

MOHAMED AMINE HACHANI

## Entire functions of exponential type not vanishing in the half-plane $\Re z > k$ , where $k > 0$

ABSTRACT. Let  $P(z)$  be a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k \leq 1$ , and let  $Q(z) := z^n \overline{P(1/\bar{z})}$ . It was shown by Govil that if  $\max_{|z|=1} |P'(z)|$  and  $\max_{|z|=1} |Q'(z)|$  are attained at the same point of the unit circle  $|z| = 1$ , then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|.$$

The main result of the present article is a generalization of Govil's polynomial inequality to a class of entire functions of exponential type.

### 1. Introduction and statement of results.

**1.1. Bernstein's inequality for trigonometric polynomials.** Let  $\mathcal{P}_m$  denote the class of all polynomials of degree at most  $m$  and let  $Q(z) := z^m \overline{P(1/\bar{z})}$ . It is well known that if  $P \in \mathcal{P}_n$  and  $|P(z)| \leq M$  for  $|z| = 1$ , then (see [9, p. 524])

$$|Q'(z)| + |P'(z)| \leq Mn \quad (|z| = 1).$$

This result includes Bernstein's inequality for polynomials

$$(1.1) \quad |P'(z)| \leq Mn \quad (|z| = 1),$$

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and yields to

$$\left| \frac{d}{d\theta} P(e^{i\theta}) \right| + \left| -inP(e^{i\theta}) + \frac{d}{d\theta} P(e^{i\theta}) \right| \leq Mn \quad (\theta \in \mathbb{R}).$$

If  $t(\theta) := \sum_{\nu=-n}^n a_\nu e^{i\nu\theta}$  is a trigonometric polynomial and  $|t(\theta)| \leq M$  for all real  $\theta \in \mathbb{R}$ , then  $e^{in\theta}t(\theta) = P(e^{i\theta})$ , where  $P \in \mathcal{P}_{2n}$  and  $|P(z)| \leq M$  for  $|z| = 1$ . Applying the preceding inequality with  $2n$  instead of  $n$ , we obtain

$$(1.2) \quad |\operatorname{int}(\theta) + t'(\theta)| + |-\operatorname{int}(\theta) + t'(\theta)| \leq 2Mn \quad (\theta \in \mathbb{R}).$$

In particular, we have

$$|2t'(\theta)| \leq |\operatorname{int}(\theta) + t'(\theta)| + |-\operatorname{int}(\theta) + t'(\theta)| \leq 2Mn \quad (\theta \in \mathbb{R}),$$

that is,

$$(1.3) \quad |t'(\theta)| \leq Mn \quad (\theta \in \mathbb{R}).$$

This is the famous inequality of S. Bernstein for trigonometric polynomials. It is sharp and in (1.2), equality can hold at any point  $\theta \in \mathbb{R}$ .

From (1.2) it follows that if  $t(\theta)$  is real for all real  $\theta$ , then

$$(1.4) \quad n^2 t^2(\theta) + (t'(\theta))^2 \leq M^2 n^2 \quad (\theta \in \mathbb{R}).$$

**Remark.** Bernstein had proved (1.3) for cosine polynomials and also for sine polynomials. M. Riesz [10] seems to have been the first to prove it in its full generality. Inequality (1.4) is a result of J. G. van der Corput and G. Shaake [5].

## 2. Functions of exponential and Bernstein's inequality.

**Basic properties of functions of exponential type.** A trigonometric polynomial  $t(\theta) := \sum_{\nu=-n}^n a_\nu e^{i\nu\theta}$  is well defined for any  $\theta$  in the complex plane  $\mathbb{C}$  and not only when  $\theta$  is restricted to the real line. Replacing  $\theta$  by  $z$ , we obtain  $t(z) := \sum_{\nu=-n}^n a_\nu e^{i\nu z}$ , which is holomorphic throughout the complex plane. Thus, a trigonometric polynomial  $t(\theta)$  can be seen as the restriction of an entire function to the real axis. Unless all the coefficients  $a_\nu$  except  $a_0$  are zero,  $t(z)$  is an entire function of order 1 and of type  $T \leq n$ . Clearly, there exists a constant  $C$  such that  $|t(z)| < C e^{n|z|}$  for all  $z \in \mathbb{C}$ . In other words,  $t(z)$  is an entire function of exponential type  $n$ . Let us recall that a function  $f(z)$  holomorphic in an unbounded domain  $\mathcal{D} \subseteq \mathbb{C}$  is said to be of exponential type  $\tau$  in  $\mathcal{D}$  if for any  $\varepsilon > 0$ , there exists a constant  $K(\varepsilon)$  such that  $|f(z)| < K(\varepsilon) e^{(\tau+\varepsilon)|z|}$  for all  $z \in \mathcal{D}$ . In the present context, an interesting example of an unbounded domain is the sector

$$A(\alpha, \beta) := \{z = re^{i\theta} : 0 < r < \infty, \alpha \leq \theta \leq \beta\},$$

where  $\beta \in (\alpha, \alpha + 2\pi)$ , and half-planes have special significance. Some of the important results about functions of exponential type are to be found in what follows.

We know that trigonometric polynomials are  $2\pi$ -periodic, but an entire function of exponential type may not be periodic at all;  $(\sin \tau z)/z$  is such a function. As another example, we wish to mention

$$(2.1) \quad f(z) := \sum_{\nu=0}^n a_\nu e^{i\lambda_\nu z}, \quad \lambda_0 < \dots < \lambda_n,$$

which is an entire function of exponential type  $\tau := \max\{|\lambda_0|, |\lambda_n|\}$  but is generally not periodic.

It is known (see [2, Theorem 6.10.1]) that if  $f(z)$  is an entire function of exponential type  $\tau$  which is periodic on the real axis with period  $\Delta$ , then it must be of the form  $f(z) = \sum_{\nu=-n}^n a_\nu e^{2\pi i \nu z / \Delta}$  with  $n \leq \lfloor \Delta \tau / (2\pi) \rfloor$ .

To characterize the dependence of the growth of a function  $f$  of exponential type  $\tau$  in a sector  $A(\alpha, \beta)$  on the direction in which  $z$  tends to infinity, Phragmén and Lindelöf introduced the function

$$h_f(\theta) := \limsup_{r \rightarrow \infty} \frac{\log |f(r e^{i\theta})|}{r} \quad (\alpha \leq \theta \leq \beta),$$

called the *indicator function* of  $f$ . It is known that unless  $h_f(\theta) \equiv -\infty$ ,  $h_f(\theta)$  is continuous in  $\alpha < \theta < \beta$  and that if  $\alpha \leq \theta < \theta + \pi \leq \beta$ , then

$$(2.2) \quad h_f(\theta) + h_f(\theta + \pi) \geq 0.$$

If  $f$  is an entire function of order 1 whose type is  $\tau$ , then,  $h_f(\theta) \leq \tau$  for all  $\theta$  and so, by (2.2),  $h_f(\theta) \geq -\tau$ . See [2, Chapter 5] for these and many other properties of the indicator function.

### Bernstein's inequality for entire functions of exponential type.

Bernstein himself was the first to extend inequality (1.1) to entire functions of exponential type. The extended version may be stated as follows.

**Theorem A** (S. Bernstein [9, p. 513]). *Let  $f(z)$  be an entire function of exponential type  $\tau$  such that  $|f(x)| \leq M$  on the real axis. Then*

$$(2.3) \quad |f'(x)| \leq M\tau \quad (x \in \mathbb{R}).$$

In (2.3) equality holds if and only if  $f(z)$  is of the form  $a e^{-i\tau z} + b e^{i\tau z}$ , where  $|a| + |b| = M$ .

If  $P(z)$  is a polynomial of degree at most  $n$ , then  $f(z) := P(e^{iz})$  is an entire function of exponential type  $n$ . Besides,  $|f(x)| \leq M$  on the real axis if  $|P(z)| \leq M$  on the unit circle. Hence, inequality (1.1) is covered by (2.3).

**A basic lemma.** The following lemma [2, Theorem 6.2.4] serves as a basic tool in the study of functions of exponential type. In [8] the reader will find a proof of this result, which contains a thorough discussion of the case of equality.

**Lemma A.** *Let  $f$  be a function of exponential type in the open upper half-plane such that  $h_f(\pi/2) \leq c$ . Furthermore, let  $f$  be continuous in the closed upper half-plane and suppose that  $|f(x)| \leq M$  on the real axis. Then*

$$(2.4) \quad |f(x + iy)| < M e^{cy} \quad (-\infty < x < \infty, y > 0)$$

unless  $f(z) \equiv M e^{i\gamma} e^{-icz}$  for some real  $\gamma$ .

**3. Entire functions of exponential type satisfying  $f(z) \neq 0$  in  $\Im z > k$ ,  $k > 0$ .** Now, we shall formulate and prove an extension of a theorem of N. K. Govil to entire functions of exponential type. Govil's theorem [6, p. 52] may be stated as follows.

**Theorem B.** *Let  $P(z)$  be a polynomial of degree  $n$  having no zeros in the open disk  $|z| < k$ ,  $k \leq 1$ , and let  $Q(z) := z^n \overline{P(1/\bar{z})}$ . If  $\max_{|z|=1} |P'(z)|$  and  $\max_{|z|=1} |Q'(z)|$  are attained at the same point of the unit circle  $|z| = 1$ , then*

$$(3.1) \quad \max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^n} \max_{|z|=1} |P(z)|.$$

The bound is attained for the polynomial  $P(z) := c(z^n + k^n)$ ,  $c \in \mathbb{C}$ .

Our result may be stated as follows.

**Theorem 1.** *Let  $f(z)$  be an entire function of order 1 and type  $\tau$  having no zeros in the half-plane  $\Im z > k$  for some  $k > 0$ . In addition, let  $h_f(\pi/2) = 0$  and  $|f(x)| \leq M$  on the real axis. Define  $\omega_f(z) := e^{i\tau z} \overline{f(\bar{z})}$  and suppose that*

$$\sup_{-\infty < x < \infty} |f'(x)| \quad \text{and} \quad \sup_{-\infty < x < \infty} |\omega'_f(x)|$$

are both attained at the same point of the real axis. Then

$$(3.2) \quad |f'(x)| \leq \frac{M\tau}{1 + e^{-\tau k}} \quad (-\infty < x < \infty).$$

The following special case of Theorem 1 deserves to be mentioned explicitly. For basic facts about uniformly almost periodic functions, we refer the reader to [1] or [4].

**Corollary 1.** *Let  $f(z)$  be a uniformly almost periodic entire function of exponential type  $\tau$  having no zeros in the half-plane  $\Im z > k$  for some  $k > 0$  and let  $h_f(\pi/2) = 0$ . In addition, let  $|f(x)| \leq M$  on the real axis and suppose that the Fourier coefficients of  $f$  are all non-negative. Then (3.2) holds.*

**4. Some more auxiliary results.** The proof of Theorem 1 requires some preparation which consists in recalling certain notions and additional results about entire functions of exponential type.

**Definition 1.** An entire function  $f$  of exponential type is said to belong to the class  $\mathbf{P}$  if it has no zeros in the open lower half-plane and  $h_f(-\pi/2) \geq h_f(\pi/2)$ .

**Note.** From (2.2) it follows that if  $f \not\equiv 0$  is an entire function of exponential type 0, then  $h_f(\theta) = 0$  for all  $\theta$ . Hence, any entire function of exponential type 0 having all its zeros in the closed upper half-plane belongs to the class  $\mathbf{P}$ .

It is known (see [7] or [2, Theorem 7.8.3]) that the Hadamard factorization of a function  $f$  belonging to the class  $\mathbf{P}$  has the form

$$(4.1) \quad f(z) = A z^m e^{cz} \prod_{k=1}^{\infty} \left\{ \left( 1 - \frac{z}{z_k} \right) e^{z \Re(1/z_k)} \right\},$$

where  $z_k \neq 0$ ,  $\Im z_k \geq 0$  and  $2\Im c = h_f(-\pi/2) - h_f(\pi/2) \geq 0$ .

It is also known [2, p. 129, Theorem 7.8.1] that if  $f$  belongs to  $\mathbf{P}$ , then

$$(4.2) \quad |f(z)| \geq |f(\bar{z})| \quad (\Im z < 0).$$

From (4.2) it follows that if  $f$  belongs to  $\mathbf{P}$ , then  $h_f(-\alpha) \geq h_f(\alpha)$  for all  $\alpha \in (0, \pi)$ .

The following result (see [7, p. 59, Lemma 3] or [2, p. 130, Theorem 7.8.6]) is of fundamental importance. Its significance in the present context cannot be over-emphasized.

**Lemma B.** Let  $f$  be an entire function of order 1 and type  $\tau$  belonging to the class  $\mathbf{P}$ . Furthermore, let  $g$  be an entire function of exponential type  $\sigma \leq \tau$  such that

$$(4.3) \quad |g(x)| \leq |f(x)| \quad \text{for all } x \in \mathbb{R}.$$

Then  $\phi_\lambda(z) := g(z) - \lambda f(z)$  belongs to  $\mathbf{P}$  for any  $\lambda \in \mathbb{C}$ ,  $|\lambda| > 1$ .

**Definition 2.** An additive homogeneous operator  $B[f(z)]$  which carries entire functions of exponential type into entire functions of exponential type and leaves the class  $\mathbf{P}$  invariant is called (see [7, p. 60] or [2, p. 225, Definition 11.7.1]) a  $B$ -operator.

It may be added that an operator  $B$  is *additive* if  $B[f + g] = B[f] + B[g]$  and *homogeneous* if  $B[cf] = cB[f]$ .

Using the representation (4.1), it can be easily shown that *differentiation* is also a  $B$ -operator (see [2, p. 226]).

Let  $f(z)$  be an entire function of order 1 and type  $\tau$ . Suppose that  $|f(x)| \leq M$  on the real axis and that  $h_f(\pi/2) \leq 0$ . Then by Lemma A,  $|f(z)| < M$  in the open upper half-plane. Hence,  $\phi(z) := f(z) - Me^{-i\alpha}$ ,  $\alpha \in \mathbb{R}$ , is an entire function of order 1 and type  $\tau$  which has no zeros in the open upper half-plane. Consequently, the function

$$\omega_\phi(z) := e^{i\tau z} \overline{\phi(\bar{z})} = \omega_f(z) - Me^{i\alpha} e^{i\tau z}$$

belongs to the class **P** and  $|\phi(x)| = |\omega_\phi(x)|$  for all real  $x$ . By Lemma B, the function  $\phi(z) - \lambda\omega_\phi(z)$  belongs to the class **P** for any  $\lambda \in \mathbb{C}$  with  $|\lambda| > 1$ . Since differentiation is a  $B$ -operator, the function  $\phi'(z) - \lambda\omega'_\phi(z)$  also belongs to the class **P** for any  $\lambda \in \mathbb{C}$  with  $|\lambda| > 1$ . In particular,  $\phi'(z) - \lambda\omega'_\phi(z) \neq 0$  in the lower half-plane for any  $\lambda \in \mathbb{C}$  with  $|\lambda| > 1$ . In other words,

$$(4.4) \quad f'(z) - \lambda (\omega'_f(z) - M i \tau e^{i\alpha} e^{i\tau z}) \neq 0$$

for any  $z$  with  $\Im z < 0$ , for any  $\alpha \in \mathbb{R}$  and for any  $\lambda \in \mathbb{C}$  with  $|\lambda| > 1$ . Now, note that  $f$  is not a constant and so  $\omega_f(z)$  cannot be of the form  $M e^{i\gamma} e^{i\tau z}$ ,  $\gamma \in \mathbb{R}$ . Hence, by Theorem A and Lemma A,  $\omega'_f(z) - M i \tau e^{i\alpha} e^{i\tau z}$  is different from zero at every point of the open lower half-plane. Hence (4.4) can hold for any  $z$  with  $\Im z < 0$ , any  $\alpha \in \mathbb{R}$  and any  $\lambda \in \mathbb{C}$  with  $|\lambda| > 1$  only if

$$|f'(z)| \leq M \tau e^{-\tau \Im z} - |\omega'_f(z)|.$$

Hence, the following result holds. Thus, we have proved that *if  $f$  is an entire function of order 1 and type  $\tau$  such that  $|f(x)| \leq M$  on the real axis and  $h_f(\pi/2) \leq 0$ , then*

$$|f'(z)| + |\omega'_f(z)| \leq M \tau e^{-\tau \Im z} \quad (\Im z < 0).$$

By continuity, the same must be true for  $z$  belonging to the real axis. In other words, the following result holds.

**Lemma B.** *Let  $f$  be an entire function of order 1 and type  $\tau$ . Suppose, in addition, that  $|f(x)| \leq M$  on the real axis and that  $h_f(\pi/2) \leq 0$ . Then*

$$(4.5) \quad |f'(z)| + |\omega'_f(z)| \leq M \tau e^{-\tau \Im z} \quad (\Im z \leq 0).$$

**5. Proof of Theorem 1.** As the first step towards the proof of Theorem 1, we prove the following proposition.

**Proposition 1.** *Let  $F$  be an entire function of order 1 and type  $\tau$  having all its zeros in the half-plane  $\{z \in \mathbb{C} : \Im z \geq -k\}$  for some  $k > 0$ . Suppose that  $|F(x)|$  is bounded on the real axis and that  $h_F(\pi/2) \leq 0$ . In addition, let  $\omega_F(z) := e^{i\tau z} \overline{F(\bar{z})}$ . Then*

$$(5.1) \quad \sup_{-\infty < x < \infty} |\omega'_F(x)| \leq e^{\tau k} \sup_{-\infty < x < \infty} |F'(x)|.$$

**Proof.** Suppose that  $|F(x)| \leq M$  on the real axis. The function defined by  $g(z) := F(z - ik)$  is of order 1 and type  $\tau$ . Besides, by Lemma A,  $|g(x)| \leq M e^{\tau k}$  for all real  $x$ . We claim that  $g$  belongs to the class **P** introduced in Definition 1. Clearly,  $g$  has no zeros in the open lower half-plane. Hence, it is sufficient to check that  $h_g(-\pi/2) \geq h_g(\pi/2)$ .

Since  $|g(x)|$  is bounded on the real axis and  $h_g(\pi/2) = h_F(\pi/2) \leq 0$ , we must necessarily have

$$h_g\left(-\frac{\pi}{2}\right) = h_F\left(-\frac{\pi}{2}\right) = \tau,$$

otherwise, by Lemma A,  $g$  and so  $F$  would not be of order 1 and type  $\tau$ . Note that  $\tau$  must be positive because a function of order 1 that is bounded on the real axis or on any line cannot be of type 0. Thus,  $h_g(-\pi/2) > 0$  whereas  $h_g(\pi/2) \leq 0$ . Hence in fact,  $h_g(-\pi/2) > h_g(\pi/2)$  and so  $g$  belongs to **P**.

Let  $\omega_g(z) := e^{i\tau z} \overline{g(\bar{z})}$ . Then,  $|\omega_g(x)| = |g(x)| \leq M e^{\tau k}$  for all real  $x$ . Besides,  $h_{\omega_g}(\pi/2) = -\tau + h_g(-\pi/2) = 0$ . Hence, by Lemma A,  $|\omega_g(z)| \leq M e^{\tau k}$  in the upper half-plane. Since

$$\begin{aligned} \omega_g(z) &= e^{i\tau z} \overline{F(\bar{z} - ik)} \\ &= e^{\tau k} e^{i\tau(z+ik)} \overline{F(z + ik)} = e^{\tau k} \omega_F(z + ik) \end{aligned}$$

we see that

$$h_{\omega_g}\left(-\frac{\pi}{2}\right) = \tau + h_g\left(\frac{\pi}{2}\right) \leq \tau$$

and so, by Lemma A,  $|\omega_g(z)| \leq M e^{\tau(k+|\Im z|)}$  in the lower half-plane. In particular,  $\omega_g(z)$  is an entire function of exponential type at most  $\tau$ .

We have a function  $g$  of order 1 and type  $\tau$  which belongs to the class **P**. Besides, we have a function  $\omega_g(z)$  of exponential type  $\tau$  such that  $|\omega_g(x)| = |g(x)|$  for all real  $x$ . So, Lemma B may be applied with  $g$  in place of  $f$  and  $\omega_g$  in place of  $g$  to conclude that for any  $\lambda$  such that  $|\lambda| > 1$ , the function  $\omega_g(z) - \lambda g(z)$  belongs to the class **P**. Since differentiation is a  $B$ -operator, the function  $\omega'_g(z) - \lambda g'(z)$  also belongs to the class **P** for any  $\lambda \in \mathbb{C}$  such that  $|\lambda| > 1$ . In particular,  $\omega'_g(z) - \lambda g'(z) \neq 0$  if  $\Im z < 0$  for any  $\lambda \in \mathbb{C}$  such that  $|\lambda| > 1$ . This is possible only if  $|\omega'_g(z)| \leq |g'(z)|$  for any  $z$  in the open lower half-plane. By continuity, the same must be true for any real  $z$  also. Thus,  $|\omega'_g(z)| \leq |g'(z)|$  for  $\Im z \leq 0$ , which means that

$$e^{\tau k} |\omega'_F(z + ik)| \leq |F'(z - ik)| \quad (\Im z \leq 0).$$

Taking  $z = x - ik$ , in this inequality, we obtain

$$(5.2) \quad e^{\tau k} |\omega'_F(x)| \leq |F'(x - 2ik)| \quad (-\infty < x < \infty).$$

Since  $F$  is an entire function of order 1 and type  $\tau$ , the same can be said about the function  $F'$ . Hence, by Lemma A, applied to the function  $\overline{F'(\bar{z})}$ , we obtain

$$|F'(x - 2ik)| \leq e^{2\tau k} \sup_{-\infty < x < \infty} |F'(x)|$$

for any real  $x$ . Combining this with (5.2), we find that

$$|\omega'_F(x)| \leq e^{\tau k} \sup_{-\infty < x < \infty} |F'(x)|$$

for any real  $x$ , which is equivalent to (5.1).  $\square$

**Proposition 2.** *Let  $f$  be an entire function of order 1 and type  $\tau$  having no zeros in the half-plane  $\Im z > k$  where  $k > 0$ . Besides, let  $h_f(\pi/2) = 0$  and suppose that  $|f(x)|$  is bounded on the real axis. In addition, let  $\omega_f(z) := e^{i\tau z} \overline{f(\bar{z})}$ . Then*

$$(5.3) \quad e^{-\tau k} \sup_{-\infty < x < \infty} |f'(x)| \leq \sup_{-\infty < x < \infty} |\omega'_f(x)|.$$

**Proof.** Lemma A can be used to see that  $h_f(-\pi/2) = \tau$ . Hence,  $\omega_f(z) := e^{i\tau z} \overline{f(\bar{z})}$  is an entire function of order 1 and type  $\tau$  having all its zeros in the half-plane  $\Im z > -k$ . Besides,  $h_{\omega_f}(\pi/2) = 0$  and  $|\omega_f(x)|$  is bounded on the real axis. Hence,  $\omega_f$  satisfies all the conditions of Proposition 1. So, let us apply Proposition 1 taking  $F = \omega_f$ . Clearly, then  $\omega_F = \omega_{\omega_f} = f$  and so by (5.1), we have

$$\sup_{-\infty < x < \infty} |f'(x)| \leq e^{\tau k} \sup_{-\infty < x < \infty} |\omega'_f(x)|,$$

which proves (5.3).  $\square$

**Proof of Theorem 1.** Suppose that

$$\sup_{-\infty < x < \infty} |f'(x)| \quad \text{and} \quad \sup_{-\infty < x < \infty} |\omega'_f(x)|$$

are both attained at the same point  $x_0$  of the real axis. Combining (4.5) and (5.3), we obtain that

$$\begin{aligned} (1 + e^{-\tau k}) \sup_{-\infty < x < \infty} |f'(x)| &\leq \sup_{-\infty < x < \infty} |f'(x)| + \sup_{-\infty < x < \infty} |\omega'_f(x)| \\ &\leq |f'(x_0)| + |\omega'_f(x_0)| \\ &\leq M\tau. \end{aligned}$$

Then

$$\sup_{-\infty < x < \infty} |f'(x)| \leq \frac{\tau}{1 + e^{-\tau k}} M,$$

which proves the theorem.  $\square$

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Mohamed Amine Hachani  
Département de Mathématiques et de Statistique  
Université de Montréal  
Montréal, Québec H3C 3J7  
Canada  
e-mail: [hachani@dms.umontreal.ca](mailto:hachani@dms.umontreal.ca)

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