geqslant 1 - \frac{E\{\sup_{(x,t)\in R^n\times T_0}\|g(x,t,\omega)\|\}}{\bar\eta}$$

$$> 1 - \eta/\bar\eta > 1 - (1-p)\bar\eta/\bar\eta = p.$$

Hence

$$\|x_0\| < \delta,\ g \in M,\ \bigvee_{\substack{\bar\Omega\in\mathfrak{A}\\ P(\bar\Omega)>p}}\bigwedge_{\omega\in\bar\Omega}\sup_{(x,t)\in R^n\times T_0}\|g(x,t,\omega)\| < \bar\eta.$$

And comparing it with (7) we get

$$\bigwedge_{\omega\in\bar{\Omega}\cap\Omega^*} (\|X_t(\omega)\| < \varepsilon,\ t \geqslant t_0).$$

Certainly the set $\bar{\Omega}\cap\Omega^*$ is measurable and $P(\bar{\Omega}\cap\Omega^*) > p$. So for arbitrary $t_0 \geqslant 0$, $\varepsilon > 0$, $p \in (0, 1)$ we found $\delta > 0$ and $\eta > 0$ such that (8) implies

$$P\{\|X_t(\omega)\| < \varepsilon,\ t \geqslant t_0\} > p,$$

q.e.d.

Theorem 2: *Suppose f satiefiss the same conditions as in Theorem 1. If the trivial solution of the equation* (1) *is uniformly asymptotically stable then it is W-stable under persistent random disturbances from M.*

The proof is a simple consequence of Theorem 1 and our Lemma.

Remark: In the same manner we can prove that the uniformly asymptotic stability implies even the uniform equal sample stability under persistent random disturbances from M. Also the uniformly asymptotic stability implies uniform W-stability under persistent random disturbances from M. To prove it it is enough to notice that the dependence on t_0 is easy to eliminate.

REFERENCES

[1] Bunke, H., *On the stability of ordinary differential equations under persistent random disturbances*, Zeitschr. Angew, Math. Mech., 51 (1971), 543-546.

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[3] Barbashin, E. A. *Introduction, to stability theory* (Russian), Moscow 1967.

STRESZCZENIE

W pracy dowiedziono, że jednostajnie asymptotyczna stabilność zerowego rozwiązania równania różniczkowego zwyczajnego (1) jest warunkiem dostatecznym (jednostajnej) W-stabilności względem stale działających losowych zakłóceń z M i (jednostajnej) jednakowej dla realizacji stabilności względem stale działających losowych zakłóceń z M.

РЕЗЮМЕ

В работе доказано, что равномерно асимптотическая устойчивость нулевого решения обыкновенного дифференциального уравнения (1) является достаточным условием (равномерной) W-устойчивости относительно постоянно действующих случайных возмущений из M и (равномерной) равностепенной по траекториям устойчивости относительно постоянно действующих случайных возмущений из M.