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MAGDALENA JASTRZĘBSKA and DARIUSZ PARTYKA

Exponential representations of injective continuous mappings in radial sets

Abstract. By a radial set we understand a non-empty set $A \subset \mathbb{C} \setminus \{0\}$ such that for every point $z \in A$ the circle with centre at the origin and passing through z is included in A. We show in a detailed manner that every continuous and injective function $F : A \to \mathbb{C} \setminus \{0\}$ can be represented by means of the natural exponential function exp and a certain continuous function $\Phi : \mathrm{Ei}(A) \to \mathbb{C}$, where $\mathrm{Ei}(A)$ is the set of all $z \in \mathbb{C}$ with the property $\exp(\mathrm{i}z) \in A$. The representation is given by $F(\exp(\mathrm{i}z)) = \exp(\mathrm{i}\Phi(z))$ for $z \in \mathrm{Ei}(A)$. We also touch the problem of the injectivity of Φ.

1. Introduction. Unless otherwise stated we assume throughout the paper that all topological notions are relevant to the Euclidean topology in the set of complex numbers $\mathbb{C}$. In the paper [7], J. Krzyż characterized quasiconformal mappings of the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ onto itself by means of quasisymmetric homeomorphisms of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ onto itself. To this aim he assigned to a quasiconformal mapping F of $\mathbb{D}$ onto itself and keeping the origin fixed a quasiconformal mapping Φ of the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ onto itself, related to F by

$$F\left(\mathrm{e}^{2\pi\mathrm{i}z}\right) = \mathrm{e}^{2\pi\mathrm{i}\Phi(z)}, \quad z \in \mathbb{C}_+. \tag{1.1}$$

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Then he carried the classical Beurling–Ahlfors quasisymmetric boundary characterization of Φ on the mapping F by means of (1.1). He defined Φ by the formula

$$\mathbb{C}_+ \ni z \mapsto \Phi(z) := \frac{1}{2\pi \mathrm{i}} \log F\left(\mathrm{e}^{2\pi \mathrm{i} z}\right),$$

cf. [7, (2.9)]. However, the interpretation of the formula as the composition of the logarithmic function log with F does not make any sense, because F takes values in the doubly connected domain $\mathbb{D} \setminus \{0\}$, where log is not defined. Unfortunately, Krzyż did not write how to interpret the expression $\log F\left(\mathrm{e}^{2\pi \mathrm{i} z}\right)$ and did not give any references, where this problem was explained. The existence of Φ satisfying the condition (1.1) seems to be an intuitive fact. However, the precise proof of this fact is not obvious. The aim of this paper is to prove the existence of a continuous mapping Φ satisfying the condition (2.2), provided F is a continuous and injective mapping in a radial set A (cf. Definition 2.1) and $F(A)$ does not contain the origin. The main result is Theorem 3.2. The crucial role in its proof plays Lemma 3.1. We also present a few auxiliary facts collected in Section 2. In Section 4, we touch the problem of the injectivity of Φ. It is worth noting that Theorem 3.2 can be proved alternatively by using the analytic continuation method and the monodromy theorem. An outline of the proof is presented in Remark 3.3. However, the first proof seems to be more elementary and constructive.

In this way we complete gaps in the proof of [7, Theorem 1], which was a motivation for our considerations. Theorem 3.2 will be very helpful for generalizations of Krzyż's results [7, Theorems 1 and 2] to quasiregular mappings instead of quasiconformal ones. However, this subject will be developed in a separate paper.

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2. Auxiliary facts. Given $a \in \mathbb{C}$ and $r, R \in \mathbb{R}$ write $\mathbb{D}(a; r, R) := \{z \in \mathbb{C} : r < |z - a| < R\}$ and $\overline{\mathbb{D}}(a; r, R) := \{z \in \mathbb{C} : r \le |z - a| \le R\}$ as well as $\mathbb{T}(a, r) := \{z \in \mathbb{C} : |z - a| = r\}$. In particular, $\mathbb{T} := \mathbb{T}(0, 1)$ is the unit circle.

Definition 2.1. A set A is said to be radial if $A \subset \mathbb{C} \setminus \{0\}$ and the following condition holds:

$$z \in A \Rightarrow \mathbb{T}(0, |z|) \subset A, \quad z \in \mathbb{C}. \tag{2.1}$$

Following the relationship (1.1) we define for every radial set A and every continuous function $F : A \to \mathbb{C} \setminus \{0\}$ the class $\mathrm{Log}(F)$ of all continuous functions $\Phi : \mathrm{Ei}(A) \to \mathbb{C}$ satisfying the following condition:

$$F(\mathrm{e}^{\mathrm{i}z}) = \mathrm{e}^{\mathrm{i}\Phi(z)}, \quad z \in \mathrm{Ei}(A), \tag{2.2}$$

where

$$\mathrm{Ei}(A) := \{z \in \mathbb{C} : \mathrm{e}^{\mathrm{i}z} \in A\}. \tag{2.3}$$

In particular, for all $r, R \in \mathbb{R}$, if $0 < r \leq R$, then

$$\mathrm{Ei}(\mathbb{D}(0; r, R)) = \{z \in \mathbb{C} : -\log R < \operatorname{Im} z < -\log r\} \tag{2.4}$$

and

$$\mathrm{Ei}(\overline{\mathbb{D}}(0; r, R)) = \{z \in \mathbb{C} : -\log R \leq \operatorname{Im} z \leq -\log r\}. \tag{2.5}$$

Definition 2.2. For any radial set A and continuous function $F : A \to \mathbb{C} \setminus \{0\}$ an object Φ is said to be *an exponential representation of* F provided $\Phi \in \mathrm{Log}(F)$.

Lemma 2.3. *Let A be a connected radial set, $F : A \to \mathbb{C} \setminus \{0\}$ be a continuous function and $\Phi \in \mathrm{Log}(F)$. Then for every function $\Psi : B \to \mathbb{C}$, where $B := \mathrm{Ei}(A)$, $\Psi \in \mathrm{Log}(F)$ if and only if Ψ is a continuous function and there exists $n \in \mathbb{Z}$ such that*

$$\Psi(z) - \Phi(z) = 2\pi n, \quad z \in B. \tag{2.6}$$

Proof. Fix A, F and Φ satisfying the assumptions of the lemma. Suppose first that $\Psi \in \mathrm{Log}(F)$. Then

$$\mathrm{e}^{\mathrm{i}\Phi(z)} = F(\mathrm{e}^{\mathrm{i}z}) = \mathrm{e}^{\mathrm{i}\Psi(z)}, \quad z \in B,$$

and consequently,

$$\mathrm{e}^{\mathrm{i}(\Psi(z)-\Phi(z))} = \mathrm{e}^{\mathrm{i}\Psi(z)}/\mathrm{e}^{\mathrm{i}\Phi(z)} = 1, \quad z \in B.$$

Hence

$$\frac{1}{2\pi}(\Psi(z) - \Phi(z)) \in \mathbb{Z}, \quad z \in B. \tag{2.7}$$

Since the functions Ψ and Φ are continuous in the connected set B, $\frac{1}{2\pi}(\Psi - \Phi)$ is a constant function, which yields the property (2.6) for a certain $n \in \mathbb{Z}$. Conversely, assume that Ψ is a continuous function satisfying the condition (2.6) for a certain $n \in \mathbb{Z}$. Then for every $z \in B$,

$$\mathrm{e}^{\mathrm{i}\Psi(z)} = \mathrm{e}^{\mathrm{i}(\Phi(z)+2\pi n)} = \mathrm{e}^{\mathrm{i}\Phi(z)}\mathrm{e}^{2\pi \mathrm{i}n} = \mathrm{e}^{\mathrm{i}\Phi(z)} = F(\mathrm{e}^{\mathrm{i}z}),$$

which means that $\Psi \in \mathrm{Log}(F)$. □

Lemma 2.4. *Let A be a connected radial set and let $p, q \in B := \mathrm{Ei}(A)$. Then for every continuous function $F : A \to \mathbb{C} \setminus \{0\}$, if $\mathrm{Log}(F) \neq \emptyset$ and*

$$F(\mathrm{e}^{\mathrm{i}p}) = \mathrm{e}^{\mathrm{i}q}, \tag{2.8}$$

then there exists the unique function $\Phi \in \mathrm{Log}(F)$ such that $\Phi(p) = q$.

Proof. Fix A, p, q and F satisfying the assumptions of the lemma. Then there exists $\tilde{\Phi} \in \mathrm{Log}(F)$, which together with (2.8) gives

$$\mathrm{e}^{\mathrm{i}\tilde{\Phi}(p)} = F(\mathrm{e}^{\mathrm{i}p}) = \mathrm{e}^{\mathrm{i}q}, \quad z \in B.$$

Hence $q = \tilde{\Phi}(p) + 2\pi n$ for a certain $n \in \mathbb{Z}$. Setting $\Phi := \tilde{\Phi} + 2\pi n$, we conclude from Lemma 2.3 that $\Phi \in \mathrm{Log}(F)$ and $\Phi(p) = q$.

To prove the uniqueness of Φ let us consider an arbitrarily fixed $\Psi \in \operatorname{Log}(F)$ satisfying $\Psi(p) = q$. Applying Lemma 2.3 once more, we see that the condition (2.6) holds for a certain $n \in \mathbb{Z}$, and consequently $\Psi - \Phi$ is a constant function. Therefore,

$$\Psi(z) - \Phi(z) = \Psi(p) - \Phi(p) = q - q = 0, \quad z \in B,$$

and so $\Psi = \Phi$, which completes the proof. □

Remark 2.5. It is well known that for every continuous function $f : \mathbb{T} \to \mathbb{T}$ there exists a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

$$f(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}\varphi(t)}, \quad t \in \mathbb{R}. \tag{2.9}$$

This is a classical result of algebraic topology relevant to the fundamental group of the unit circle, cf. e.g., [5, Chap. 1], [6, Chap. 16]. This result is also very useful in complex analysis, cf. e.g., [3, Sec. 3.3], [11, Sec. 2.1 and 3.1]. By Definition 2.2, each continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying the condition (2.9) is an exponential representation of a given continuous function $f : \mathbb{T} \to \mathbb{T}$, which means that $\varphi \in \operatorname{Log}(f)$, and so $\operatorname{Log}(f) \neq \emptyset$. Since $\mathbb{T}$ is a connected radial set, Lemmas 2.3 and 2.4 are applicable to $A := \mathbb{T}$. By the formula (2.3), $\operatorname{Ei}(A) = \mathbb{R}$, while from the condition (2.9) we see that $\varphi : \mathbb{R} \to \mathbb{R}$ for every $\varphi \in \operatorname{Log}(f)$. Moreover, from Lemma 2.4 it follows that there exists the unique $\varphi \in \operatorname{Log}(f)$ such that

$$0 \leq \varphi(0) < 2\pi. \tag{2.10}$$

Thus an exponential representation φ of f is uniquely determined by the condition (2.10).

Lemma 2.6. *For any continuous functions $f, g : \mathbb{T} \to \mathbb{T}$, if*

$$|f(u) - g(u)| < \sqrt{2}, \quad u \in \mathbb{T}, \tag{2.11}$$

then for all $\varphi \in \operatorname{Log}(f)$ and $\psi \in \operatorname{Log}(g)$ there exists $n \in \mathbb{Z}$ such that

$$\varphi(t) - \psi(t) = \arcsin\left(\operatorname{Im}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right) + 2\pi n, \quad t \in \mathbb{R}. \tag{2.12}$$

Proof. Given continuous functions $f, g : \mathbb{T} \to \mathbb{T}$ satisfying the condition (2.11) we have

$$\left|\frac{f(u)}{g(u)} - 1\right| = \frac{|f(u) - g(u)|}{|g(u)|} = |f(u) - g(u)| < \sqrt{2}, \quad u \in \mathbb{T},$$

and $f(u)/g(u) \in \mathbb{T}$ for $u \in \mathbb{T}$. Hence

$$\operatorname{Re}\frac{f(u)}{g(u)} > 0 \quad \text{and} \quad \left|\operatorname{Im}\frac{f(u)}{g(u)}\right| < 1, \quad u \in \mathbb{T}. \tag{2.13}$$

Let us consider the function

$$\mathbb{R} \ni t \mapsto \alpha(t) := \arcsin\left(\operatorname{Im}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right).$$

Then for an arbitrarily fixed $t \in \mathbb{R}$, $\alpha(t) \in (-\pi/2; \pi/2)$ as well as

$$\mathrm{e}^{\mathrm{i}\alpha(t)} = \cos(\alpha(t)) + \mathrm{i}\sin(\alpha(t)) = \cos(\alpha(t)) + \mathrm{i}\operatorname{Im}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})},$$

which implies

$$1 = |\mathrm{e}^{\mathrm{i}\alpha(t)}|^2 = \big(\cos(\alpha(t))\big)^2 + \left(\operatorname{Im}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right)^2.$$

On the other hand,

$$1 = \left|\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right|^2 = \left(\operatorname{Re}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right)^2 + \left(\operatorname{Im}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right)^2.$$

Hence and by the first inequality in (2.13),

$$\operatorname{Re}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})} = \left|\operatorname{Re}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\right| = |\cos(\alpha(t))| = \cos(\alpha(t)).$$

Therefore,

$$\mathrm{e}^{\mathrm{i}\alpha(t)} = \cos(\alpha(t)) + \mathrm{i}\sin(\alpha(t)) = \operatorname{Re}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})} + \mathrm{i}\operatorname{Im}\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})} = \frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}, \tag{2.14}$$

$t \in \mathbb{R}$. Since f and g are continuous functions, the function α is continuous, and thereby for given $\varphi \in \operatorname{Log}(f)$ and $\psi \in \operatorname{Log}(g)$, the function $\lambda := \varphi - \psi - \alpha$ is also continuous. On the other hand, by (2.14) we obtain

$$\mathrm{e}^{\mathrm{i}\lambda(t)} = \frac{\mathrm{e}^{\mathrm{i}\varphi(t)}}{\mathrm{e}^{\mathrm{i}\psi(t)}}\Big/\mathrm{e}^{\mathrm{i}\alpha(t)} = \frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})}\Big/\frac{f(\mathrm{e}^{\mathrm{i}t})}{g(\mathrm{e}^{\mathrm{i}t})} = 1, \quad t \in \mathbb{R},$$

and so $\frac{1}{2\pi}\lambda(t) \in \mathbb{Z}$ for $t \in \mathbb{R}$. Therefore, there exists $n \in \mathbb{Z}$ such that $\lambda(t) = 2\pi n$ for $t \in \mathbb{R}$, which proves the property (2.12). □

3. Main results. The main aim of this section is to show that $\operatorname{Log}(F) \neq \emptyset$, provided $F : A \to \mathbb{C} \setminus \{0\}$ is a continuous and injective function on a connected radial set A. We will start with the following auxiliary lemma.

Lemma 3.1. *Given $r, R \in \mathbb{R}$ and $p, q \in \mathbb{C}$ assume that $0 < r \leq R$ and $p, q \in B := \operatorname{Ei}(A)$, where $A := \overline{\mathbb{D}}(0; r, R)$. Let $F : A \to \mathbb{C} \setminus \{0\}$ be a continuous and injective function such that*

$$F(\mathrm{e}^{\mathrm{i}p}) = \mathrm{e}^{\mathrm{i}q}. \tag{3.1}$$

Then there exists the unique function $\Phi \in \operatorname{Log}(F)$ satisfying the equality $\Phi(p) = q$.

Proof. Fix r, R, p, q and F satisfying the assumptions. Since A is a compact set and F is a continuous function in A, we conclude from the extreme value theorem that there exist $\zeta, \xi \in A$ satisfying the following property:

$$|F(\zeta)| \leq |F(u)| \leq |F(\xi)|, \quad u \in A. \tag{3.2}$$

We will consider two complementary cases.
Case I. Assume first that $\{\zeta, \xi\} \cap [r; R] = \emptyset$. Then there exists a connected set $\Gamma \subset A \setminus [r; R]$ containing the points ζ and ξ. Setting

$$E := \{tF(\zeta) : t \in [0; 1]\} \cup F(\Gamma) \cup \{tF(\xi) : t \in [1; +\infty)\} \cup \{\infty\}, \tag{3.3}$$

we see that $0, \infty \in E \subset \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. By the continuity of F, the image $F(\Gamma)$ is a connected set. Since $F(\zeta), F(\xi) \in F(\Gamma)$, we conclude from the formula (3.3) that E is a connected set in $(\hat{\mathbb{C}}, \rho)$ where ρ is the chordal metric in $\hat{\mathbb{C}}$. By the injectivity of F, $F([r; R]) \cap F(\Gamma) = \emptyset$. Combining this with (3.2) and (3.3), we see that $E \cap F([r; R]) = \emptyset$. Therefore, the points 0 and ∞ do not belong to different connected components of the set $\hat{\mathbb{C}} \setminus F([r; R])$ in the extended complex plane $(\hat{\mathbb{C}}, \rho)$. Since the interval $[r; R]$ is a compact set and the mapping F is continuous, $F([r; R])$ is a compact set. Thus there exists a continuous function $L : F([r; R]) \to \mathbb{C}$ satisfying the following condition:

$$\mathrm{e}^{L(u)} = u, \quad u \in F([r; R]), \tag{3.4}$$

which is a consequence of the classical Eilenberg's theorem, cf. [4, Théorème 1 on p. 75], [8, Chap. XXI §3]. Since $\mathrm{e}^{-t} \in [r; R] \subset A$ for $t \in I := [-\log R; -\log r]$, we see that the function

$$I \ni t \mapsto \lambda(t) := \frac{1}{\mathrm{i}}\big(L(F(\mathrm{e}^{-t})) - \log(|F(\mathrm{e}^{-t})|)\big) \tag{3.5}$$

is well defined and continuous. For every $y \in I$ the function

$$\mathbb{T} \ni u \mapsto f_y(u) := \frac{F(\mathrm{e}^{-y}u)}{|F(\mathrm{e}^{-y}u)|} \tag{3.6}$$

is continuous and $f_y(\mathbb{T}) \subset \mathbb{T}$. From Remark 2.5 it follows that for a given $y \in I$ there exists $\tilde{\varphi}_y \in \mathrm{Log}(f_y)$. Then $\mathrm{e}^{\mathrm{i}\tilde{\varphi}_y(0)} = f_y(1)$. On the other hand, combining (3.5) with (3.4), we have

$$\mathrm{e}^{\mathrm{i}\lambda(y)} = \mathrm{e}^{L(F(\mathrm{e}^{-y})) - \log|F(\mathrm{e}^{-y})|} = \frac{F(\mathrm{e}^{-y})}{|F(\mathrm{e}^{-y})|} = f_y(1).$$

Thus $\mathrm{e}^{\mathrm{i}\tilde{\varphi}_y(0)} = \mathrm{e}^{\mathrm{i}\lambda(y)}$ for $y \in I$, and so there exists a function $\mu : I \to \mathbb{Z}$ such that $\tilde{\varphi}_y(0) - \lambda(y) = 2\pi\mu(y)$ for $y \in I$. Setting now

$$\varphi_y := \tilde{\varphi}_y - 2\pi\mu(y), \quad y \in I, \tag{3.7}$$

we obtain

$$\varphi_y(0) = \lambda(y), \quad y \in I, \tag{3.8}$$

and applying Lemma 2.3, we see that $\varphi_y \in \operatorname{Log}(f_y)$ for $y \in I$. From the formula (3.6) we conclude that

$$\begin{aligned} F(\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)}) &= F(\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}x}) = f_y(\mathrm{e}^{\mathrm{i}x})|F(\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}x})| \\ &= \mathrm{e}^{\mathrm{i}\varphi_y(x)}\mathrm{e}^{\log|F(\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}x})|} = \mathrm{e}^{\mathrm{i}(\varphi_y(x)-\mathrm{i}\log|F(\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}x})|)}, \end{aligned} \tag{3.9}$$

$x \in \mathbb{R}$, $y \in I$. We proceed to show that the function

$$B \ni x+\mathrm{i}y \mapsto \Psi(x+\mathrm{i}y) := \varphi_y(x) - \mathrm{i}\log|F(\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}x})| \tag{3.10}$$

is continuous. Since the function $B \ni z \mapsto \log|F(\mathrm{e}^{\mathrm{i}z})|$ is continuous, it is enough to show that the function $B \ni x+\mathrm{i}y \mapsto \varphi_y(x)$ is continuous. Since F is a continuous function and $0 \notin F(A)$, it follows that $F/|F|$ is a continuous function. Therefore, $F/|F|$ is uniformly continuous, because A is a compact set. Then for a given $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$|u-v| < \delta_0 \Rightarrow \left|\frac{F(u)}{|F(u)|} - \frac{F(v)}{|F(v)|}\right| < \min(\{\varepsilon, \sqrt{2}\}), \quad u, v \in A. \tag{3.11}$$

By Lagrange's Mean Value Theorem applied to the function $I \ni t \mapsto \mathrm{e}^{-t}$ we have

$$|\mathrm{e}^{-t_1} - \mathrm{e}^{-t_2}| \le \sup_{t\in I} |-\mathrm{e}^{-t}| \cdot |t_1 - t_2| \le R|t_1 - t_2|, \quad t_1, t_2 \in I,$$

because $-t \le \log R$ for $t \in I$. Given $y_1, y_2 \in I$ assume that $|y_1 - y_2| < \delta_0/R$. Then for each $x \in \mathbb{R}$, $\mathrm{e}^{-y_1}\mathrm{e}^{\mathrm{i}x} \in A$ and $\mathrm{e}^{-y_2}\mathrm{e}^{\mathrm{i}x} \in A$ as well as

$$|\mathrm{e}^{-y_1}\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-y_2}\mathrm{e}^{\mathrm{i}x}| = |\mathrm{e}^{\mathrm{i}x}| \cdot |\mathrm{e}^{-y_1} - \mathrm{e}^{-y_2}| < \delta_0.$$

Combining this with (3.11) and the formula (3.6), we obtain

$$|f_{y_1}(\mathrm{e}^{\mathrm{i}x}) - f_{y_2}(\mathrm{e}^{\mathrm{i}x})| < \sqrt{2}, \quad x \in \mathbb{R}.$$

By Lemma 2.6, there exists a function $I \times I \ni (y_1, y_2) \mapsto \nu(y_1, y_2) \in \mathbb{Z}$ such that

$$\varphi_{y_1}(x) - \varphi_{y_2}(x) = \arcsin\left(\operatorname{Im}\frac{f_{y_1}(\mathrm{e}^{\mathrm{i}x})}{f_{y_2}(\mathrm{e}^{\mathrm{i}x})}\right) + 2\pi\nu(y_1, y_2), \tag{3.12}$$

$x \in \mathbb{R}$, $y_1, y_2 \in I$. In particular, for $x := 0$ we deduce from (3.8) that

$$2\pi|\nu(y_1, y_2)| \le |\lambda(y_1) - \lambda(y_2)| + \left|\arcsin\left(\operatorname{Im}\frac{f_{y_1}(1)}{f_{y_2}(1)}\right)\right|, \quad y_1, y_2 \in I. \tag{3.13}$$

Since λ is a continuous function on the compact set I, λ is uniformly continuous on I. Therefore, there exists $\delta_1 > 0$ such that

$$|y_1 - y_2| < \delta_1 \Rightarrow |\lambda(y_1) - \lambda(y_2)| < 1, \quad y_1, y_2 \in I.$$

Setting $\delta_2 := \min(\{\delta_0/R, \delta_1\})$, we conclude from (3.13) that for all $y_1, y_2 \in I$, if $|y_1 - y_2| < \delta_2$, then

$$|\nu(y_1, y_2)| < \frac{1}{2\pi} + \frac{1}{2\pi}\cdot\frac{\pi}{2} < \frac{1}{2},$$

and so $\nu(y_1, y_2) = 0$. Combining this with (3.12), we get

$$|y_1 - y_2| < \delta_2 \Rightarrow |\varphi_{y_1}(x) - \varphi_{y_2}(x)| = \left|\arcsin\left(\operatorname{Im}\frac{f_{y_1}(\mathrm{e}^{\mathrm{i}x})}{f_{y_2}(\mathrm{e}^{\mathrm{i}x})}\right)\right|, \tag{3.14}$$

$x \in \mathbb{R}$, $y_1, y_2 \in I$. Fix $z_0 = x_0 + \mathrm{i}y_0 \in B$. Then for all $x \in \mathbb{R}$ and $y \in I$, we see that $u := \mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)} \in A$ and $v := \mathrm{e}^{\mathrm{i}(x+\mathrm{i}y_0)} \in A$, and using the condition (3.11), we obtain

$$\begin{aligned}\left|\operatorname{Im}\frac{f_y(\mathrm{e}^{\mathrm{i}x})}{f_{y_0}(\mathrm{e}^{\mathrm{i}x})}\right| &= \left|\operatorname{Im}\left(\frac{f_y(\mathrm{e}^{\mathrm{i}x})}{f_{y_0}(\mathrm{e}^{\mathrm{i}x})} - 1\right)\right| \\ &\le \left|\frac{f_y(\mathrm{e}^{\mathrm{i}x})}{f_{y_0}(\mathrm{e}^{\mathrm{i}x})} - 1\right| \\ &= \left|\frac{F(u)}{|F(u)|}\Big/\frac{F(v)}{|F(v)|} - 1\right| \\ &= \left|\frac{F(u)}{|F(u)|} - \frac{F(v)}{|F(v)|}\right| < \varepsilon,\end{aligned}$$

provided $|y - y_0| < \delta_2$. Hence and by (3.14),

$$\begin{aligned}|\varphi_y(x) - \varphi_{y_0}(x_0)| &\le |\varphi_y(x) - \varphi_{y_0}(x)| + |\varphi_{y_0}(x) - \varphi_{y_0}(x_0)| \\ &\le \left|\arcsin\left(\operatorname{Im}\frac{f_y(\mathrm{e}^{\mathrm{i}x})}{f_{y_0}(\mathrm{e}^{\mathrm{i}x})}\right)\right| + |\varphi_{y_0}(x) - \varphi_{y_0}(x_0)| \\ &\le \frac{\pi}{2}\varepsilon + |\varphi_{y_0}(x) - \varphi_{y_0}(x_0)|,\end{aligned}$$

provided $|y - y_0| < \delta_2$. Since the function φ_{y_0} is continuous, there exists $\delta_3 > 0$ such that

$$|x - x_0| < \delta_3 \Rightarrow |\varphi_{y_0}(x) - \varphi_{y_0}(x_0)| < \varepsilon, \quad x \in \mathbb{R}.$$

Thus setting $\delta := \min(\{\delta_2, \delta_3\})$, we obtain

$$|z - z_0| < \delta \Rightarrow |\varphi_y(x) - \varphi_{y_0}(x_0)| < 3\varepsilon, \quad z := x + \mathrm{i}y \in B,$$

and so the function $B \ni x + \mathrm{i}y \mapsto \varphi_y(x)$ is continuous at each point $z_0 \in B$. Therefore, the function Ψ defined by the formula (3.10) is continuous. Thus from (3.9) and (3.10) it follows that $\Psi \in \operatorname{Log}(F)$, and so $\operatorname{Log}(F) \neq \emptyset$.

Case II. Assume now that $\{\zeta, \xi\} \cap [r; R] \neq \emptyset$. Then there exists $\alpha \in \mathbb{R}$ such that $\{\mathrm{e}^{\mathrm{i}\alpha}\zeta, \mathrm{e}^{\mathrm{i}\alpha}\xi\} \cap [r; R] = \emptyset$. Setting

$$A \ni u \mapsto \tilde{F}(u) := F(\mathrm{e}^{-\mathrm{i}\alpha}u), \tag{3.15}$$

we see that $\tilde{F} : A \to \mathbb{C} \setminus \{0\}$ is a continuous and injective function, and by (3.2),

$$|\tilde{F}(\mathrm{e}^{\mathrm{i}\alpha}\zeta)| \le |F(u)| \le |\tilde{F}(\mathrm{e}^{\mathrm{i}\alpha}\xi)|, \quad u \in A,$$

which gives

$$|\tilde{F}(\mathrm{e}^{\mathrm{i}\alpha}\zeta)| \le |\tilde{F}(u)| \le |\tilde{F}(\mathrm{e}^{\mathrm{i}\alpha}\xi)|, \quad u \in A.$$

Thus we can appeal to the first case which has been already proved with F, ζ and ξ replaced by $\tilde{F}$, $\mathrm{e}^{\mathrm{i}\alpha}\zeta$ and $\mathrm{e}^{\mathrm{i}\alpha}\xi$, respectively. As a result there exists $\tilde{\Psi} \in \mathrm{Log}(\tilde{F})$. Applying the formula (3.15), we see that for every $z \in B$,

$$F(\mathrm{e}^{\mathrm{i}z}) = F(\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}z}) = \tilde{F}(\mathrm{e}^{\mathrm{i}(z+\alpha)}) = \mathrm{e}^{\mathrm{i}\tilde{\Psi}(z+\alpha)} = \mathrm{e}^{\mathrm{i}\Psi(z)},$$

where $B \ni z \mapsto \Psi(z) := \tilde{\Psi}(z+\alpha)$. Hence $\Psi \in \mathrm{Log}(F)$, because $\tilde{\Psi}$ is a continuous function.

Both cases lead to $\mathrm{Log}(F) \neq \emptyset$. Moreover, by the assumption (3.1) the condition (2.8) holds. Lemma 2.4 now shows that there exists the unique function $\Phi \in \mathrm{Log}(F)$ satisfying the equality $\Phi(p) = q$, which is the desired conclusion. □

The following theorem extends Lemma 3.1 to an arbitrary connected radial set.

Theorem 3.2. *Let A be a connected radial set and let $p, q \in B := \mathrm{Ei}(A)$. Then for every continuous and injective function $F : A \to \mathbb{C} \setminus \{0\}$ satisfying the condition* (3.1) *there exists the unique $\Phi \in \mathrm{Log}(F)$ such that $\Phi(p) = q$.*

Proof. Fix a connected radial set A. If $A = \overline{\mathbb{D}}(0; r, R)$ for some $r, R \in \mathbb{R}$ with $0 < r \leq R$, then the theorem reduces to Lemma 3.1. Otherwise there exist $r, R \in \mathbb{R}$ such that $0 < r < R$ and one of the following three possibilities hold:

(3.16) $A = \mathbb{D}(0; r, R) \cup \mathbb{T}(0, R)$ or $A = \mathbb{D}(0; r, R) \cup \mathbb{T}(0, r)$ or $A = \mathbb{D}(0; r, R)$.

If the first equality in (3.16) holds, then we put

$$A_n := \overline{\mathbb{D}}(0; r + (R-r)/n, R), \quad n \in \mathbb{N}. \tag{3.17}$$

If the second equality in (3.16) holds, then we put

$$A_n := \overline{\mathbb{D}}(0; r, R - (R-r)/n), \quad n \in \mathbb{N}. \tag{3.18}$$

If the last equality in (3.16) holds, then we put

$$A_n := \overline{\mathbb{D}}(0; r + (R-r)/(n+1), R - (R-r)/(n+1)), \quad n \in \mathbb{N}. \tag{3.19}$$

By the formulas (3.17), (3.18) and (3.19) we have

$$A_n \subset A_{n+1}, \quad n \in \mathbb{N}, \tag{3.20}$$

as well as

$$\bigcup_{n \in \mathbb{N}} A_n = A. \tag{3.21}$$

Setting now $B_n := \mathrm{Ei}(A_n)$ for $n \in \mathbb{N}$, we deduce from (3.20) and (3.21) that

$$B_n \subset B_{n+1}, \quad n \in \mathbb{N} \tag{3.22}$$

as well as

$$\bigcup_{n \in \mathbb{N}} B_n = B. \tag{3.23}$$

Write $\mathbb{Z}_j := \{n \in \mathbb{Z} : n \geq j\}$ for $j \in \mathbb{Z}$. From (3.22) and (3.23) it follows that there exists $j \in \mathbb{N}$ such that

$$p, q \in B_n, \quad n \in \mathbb{Z}_j. \tag{3.24}$$

By the formulas (3.17), (3.18) and (3.19), A_n is a closed annulus for $n \in \mathbb{N}$. Moreover, by the assumption, the restriction $F_n := F|_{A_n}$ is a continuous function for $n \in \mathbb{N}$. Then Lemma 3.2 shows, by (3.24), that for each $n \in \mathbb{Z}_j$ there exists the unique $\Phi_n \in \mathrm{Log}(F_n)$ satisfying the equality $\Phi_n(p) = q$. From (3.22) it follows that $B_n \subset B_m$ for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}_n$. Thus for all $n \in \mathbb{Z}_j$, $m \in \mathbb{Z}_n$ and $z \in B_n$ we have

$$\mathrm{e}^{\mathrm{i}\Phi_m|_{B_n}(z)} = \mathrm{e}^{\mathrm{i}\Phi_m(z)} = F_m(\mathrm{e}^{\mathrm{i}z}) = F(\mathrm{e}^{\mathrm{i}z}) = F_n(\mathrm{e}^{\mathrm{i}z}),$$

and consequently $\Phi_m|_{B_n} \in \mathrm{Log}(F_n)$. Since $\Phi_n(p) = q = \Phi_m(p)$ for $n, m \in \mathbb{Z}_j$, we conclude from Lemma 2.4 that

$$\Phi_m|_{B_n} = \Phi_n, \quad n \in \mathbb{Z}_j,\, m \in \mathbb{Z}_n. \tag{3.25}$$

Using the properties (3.22) and (3.23), we can define the function $\Phi : B \to \mathbb{C}$ by the following formulas:

$$\begin{aligned} &\Phi(z) := \Phi_j(z),\ z \in B_j;\\ &\Phi(z) := \Phi_n(z),\ n \in \mathbb{Z}_{j+1},\, z \in B_n \setminus B_{n-1}. \end{aligned} \tag{3.26}$$

In particular, $\Phi|_{B_j} = \Phi_j$. Suppose that $\Phi|_{B_n} = \Phi_n$ for a fixed $n \in \mathbb{Z}_j$. Given $z \in B_{n+1}$ we see by (3.22) that $z \in B_{n+1} \setminus B_n$ or $z \in B_n$. If $z \in B_{n+1} \setminus B_n$, then by the second formula in (3.26) we have $\Phi(z) = \Phi_{n+1}(z)$. If $z \in B_n$, then by (3.25) we see that

$$\Phi(z) = \Phi|_{B_n}(z) = \Phi_n(z) = \Phi_{n+1}|_{B_n}(z) = \Phi_{n+1}(z).$$

Therefore, $\Phi(z) = \Phi_{n+1}(z)$ for $z \in B_{n+1}$, which implies $\Phi|_{B_{n+1}} = \Phi_{n+1}$. Then by the Principle of Mathematical Induction we obtain

$$\Phi|_{B_n} = \Phi_n, \quad n \in \mathbb{Z}_j. \tag{3.27}$$

From the formulas (3.17), (3.18), (3.19) and the equality (3.23) it follows that for every $z \in B$ there exists $n \in \mathbb{Z}_j$ such that z is an inner point of B_n. Since each function Φ_n, $n \in \mathbb{Z}_j$, is continuous, we conclude from (3.27) that Φ is a continuous function. Moreover, by (3.27) we obtain

$$\mathrm{e}^{\mathrm{i}\Phi(z)} = \mathrm{e}^{\mathrm{i}\Phi_n(z)} = F_n(\mathrm{e}^{\mathrm{i}z}) = F(\mathrm{e}^{\mathrm{i}z}), \quad n \in \mathbb{Z}_j,\, z \in B_n,$$

which together with (3.23) yields $\Phi \in \mathrm{Log}(F)$. By (3.24) we also have $\Phi(p) = \Phi_j(p) = q$. Lemma 2.4 now shows that Φ is the unique function with this property, which proves the theorem. □

Write $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{\mathbb{D}}(a, r) := \{z \in \mathbb{C} : |z - a| \leq r\}$ for $a \in \mathbb{C}$ and $r \in \mathbb{R}$.

Remark 3.3. We outline now an alternative proof of Theorem 3.2 based on the analytic continuation technique and the monodromy theorem, cf. [2, Chap. 8] or [12, Chap. 16]. Assume that A, p, q and F satisfy the assumptions of Theorem 3.2. By the properties of the function exp we know that for every $\theta \in \mathbb{R}$ the function $D_\theta \ni w \mapsto T_\theta(w) := \exp(\mathrm{i}w)$ is holomorphic and injective, where $D_\theta := \{w \in \mathbb{C} : |\operatorname{Re} w - \theta| < \pi\}$. Hence $(T_\theta^{-1}, \mathbb{D}(r\mathrm{e}^{\mathrm{i}\theta}, r))$ is a function element for all $\theta \in \mathbb{R}$ and $r \in (0;+\infty)$, cf. [12, Definition 16.9]. Given $z \in B$ let γ be a path in B joining p with z, i.e., $\gamma : [0;1] \to B$ is a continuous function satisfying $\gamma(0) = p$ and $\gamma(1) = z$. Then there exists $\theta_z \in \mathbb{R}$ such that $\tilde{F}(z) = |\tilde{F}(z)|\mathrm{e}^{\mathrm{i}\theta_z}$ and $(T_{\theta_z}^{-1}, \mathbb{D}(\tilde{F}(z), |\tilde{F}(z)|))$ is the unique analytic continuation of the function element $(T_{\theta_p}^{-1}, \mathbb{D}(\mathrm{e}^{\mathrm{i}q}, |\mathrm{e}^{\mathrm{i}q}|))$ along the path $\tilde{F} \circ \gamma$, where $B \ni w \mapsto \tilde{F}(w) := F(\mathrm{e}^{\mathrm{i}w})$ and $\theta_p := \operatorname{Re} q$, cf. [12, Theorem 16.11]. Moreover, since B is a simply connected set, the function element $(T_{\theta_z}^{-1}, \mathbb{D}(\tilde{F}(z), |\tilde{F}(z)|))$ does not depend on a path joining p with z, cf. [12, Theorems 16.13 and 16.14]. In this manner we define a function $B \ni z \mapsto \Phi(z) := T_{\theta_z}^{-1}(\tilde{F}(z))$. By definition, Φ is a continuous function such that

$$\mathrm{e}^{\mathrm{i}\Phi(z)} = T_{\theta_z}(\Phi(z)) = \tilde{F}(z) = F(\mathrm{e}^{\mathrm{i}z}), \quad z \in B.$$

Thus $\Phi \in \operatorname{Log}(F)$ and $\Phi(p) = q$, which implies the existential part of Theorem 3.2. The detailed proof according to the above sketch is of course much longer.

4. Complementary remarks. By Theorem 3.2 we know that $\operatorname{Log}(F) \neq \emptyset$, provided $F : A \to \mathbb{C} \setminus \{0\}$ is a continuous and injective function on a connected radial set A. A natural question arises about injectivity of $\Phi \in \operatorname{Log}(F)$. This seems to be rather a difficult problem in general. Therefore we restrict further considerations to a few simple but useful cases.

Lemma 4.1. *Let A be a connected radial set and $F : A \to \mathbb{C} \setminus \{0\}$ be a continuous and injective function. Assume that there exist $r, R > 0$ such that*

$$\mathbb{T}(0,r) \subset A \quad \textit{and} \quad \mathbb{T}(0,R) = F(\mathbb{T}(0,r)). \tag{4.1}$$

Then there exists $n_F \in \{-1, 1\}$ such that for every $\Phi \in \operatorname{Log}(F)$,

$$\Phi(z + 2\pi) - \Phi(z) = 2\pi n_F, \quad z \in B := \operatorname{Ei}(A). \tag{4.2}$$

Proof. Given A and F satisfying the assumption let $\Phi \in \operatorname{Log}(F)$ be arbitrarily fixed. Setting

$$B \ni z \mapsto \Psi(z) := \Phi(z + 2\pi), \tag{4.3}$$

we see that Ψ is a continuous function and for every $z \in B$,

$$\mathrm{e}^{\mathrm{i}\Psi(z)} = \mathrm{e}^{\mathrm{i}\Phi(z+2\pi)} = F\big(\mathrm{e}^{\mathrm{i}(z+2\pi)}\big) = F(\mathrm{e}^{\mathrm{i}z}).$$

Therefore, $\Psi \in \operatorname{Log}(F)$. By Lemma 2.3 there exists $n \in \mathbb{Z}$ such that

$$\Phi(z+2\pi) - \Phi(z) = \Psi(z) - \Phi(z) = 2\pi n, \quad z \in B. \tag{4.4}$$

Suppose now that $|n| > 1$. By (4.4) we have

$$\frac{1}{n}\Phi(z+2\pi) - \frac{1}{n}\Phi(z) = 2\pi, \quad z \in B, \tag{4.5}$$

and so there exists a function $G : A \to \mathbb{C}$ satisfying the following condition:

$$G(\mathrm{e}^{\mathrm{i}z}) = \mathrm{e}^{\mathrm{i}\Phi(z)/n}, \quad z \in B. \tag{4.6}$$

Since $\Phi \in \operatorname{Log}(F)$, we conclude from (4.6) that

$$G(\mathrm{e}^{\mathrm{i}z})^n = \mathrm{e}^{\mathrm{i}\Phi(z)} = F(\mathrm{e}^{\mathrm{i}z}), \quad z \in B. \tag{4.7}$$

Hence and by the assumption (4.1) we obtain the inclusion

$$G(\mathbb{T}(0,r)) \subset \mathbb{T}(0, \sqrt[n]{R}). \tag{4.8}$$

Setting

$$\mathbb{R} \ni t \mapsto \varphi(t) := \frac{1}{n}\operatorname{Re}\big(\Phi(t - \mathrm{i}\log r)\big), \tag{4.9}$$

we see that φ is a real valued and continuous function, and by (4.4), $\varphi(2\pi) = \varphi(0) + 2\pi$. Using now the Darboux principle, we obtain the inclusion $[\varphi(0); \varphi(0) + 2\pi] \subset \varphi([0; 2\pi])$. Combining the properties (4.6) and (4.8), we get

$$G(r\mathrm{e}^{\mathrm{i}t}) = G\big(\mathrm{e}^{\mathrm{i}(t-\mathrm{i}\log r)}\big) = \mathrm{e}^{\mathrm{i}\Phi(t-\mathrm{i}\log r)/n} = \sqrt[n]{R}\mathrm{e}^{\mathrm{i}\varphi(t)}, \quad t \in \mathbb{R}.$$

Thus $G(\mathbb{T}(0,r)) = \mathbb{T}(0, \sqrt[n]{R})$, and consequently for each $u \in \mathbb{T}(0,r)$ there exists $v \in \mathbb{T}(0,r)$ such that $G(v) = \mathrm{e}^{2\pi\mathrm{i}/n}G(u)$. Hence $u \neq v$ and, by (4.7),

$$F(v) = G(v)^n = \mathrm{e}^{2\pi\mathrm{i}}G(u)^n = F(u).$$

This contradicts the injectivity of F. Therefore, $|n| \leq 1$. Assume that $n = 0$. Then condition (4.4) takes the following form:

$$\Phi(z+2\pi) = \Phi(z), \quad z \in B. \tag{4.10}$$

Setting

$$\mathbb{R} \ni t \mapsto \psi(t) := \operatorname{Re}\big(\Phi(t - \mathrm{i}\log r)\big), \tag{4.11}$$

we see that ψ is a real valued and continuous function, and by (4.10), $\psi(0) = \psi(2\pi)$. Applying the Darboux principle once more, we conclude that $0 < t_2 - t_1 < 2\pi$ and $\psi(t_1) = \psi(t_2)$ for some $t_1, t_2 \in [0; 2\pi]$. Setting now $z_1 := t_1 - \mathrm{i}\log r$ and $z_2 := t_2 - \mathrm{i}\log r$, we have

$$\mathrm{e}^{\mathrm{i}z_1} = r\mathrm{e}^{\mathrm{i}t_1} \neq r\mathrm{e}^{\mathrm{i}t_2} = \mathrm{e}^{\mathrm{i}z_2}, \tag{4.12}$$

and thereby $\mathrm{e}^{\mathrm{i}z_1}, \mathrm{e}^{\mathrm{i}z_2} \in \mathbb{T}(0,r)$. Hence and by the assumption (4.1) we have

$$F(\mathrm{e}^{\mathrm{i}z_1}) = \mathrm{e}^{\mathrm{i}\Phi(t_1-\mathrm{i}\log r)} = R\mathrm{e}^{\mathrm{i}\psi(t_1)} = R\mathrm{e}^{\mathrm{i}\psi(t_2)} = \mathrm{e}^{\mathrm{i}\Phi(t_2-\mathrm{i}\log r)} = F(\mathrm{e}^{\mathrm{i}z_2}).$$

This together with (4.12) contradicts the injectivity of F. Thus $0 < |n| \leq 1$, and setting $n_F := n$, we see that $n_F \in \{-1, 1\}$ and the condition (4.2) holds. Taking into account Lemma 2.3, we finally see that the condition (4.2) holds for every $\Phi \in \operatorname{Log}(F)$, which completes the proof. □

Theorem 4.2. *Let A be a connected radial set and $F : A \to \mathbb{C} \setminus \{0\}$ be a continuous and injective function. Assume that the condition* (4.1) *holds for some $r, R > 0$. Then each function $\Phi \in \operatorname{Log}(F)$ is injective.*

Proof. Given A and F satisfying the assumption let $\Phi \in \operatorname{Log}(F)$ be arbitrarily fixed. From Lemma 4.1 it follows that the condition (4.2) holds with the constant $n_F \in \{-1, 1\}$. Suppose that $\Phi(z_1) = \Phi(z_2)$ for arbitrarily fixed $z_1, z_2 \in B$. Then

$$F(\mathrm{e}^{\mathrm{i}z_1}) = \mathrm{e}^{\mathrm{i}\Phi(z_1)} = \mathrm{e}^{\mathrm{i}\Phi(z_2)} = F(\mathrm{e}^{\mathrm{i}z_2}),$$

and by the injectivity of F we have $\mathrm{e}^{\mathrm{i}z_1} = \mathrm{e}^{\mathrm{i}z_2}$. Hence $z_2 - z_1 = 2\pi m$ for a certain $m \in \mathbb{Z}$, and from the condition (4.2) we deduce that

$$0 = \Phi(z_2) - \Phi(z_1) = \Phi(z_1 + 2\pi m) - \Phi(z_1) = 2\pi m n_F,$$

which implies $m = 0$, and thereby $z_1 = z_2$. This proves the injectivity of Φ, and the proof is complete. □

Corollary 4.3. *Given $r, R > 0$ let F be a homeomorphism of $\overline{\mathbb{D}}(0, r)$ onto $\overline{\mathbb{D}}(0, R)$ keeping the origin fixed. Then each mapping $\Phi \in \operatorname{Log}(F|_A)$ is a homeomorphism of $B := \operatorname{Ei}(A)$ onto $B' := \operatorname{Ei}(A')$, where $A := \overline{\mathbb{D}}(0, r) \setminus \{0\}$ and $A' := \overline{\mathbb{D}}(0, R) \setminus \{0\}$.*

Proof. Under the assumptions of the corollary we see that A is a connected radial set, $F|_A : A \to \mathbb{C} \setminus \{0\}$ is a continuous and injective function as well as $F|_A(\mathbb{T}(0, r)) = \mathbb{T}(0, R)$. From Theorem 4.2 it follows that Φ is an injective mapping. Since $F(A) = A'$, we conclude from the property (4.2) that $\Phi(B) = B'$. By the properties of the exponential function exp we know that for every $p \in \mathbb{C}$,

$$\Omega_p \ni z \mapsto \lambda_p(z) := \mathrm{e}^{\mathrm{i}z} \tag{4.13}$$

is a conformal mapping in the domain $\Omega_p := \{z \in \mathbb{C} : |\operatorname{Re} z - \operatorname{Re} p| < \pi\}$. From the condition (2.2) it follows that for every $p \in B$ there exists $r_p > 0$ such that Φ has the following local representation:

$$\Phi(z) = \left(\lambda_{\Phi(p)}\right)^{-1} \circ F \circ \lambda_p(z), \quad z \in \mathbb{D}(p, r_p) \cap B. \tag{4.14}$$

Hence Φ^{-1} is a continuous mapping in B'. By definition, Φ is a continuous mapping in B. Thus Φ is a homeomorphism of B onto B', which is the desired conclusion. □

Corollary 4.4. *Given $r, R > 0$ and $K \geq 1$ let F be a K-quasiconformal mapping of $\mathbb{D}(0, r)$ onto $\mathbb{D}(0, R)$ keeping the origin fixed. Then each $\Phi \in$*

$\mathrm{Log}(F|_A)$ *is a* K*-quasiconformal mapping of* $B := \mathrm{Ei}(A)$ *onto* $B' := \mathrm{Ei}(A')$, *where* $A := \mathbb{D}(0,r) \setminus \{0\}$ *and* $A' := \mathbb{D}(0,R) \setminus \{0\}$.

Here and later on the quasiconformality of a mapping is understood in the sense of geometric definition, cf. [9, Chap. I §3], [1, Chap. II Sec. A].

Proof. Under the assumptions of the corollary we see that F has an extension to a homeomorphism $\tilde{F}$ of $\overline{\mathbb{D}}(0,r)$ onto $\overline{\mathbb{D}}(0,R)$, cf. [10, Theorem 4], and also [1, Corollary in Chap. III Sec. C], [9, Theorem 8.2 in Chap. I §8]. By Corollary 4.3, a given $\tilde{\Phi} \in \mathrm{Log}(\tilde{F}|_{\tilde{A}})$ is a homeomorphism of $\tilde{B} := \mathrm{Ei}(\tilde{A})$ onto $\tilde{B}' := \mathrm{Ei}(\tilde{A}')$, where $\tilde{A} := \overline{\mathbb{D}}(0,r) \setminus \{0\}$ and $\tilde{A}' := \overline{\mathbb{D}}(0,R) \setminus \{0\}$. Then the restriction $\Phi_0 := \tilde{\Phi}|_B$ is a homeomorphism of B onto B'. Since $F|_A(\mathrm{e}^{\mathrm{i}z}) = \tilde{F}|_{\tilde{A}}(\mathrm{e}^{\mathrm{i}z})$ for $z \in B$, $\Phi_0 \in \mathrm{Log}(F|_A)$. Moreover, from the representation (4.14) with $\Phi := \Phi_0$ it follows that Φ_0 is a locally K-quasiconformal mapping as a composition of the K-quasiconformal mapping F with conformal mappings, cf. [9, Chap. I §3], [1, Chap. II Sec. A]. Since Φ_0 is a homeomorphism of B onto B', Φ_0 is a K-quasiconformal mapping of B onto B', cf. [9, Theorem 9.1 in Chap. I §9], [1, Theorem 1 in Chap. II Sec. A]. By Lemma 2.3, this property is valid for every $\Phi \in \mathrm{Log}(F|_A)$, which is our assertion. □

Remark 4.5. Given $K \geq 1$ assume that F is a K-quasiconformal mapping of the unit disk $\mathbb{D}$ onto itself and satisfying the equality $F(0) = 0$. Since $A := \mathbb{D} \setminus \{0\}$ is a connected radial set and $0 \notin F(A)$, it follows from Theorem 3.2 that $\mathrm{Log}(F|_A) \neq \emptyset$. Applying Corollary 4.4 with $r := 1$ and $R := 1$ we see that each $\Phi \in \mathrm{Log}(F|_A)$ is a K-quasiconformal mapping of the upper half-plane $\mathbb{C}_+$ $(= \mathrm{Ei}(A))$ onto itself. Moreover, from Lemma 4.1 it follows that the condition (4.2) holds with $n_F = 1$, because Φ as a quasiconformal mapping is a sense-preserving homeomorphism. This completes Krzyż's proof of [7, Theorem 1].

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Magdalena Jastrzębska
Department of Applied Mathematics
Lublin University of Technology
ul. Nadbystrzycka 38D
20-618 Lublin
Poland
e-mail: `m.jastrzebska@pollub.pl`

Dariusz Partyka
Department of Mathematical Analysis
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129
20-950 Lublin
Poland
e-mail: `partyka@kul.lublin.pl`

Institute of Mathematics and Information Technology
The State School of Higher Education in Chełm
Pocztowa 54
22-100 Chełm
Poland

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