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# On Periodic Solutions of a Neutral Type Equation

O okresowych rozwiązaniach równania typu neutralnego

О периодических решениях уравнения нейтрального типа.

The aim of the present paper is to find sufficient conditions for the existence of a periodic solution of the nonlinear equation

$$\dot{x}(t) = f(t, x(t), x(t-h), \dot{x}(t-h)) \tag{1}$$

where $h$ is a constant deviation.

Problems connected with investigating periodic solutions for differential equations with a deviating argument have been considered by a number of authors, as, for example, in [1], [2], [3] and [4].

We will say that conditions (A) are satisfied if:

A1. The function $f(t, x, y, z)$ is periodic w.r.t. $t$ with period $T > 0$, and has continuous first derivatives w.r.t. $(x, y, z)$ and is continuous for all $(t, x, y, z)$ from the four-dimensional space.

A2. There exists a constant $m > 0$ such that the following inequality is fulfilled:

$$\left|\int_0^T [f_x(t, \sigma_1(t), \sigma_2(t)\sigma_3(t)) + f_y(t, \sigma_1(t), \sigma_2(t), \sigma_3(t))]\,dt\right| \geqslant m \tag{2}$$

for arbitrary $T$-periodic functions $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_3(t)$.

A3. The functions $f_x(t, x, y, z)$, $f_y(t, x, y, z)$ and $f_z(t, x, y, z)$ satisfy the condition

$$|f_x(t, x, y, z)| \leqslant M, \; |f_y(t, x, y, z)| \leqslant M, \tag{3}$$
$$|f_z(t, x, y, z)| \leqslant 2M, \; |f_x(t, x, y, z) + f_y(t, x, y, z)| \leqslant N$$

where $M$, $N$ are constans, for which the following inequality holds:

$$2M + \frac{MNT^2}{m} < 1 \tag{4}$$

**Theorem 1.** *Let conditions* (A) *be satisfied. Then, equation* (1) *has a T-periodic solution* $x(t) = \lim_{n\to\infty} x_n(t)$, *where*

$$x_n(t) = a_n + \varphi_n(t), \tag{5}$$

*while the constants* $a_n$ *and the functions* $\varphi_n(t)$ *are determined as follows:*

$$\varphi_0(t) = 0, \quad -\infty < t < +\infty \tag{6}$$

$$\int_0^T f(t, a_0, a_0, 0)\,dt = 0 \tag{7}$$

$$\varphi_n(t) = \int_0^t f\big(s, a_{n-1} + \varphi_{n-1}(s), a_{n-1} + \varphi_{n-1}(s-h), \dot{\varphi}_{n-1}(s-h)\big)\,ds \tag{8}$$

*and*

$$\int_0^T f\big(s, a_n + \varphi_n(s), a_n + \varphi_n(s-h), \dot{\varphi}_n(s-h)\big)\,ds = 0 \tag{9}$$

**Proof.** From the condition A1, (6), (8) and (9) it follows that all functions $\varphi_n(t)$ are $T$-periodic.

To prove that there exists a unique solution $a_n$ of the equation (9) for every $n = 0, 1, 2, \ldots$, consider the function

$$\Gamma_n(a) = \int_0^T f\big(s, a + \varphi_n(s), a + \varphi_n(s-h), \dot{\varphi}_n(s-h)\big)\,ds$$

Further, let us calculate

$$\Gamma_n'(a) = \int_0^T \big[f_x\big(s, a + \varphi_n(s), a + \varphi_n(s-h), \dot{\varphi}_n(s-h)\big) + $$
$$+ f_{\bar{x}}\big(s, a + \varphi_n(s), a + \varphi_n(s-h), \dot{\varphi}_n(s-h)\big)\big]\,ds$$

From condition (2) it follows that $|\Gamma_n'(a)| \geqslant m > 0$. Assume for convenience that $\Gamma_n'(a) > 0$. Then, for $a > 0$, $\Gamma_n(a)$ satisfies the inequality $\Gamma_n(a) \geqslant ma + \Gamma_n(0)$, while for $a < 0$ it satisfies the inequality $\Gamma_n(a) \leqslant ma + \Gamma_n(0)$.

Hence obviously there exists a constant $R_n > 0$, such that $\Gamma_n(R_n) > 0$, $\Gamma_n(-R_n) < 0$ for every $n = 0, 1, \ldots$. Since $\Gamma_n(a)$ is a continuous monotone function, then there exists only one $a_n \in (-R_n, R_n)$, such that $\Gamma_n(a_n) = 0$. Thus, we have established that $a_n$ and $\varphi_n(t), (n = 0, 1, \ldots)$ are determined uniquely from the relations (6), (7), (8) and (9).

Now we will prove that the sequence $x_n(t)$ converges uniformly on the segment $[0, T]$.

Introduce the notations

$$\psi_n(t) = \varphi_n(t) - \varphi_{n-1}(t), \quad b_n = a_n - a_{n-1},$$

$$\varrho_n = \|\psi_n(t)\| = \max_t |\psi_n(t)| + \max_t |\dot{\varphi}_n(t)| \tag{10}$$

Estimate $|\psi_n(t)|$. Using (8), we get

$$\psi_n(t) = \int_0^t \big[f_x\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\big(\psi_{n-1}(s) + b_{n-1}\big) + f_y\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\big(\psi_{n-1}(s-h) + b_{n-1}\big) + f_z\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\psi_{n-1}(s-h)\big]\, ds$$

$$= \int_0^t \big[f_x\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\psi_{n-1}(s) + f_y\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\psi_{n-1}(s-h)\big]\, ds + b_{n-1}\int_0^t \big[f_x\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big) + f_y\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\big]\, ds + \int_0^t f_z\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\varphi_{n-1}(s-h)\, ds$$

On the other hand, from (9) we have

$$b_{n-1}\int_0^T \big[f_x\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big) + f_y\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\big]\, ds \tag{12}$$

$$= \int_0^T \big[f_x\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\psi_{n-1}(s) + f_y\big(s, \sigma_n(s), \theta_n(s)\big)\psi_{n-1}(s-h)\big]\, ds - \int_0^T f_z\big(s, \sigma_n(s), \tau_n(s), \theta_n(s)\big)\psi_{n-1}(s-h)\, ds,$$

hence we could write (11) in the form

$$|\psi_n(t)| = \left[1 - \frac{\int_0^t [f_x(s, \sigma_n, \tau_n, \theta_n) + f_y(s, \sigma_n, \tau_n, \theta_n)]\, ds}{\int_0^T [f_x(s, \sigma_n, \tau_n, \theta_n) + f_y(s, \sigma_n, \tau_n, \theta_n)]\, ds}\right] \times$$

$$\times \int_0^t [f_x(s, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(s) + f_y(s, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(s-h) + f_z(s, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(s-h)]\, ds - \frac{\int_0^t [f_x(s, \sigma_n, \tau_n, \theta_n) + f_y(s, \sigma_n, \tau_n, \theta_n)]\, ds}{\int_0^T [f_x(s, \sigma_n, \tau_n, \theta_n) + f_y(s, \sigma_n, \tau_n, \theta_n)]\, ds}$$

$$\times \int_t^T [f_x(s, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(s) + f_y(s, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(s-h) + f_z(s, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(s-h)]\, ds| \leqslant \frac{2MN}{m}[(T-t)t + t(T-t)]\|\psi_{n-1}(t)\|.$$

Since the maximum of the function $(T-t)\,t$ on the segment $[0, T]$ is equal to $T^2/4$, we obtain

$$|\psi_n(t)| \leqslant \frac{MNT^2}{m} \|\psi_{n-1}(t)\| \tag{13}$$

Estimate $|\psi_n(t)|$. From (8) we get

$$\psi_n(t) = f_x(t, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(t) + f_y(t, \sigma_n, \tau_n, \theta_n)\psi_n(t-h) + $$
$$+ f_z(t, \sigma_n, \tau_n, \theta_n)\psi_{n-1}(t-h),$$

whence

$$|\psi_n(t)| \leqslant 2M \|\psi_{n-1}(t)\| \tag{14}$$

From (13) and (14) follows the inequality

$$\varrho_n \leqslant \left(\frac{MNT^2}{m} + 2M\right)\varrho_{n-1} \tag{15}$$

From the estimation of (15) follows the uniform convergence of the $T$-periodic functions $\varphi_n(t)$.

To prove the convergence of the sequence $a_n$, note that from (12) one can easily obtain the inequality

$$|b_n| \leqslant \frac{2MT}{m} \varrho_n$$

From (4) and (15) it follows that the sequence $a_n$ is convergent.

Set

$$a = \lim_{n\to\infty} a_n, \varphi(t) = \lim_{n\to\infty} \varphi_n(t), x(t) = \lim_{n\to\infty} \left(a_n + \varphi_n(t)\right)$$

One can see from (8) that $x(t)$ is a $T$-periodic solution of equation (1). Thus, the theorem is proved.

**Remark 1.** The theorem holds if condition (6) is replaced by: $\varphi_0(t) = \varphi(t)$, where $\varphi(t)$ is an arbitrary $T$-periodic and continuously-differentiable for $t \in (-\infty, +\infty)$ function.

**Remark 2.** $T = h$, the conditions (A) could be weakened, as follows: Conditions (3) and (4) are replaced by:

$$|f_x(t, x, y, z) + f_y(t, x, y, z)| \leqslant N, \; |f_z(t, x, y, z)| \leqslant N \tag{3}$$

$$\frac{(Nh)^2}{2m} + N < 1 \tag{4}$$

**Remark 3.** Every $T$-periodic solution of the equation (1) can be considered as the limit of the sequence of the type (5), where

$$a_0 = x(0), \varphi_0(t) = \int_0^t f\left(s, x(s), x(s-h), \dot{x}(s-h)\right) ds$$

while $a_n$ and $\varphi_n(t)$ for $n \geqslant 1$ are determined by the relations (8) and (9).

Indeed, in this case we obtain

$$\varphi_n(t) = \varphi_0(t),\ a_n = a_0,\ (n = 1, 2, \ldots),\ x_n(t) = x(t)$$

and the assumption is obvious.

**Theorem 2.** *Let the conditions* (A) *be satisfied. Then the T-periodic solution of equation* (1) *is unique.*

**Proof.** Let $x(t)$ be a $T$-periodic solution of equation (1), defined by theorem 1, i.e.

$$x(t) = \lim_{n\to\infty} x_n(t)$$

where $x_n(t)$ are obtained from (5), (6), (7), (8) and (9).

By $\omega(t)$ denote an arbitrary $T$-periodic solution of equation (1). Determimine the sequence $\omega_n(t) = \tilde{a}_n + \tilde{\varphi}\ (t)$ according to remark 3.

Consider the difference $\tilde{\psi}_n(t) = \tilde{\varphi}_n - \varphi_n$. It is easy to obtain the estimation

$$|\tilde{\psi}_n(t)| \leqslant \frac{MNT^2}{m} \|\tilde{\psi}_{n-1}(t)\|$$

$$|\dot{\tilde{\psi}}_n(t)| \leqslant 2M \|\tilde{\psi}_{n-1}(t)\|$$

whence

$$\|\tilde{\psi}_n(t)\| \leqslant \left(\frac{MNT^2}{m} + 2M\right)^n \|\tilde{\psi}_0(t)\| \tag{17}$$

$$|\tilde{a}_n - a_n| \leqslant \frac{2MT}{m} \|\tilde{\psi}_n(t)\| \tag{18}$$

From (17) and (18) and using the condition (4), we get

$$\|\tilde{\psi}_n(t)\| \xrightarrow[n\to\infty]{} 0,\quad |\tilde{a}_n - a_n| \xrightarrow[n\to\infty]{} 0$$

i.e.

$$\tilde{\varphi}(t) = \varphi(t),\ \tilde{a} = a,\ x(t) = \omega(t)$$

Thus, the theorem is proved.

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## STRESZCZENIE

W pracy podano kilka warunków dostatecznych na istnienie i jednoznaczność rozwiązań okresowych dla równania typu neutralnego o stałym odchyleniu.

## РЕЗЮМЕ

В работе доказано несколько достаточных условий о существовании и единственности периодических решений уравнений нейтрального типа с постоянным отклонением.