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On generalized Mersenne hybrid numbers

Abstract. The hybrid numbers are generalization of complex, hyperbolic and dual numbers. In this paper we consider a special kind of hybrid numbers, namely the Mersenne hybrid numbers and we give some of their properties.

1. Introduction and preliminary results. Let p, q, n be integers. For $n \geq 0$, Horadam (see [2]) defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation

$$W_n = p \cdot W_{n-1} - q \cdot W_{n-2}, \tag{1}$$

for $n \geq 2$ with fixed real numbers W_0, W_1.

For the historical reasons these numbers were later called the Horadam numbers. For special W_0, W_1, p, q, the equation (1) defines the well-known numbers named as the numbers of the Fibonacci type, e.g. the Fibonacci numbers $F_n = W_n(0, 1; 1, -1)$, the Jacobsthal numbers $J_n = W_n(0, 1; 1, -2)$, the Pell numbers $P_n = W_n(0, 1; 2, -1)$ and the Mersenne numbers $M_n = W_n(0, 1; 3, 2)$. Fibonacci numbers, Jacobsthal numbers, Pell numbers and others have many different generalizations, see the list of them in [1].

In [3], Ochalik and Włoch introduced the generalized Mersenne numbers as follows. Let $k \geq 3$ be a fixed integer. For any integer $n \geq 0$, let $M(k, n)$ be the nth generalized Mersenne number defined by the second order linear recurrence relation of the form

$$M(k, n) = k \cdot M(k, n-1) - (k-1) \cdot M(k, n-2) \tag{2}$$

for $n \geq 2$ with initial conditions $M(k, 0) = 0$ and $M(k, 1) = 1$.

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Then the generalized Mersenne numbers form the generalized Mersenne sequence $0, 1, k, k^2 - k + 1, k^3 - 2k^2 + 2k, \ldots$.

If $k = 3$, then $M(3, n)$ is the nth Mersenne number denoted by M_n. Note that $M(k, n) = W_n(0, 1; k, k-1)$.

Some identities and properties of $M(k, n)$, also combinatorial interpretations and matrix generators were given in [3]. In the next part of the paper we use the following results.

Theorem 1 ([3]). *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$M(k, n) = \frac{1}{k-2}\left((k-1)^n - 1\right). \tag{3}$$

Theorem 2 ([3]). *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$M(k, n+1) - M(k, n) = (k-1)^n. \tag{4}$$

For other identities, see [3]. Moreover, for $M(k, n)$ we have the following identity.

Theorem 3. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$M(k, n+1) = (k-1) \cdot M(k, n) + 1. \tag{5}$$

Proof (by induction on n).
Let $n = 0$. Then $M(k, 1) = 1$ and $(k-1) \cdot M(k, 0) + 1 = (k-1) \cdot 0 + 1 = 1$. Assume now that the identity is true for an arbitrary $n \geq 0$. We shall show that it is true for $n+1$, i.e. $M(k, n+2) = (k-1) \cdot M(k, n+1) + 1$. Using the induction hypothesis, we have

$$\begin{aligned} M(k, n+2) &= k \cdot M(k, n+1) - (k-1) \cdot M(k, n) \\ &= k \cdot M(k, n+1) - (M(k, n+1) - 1) \\ &= k \cdot M(k, n+1) - M(k, n+1) + 1 \\ &= (k-1) \cdot M(k, n+1) + 1, \end{aligned}$$

which ends the proof. □

Remark 4. For $k = 3$ we obtain the well-known identity for the classical Mersenne numbers $M_{n+1} = 2 \cdot M_n + 1$.

In this paper, we introduce and study the generalized Mersenne hybrid numbers. It is worth to mention that the Horadam hybrid numbers were introduced in [5] and consequently, the Fibonacci hybrid numbers and the like were studied in [6, 7, 8]. This paper is a sequel of them. Firstly we give the necessary definitions.

Let us consider the set $\mathbb{K}$ of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1, \ \varepsilon^2 = 0, \ \mathbf{h}^2 = 1 \tag{6}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{7}$$

Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ be any two hybrid numbers. We define equality, addition, subtraction and multiplication by scalar in the following way:

$\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$ (equality)
$\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$ (addition)
$\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h}$ (subtraction)
$s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$ (multiplication by scalar $s \in \mathbb{R}$).

Using equalities (6) and (7), we define multiplication of hybrid numbers. Moreover, by formulas (6) and (7) the product of any two hybrid operators can be calculated, see Table 1.

Table 1. The hybrid number multiplication

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\varepsilon + \mathbf{i}$
ε	$\mathbf{h} + 1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon - \mathbf{i}$	ε	1

From the above rules the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. Note that the multiplication operation in the hybrid numbers is associative, but not commutative.

The conjugate of a hybrid number $\mathbf{Z}$ is defined by

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\varepsilon + d\mathbf{h}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

The real number

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is called the character of the hybrid number $\mathbf{Z}$.

The hybrid numbers were introduced by Özdemir in [4] as a generalization of complex, hyperbolic and dual numbers.

For $n \geq 0$, the nth Horadam hybrid number H_n is defined as

$$H_n = W_n + \mathbf{i}W_{n+1} + \varepsilon W_{n+2} + \mathbf{h}W_{n+3}. \tag{8}$$

Some interesting results for the Horadam hybrid numbers were obtained in [5]. Now we will define the generalized Mersenne hybrid numbers, which form a subclass of the Horadam hybrid numbers (defined by (8)).

Let $n \geq 0$ be an integer. We define the generalized Mersenne hybrid sequence $\{MH_n^k\}$ by the following recurrence:

$$MH_n^k = M(k,n) + \mathbf{i}M(k,n+1) + \varepsilon M(k,n+2) + \mathbf{h}M(k,n+3), \tag{9}$$

where $M(k,n)$ denotes the nth generalized Mersenne number, defined by (2). For $k = 3$ we have the Mersenne hybrid numbers denoted by MH_n.

Using the above definitions, we can write the initial generalized Mersenne hybrid numbers:

$$\begin{aligned} MH_0^k &= 0 + \mathbf{i} + \varepsilon \cdot k + \mathbf{h} \cdot (k^2 - k + 1), \\ MH_1^k &= 1 + \mathbf{i} \cdot k + \varepsilon \cdot (k^2 - k + 1) + \mathbf{h} \cdot (k^3 - 2k^2 + 2k) \end{aligned} \tag{10}$$

and the Mersenne hybrid numbers:

$$\begin{aligned} MH_0 &= 0 + \mathbf{i} + 3\varepsilon + 7\mathbf{h}, \\ MH_1 &= 1 + 3\mathbf{i} + 7\varepsilon + 15\mathbf{h}. \end{aligned}$$

2. Main results. In this section we present some properties of the generalized Mersenne hybrid numbers.

Theorem 5. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$MH_{n+2}^k = k \cdot MH_{n+1}^k - (k-1) \cdot MH_n^k,$$

where MH_0^k and MH_1^k are defined by (10).

Proof. By formula (9) we get

$$\begin{aligned} k \cdot & MH_{n+1}^k - (k-1) \cdot MH_n^k \\ &= k \cdot (M(k,n+1) + \mathbf{i}M(k,n+2) + \varepsilon M(k,n+3) + \mathbf{h}M(k,n+4)) \\ &\quad - (k-1) \cdot (M(k,n) + \mathbf{i}M(k,n+1) + \varepsilon M(k,n+2) + \mathbf{h}M(k,n+3)) \\ &= k \cdot M(k,n+1) - (k-1) \cdot M(k,n) \\ &\quad + \mathbf{i}\,(k \cdot M(k,n+2) - (k-1) \cdot M(k,n+1)) \\ &\quad + \varepsilon\,(k \cdot M(k,n+3) - (k-1) \cdot M(k,n+2)) \\ &\quad + \mathbf{h}\,(k \cdot M(k,n+4) - (k-1) \cdot M(k,n+3)) \\ &= M(k,n+2) + \mathbf{i}M(k,n+3) + \varepsilon M(k,n+4) + \mathbf{h}M(k,n+5) \\ &= MH_{n+2}^k. \end{aligned}$$

□

In [5], the character of a Horadam hybrid number was calculated.

Theorem 6 ([5]). *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} C(H_n) &= W_n^2\big(1 - p^2q^2\big) + W_nW_{n+1}\big(2q + 2p^3q - 2pq^2\big) \\ &\quad + W_{n+1}^2\big(1 - 2p - p^4 + 2p^2q - q^2\big). \end{aligned} \tag{11}$$

By formula (11) we get the following result.

Corollary 7. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} C(MH_n^k) &= \big(1 - k^2(k-1)^2\big) \cdot M^2(k,n) \\ &\quad + \big(2(k-1) + 2k^3(k-1) - 2k(k-1)^2\big) \cdot M(k,n)M(k,n+1) \\ &\quad + \big(1 - 2k - k^4 + 2k^2(k-1) - (k-1)^2\big) \cdot M^2(k,n+1). \end{aligned}$$

Remark 8. Using (5), we get another form of character of the generalized Mersenne hybrid number:

$$\begin{aligned} C(MH_n^k) &= \big(1 + (k-1)^2(-k^4 + 4k^3 - 6k^2 + 2k + 2)\big) \cdot M^2(k,n) \\ &\quad + (k-1)\big(- 2k^4 + 6k^3 - 8k^2 + 2k + 2\big) \cdot M(k,n) \\ &\quad - k^4 + 2k^3 - 3k^2. \end{aligned}$$

Remark 9. For $k = 3$, we get the character of the Mersenne hybrid numbers:

$$C(MH_n) = -75 \cdot M_n^2 - 128 \cdot M_n - 54.$$

Next theorem gives the Binet formula for the generalized Mersenne hybrid numbers.

Theorem 10. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$MH_n^k = M(k,n) \cdot (1 + \mathbf{i} + \varepsilon + \mathbf{h}) + (k-1)^n \cdot MH_0^k. \tag{12}$$

Proof. Using (4), we have

$M(k,n+1) = M(k,n) + (k-1)^n$,

$M(k,n+2) = M(k,n+1) + (k-1)^{n+1} = M(k,n) + (k-1)^n + (k-1)^{n+1}$

and

$M(k,n+3) = M(k,n) + (k-1)^n + (k-1)^{n+1} + (k-1)^{n+2}$,

so

$$\begin{aligned} MH_n^k &= M(k,n) + \mathbf{i}M(k,n+1) + \varepsilon M(k,n+2) + \mathbf{h}M(k,n+3) \\ &= M(k,n)\,(1 + \mathbf{i} + \varepsilon + \mathbf{h}) \\ &\quad + \mathbf{i}(k-1)^n + \varepsilon\left((k-1)^n + (k-1)^{n+1}\right) \\ &\quad + \mathbf{h}\left((k-1)^n + (k-1)^{n+1} + (k-1)^{n+2}\right) \\ &= M(k,n)(1 + \mathbf{i} + \varepsilon + \mathbf{h}) + (k-1)^n \left(\mathbf{i} + \varepsilon k + \mathbf{h}\big(k^2 - k + 1\big)\right) \\ &= M(k,n)(1 + \mathbf{i} + \varepsilon + \mathbf{h}) + (k-1)^n MH_0^k. \qquad \square \end{aligned}$$

Remark 11. For $k = 3$, we have $MH_n = M_n(1+\mathbf{i}+\varepsilon+\mathbf{h}) + 2^n\,(\mathbf{i} + 3\varepsilon + 7\mathbf{h})$.

Using (3), we obtain the next results.

Remark 12. Let $n \geq 0$, $k \geq 3$ be integers. Then

$$MH_n^k = \frac{(k-1)^n - 1}{k-2}(1 + \mathbf{i} + \varepsilon + \mathbf{h}) + (k-1)^n MH_0^k.$$

Remark 13. For $k=3$, we have

$$\begin{aligned} MH_n &= (2^n-1)(1+\mathbf{i}+\varepsilon+\mathbf{h})+2^n(\mathbf{i}+3\varepsilon+7\mathbf{h}) \\ &= 2^n(1+2\mathbf{i}+4\varepsilon+8\mathbf{h})-(1+\mathbf{i}+\varepsilon+\mathbf{h}). \end{aligned}$$

Using the Binet formula (12) and identity (3), one can easily derive the Cassini identities for the generalized Mersenne hybrid numbers and Mersenne hybrid numbers.

Theorem 14. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$\begin{aligned} &MH_{n+2}^k \cdot MH_n^k - \left(MH_{n+1}^k\right)^2 \\ &\quad = \left[\left(-k^3+2k^2-k-1\right)+\mathbf{i}\cdot\left(k^3-k^2-k\right)+\varepsilon\cdot(k-2) \right. \\ &\qquad \left. +\mathbf{h}\cdot\left(-k^3+2k^2-2k\right)\right]\cdot(k-1)^n. \end{aligned}$$

Theorem 15. *Let $n \geq 0$ be an integer. Then*

$$MH_{n+2}\cdot MH_n-(MH_{n+1})^2=(-13+15\mathbf{i}+\varepsilon-15\mathbf{h})\cdot 2^n.$$

Next we shall give the ordinary generating function for the generalized Mersenne hybrid numbers.

Theorem 16. *The generating function for the generalized Mersenne hybrid number sequence $\{MH_n^k\}$ is*

$$G(t)=\frac{MH_0^k+(MH_1^k-k\cdot MH_0^k)t}{1-kt+(k-1)t^2}.$$

Proof. Assume that the generating function of the generalized Mersenne hybrid number sequence $\{MH_n^k\}$ has the form $G(t)=\sum_{n=0}^{\infty} MH_n^k t^n$. Then

$$\begin{aligned} &\left(1-kt+(k-1)t^2\right)G(t) \\ &\quad = \left(1-kt+(k-1)t^2\right)\cdot\left(MH_0^k+MH_1^k t+MH_2^k t^2+\dots\right) \\ &\quad = MH_0^k+MH_1^k t+MH_2^k t^2+\dots \\ &\qquad -k\cdot MH_0^k t-k\cdot MH_1^k t^2-k\cdot MH_2^k t^3-\dots \\ &\qquad +(k-1)MH_0^k t^2+(k-1)MH_1^k t^3+(k-1)MH_2^k t^4+\dots \\ &\quad = MH_0^k+(MH_1^k-k\cdot MH_0^k)t, \end{aligned}$$

since $MH_n^k=k\cdot MH_{n-1}^k-(k-1)\cdot MH_{n-2}^k$ and the coefficients of t^n for $n\geq 2$ are equal to zero. Moreover,

$MH_0^k=\mathbf{i}+\varepsilon\cdot k+\mathbf{h}\cdot\left(k^2-k+1\right)$, $MH_1^k-k\cdot MH_0^k=1+\varepsilon\cdot(-k+1)+\mathbf{h}\cdot\left(-k^2+k\right)$. □

Remark 17. For $k=3$, we have the generating function for the Mersenne hybrid number sequence $\{MH_n\}$:

$$g(t)=\frac{(\mathbf{i}+3\varepsilon+7\mathbf{h})+(1-2\varepsilon-6\mathbf{h})t}{1-3t+2t^2}.$$

Finally, we give the matrix representation of the generalized Mersenne hybrid numbers.

Theorem 18. *Let* $n \geq 0$, $k \geq 3$ *be integers. Then*

$$\begin{bmatrix} MH^k_{n+2} & MH^k_{n+1} \\ MH^k_{n+1} & MH^k_n \end{bmatrix} = \begin{bmatrix} MH^k_2 & MH^k_1 \\ MH^k_1 & MH^k_0 \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n .$$

Proof (by induction on n).
If $n = 0$, then assuming that the matrix to the power 0 is the identity matrix, the result is obvious. Now suppose that for any $n \geq 0$ the following formula holds:

$$\begin{bmatrix} MH^k_{n+2} & MH^k_{n+1} \\ MH^k_{n+1} & MH^k_n \end{bmatrix} = \begin{bmatrix} MH^k_2 & MH^k_1 \\ MH^k_1 & MH^k_0 \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n .$$

We shall show that

$$\begin{bmatrix} MH^k_{n+3} & MH^k_{n+2} \\ MH^k_{n+2} & MH^k_{n+1} \end{bmatrix} = \begin{bmatrix} MH^k_2 & MH^k_1 \\ MH^k_1 & MH^k_0 \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^{n+1} .$$

By simple calculation using the induction hypothesis, we have

$$\begin{bmatrix} MH^k_2 & MH^k_1 \\ MH^k_1 & MH^k_0 \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} MH^k_{n+2} & MH^k_{n+1} \\ MH^k_{n+1} & MH^k_n \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} k \cdot MH^k_{n+2} - (k-1) \cdot MH^k_{n+1} & MH^k_{n+2} \\ k \cdot MH^k_{n+1} - (k-1) \cdot MH^k_n & MH^k_{n+1} \end{bmatrix} = \begin{bmatrix} MH^k_{n+3} & MH^k_{n+2} \\ MH^k_{n+2} & MH^k_{n+1} \end{bmatrix},$$

which ends the proof. □

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