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Feynman Path Integration in Optics: Numerical Example#

1. Introduction.

In the following article we want to show how to solve the problem of propagation of light in a planar waveguide structure with a graded refractive index. The method here is a consequence of Feynman path integration formalism [1]. This formalism was developed for quantum ,mechanics and can be applied to solve the typical quantum mechanical problems [1,2].

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2. Schrödinger equation in paraxial optics. ##)

In the case of light propagation in media with graded index profile the Schrödinger—like equation appears in the so called paraxial optics [3,4]. In order to have such a situation consider the planar waveguide directed along z axis. The coordinate x is perpendicular to the propagation direction and measures the lateral dimension of the waveguide. The Schrödinger equation for this case reads [4]:

$$i\,\frac{\lambda}{2\pi}\,\frac{\partial E(x,z)}{\partial z} = \left\{-\frac{(\lambda/2\pi)^2}{2n_0}\,\nabla_\perp^2 + n_1(x)\right\} E(x,z) = \mathbb{H}\,E(x,z). \tag{1}$$

Here $n_1(x)$ is the deviation of the refractive index $n(x)$ from its constant value n_0: $n(x)=n_0+n_1(x)$, λ is the wave lenght of propagated light and $E(x,z)$ is the electric field in the waveguide. In this equation the role of the Planck constant is played by the reduced wavelength $\lambda/2\pi$ of propagated light.

To solve this equation one can use the Green function $G(x,x';z)$ which is defined as a propagator of the point signal as follows:

$$\left[\mathbb{H} - i\frac{\lambda}{2\pi}\frac{\partial}{\partial z}\right] G(x,x';z-z_0) = -i\frac{\lambda}{2\pi}\delta(x-x')\,\delta(z-z'). \tag{2}$$

The Green function $G(x,x';z)$ is related to propagation constants $\{\beta_m\}$ through the relation known from the analog situation in quantum mechanics. The formula reads:

$$G(x,x';z) = \sum_{m=0}^{\infty} u_m(x)\,\exp(i\beta_m z)\,u_m^*(x'), \tag{3}$$

where $u_m(x)=\langle x|m\rangle$ is a normalized eigenfunction and β_m is the eigenvalue of the $\mathbb{H}$ operator defined in eq.(1):

$$\mathbb{H}|m\rangle=-(\lambda/2\pi)\beta_m|m\rangle. \tag{4}$$

By the use of the orthonormality of the modal eigenfunctions $u_m(x)$, one can find the useful relation between G and $\{\beta_m\}$, which does not exploit the explicit form of these functions. One has

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##) This chapter and Chapter 3 follow closely the material of paper [3] by Hawkins and serve as a guide for understanding Chapter 4.

$$\mathrm{Tr}\, G(x,x';z) \equiv \int_{-\infty}^{\infty} G(x,x;z)\, dx = \sum_{m=0}^{\infty} \exp(i\beta_m z). \tag{5}$$

In the next step we use the analitical continuation of the Green function $G(x,x';z)$ by the substitution $\tau = -iz$. If τ is large and positive then the largest contribution to the sum in equation (5) comes from the lowest eigenvalue β_0 of $\mathbb{H}$ operator. Thus one has the relation

$$\beta_0 = -\lim_{\tau\to\infty} \frac{1}{\tau} \ln \mathrm{Tr}\, G(x,x';i\tau), \tag{6}$$

which is the known Feynman–Kac formula [2]. In the same limit of large τ one can also obtain the intensity distribution function

$$I_0(x) = \lim_{\tau\to\infty} \frac{G(x,x;i\tau)}{\mathrm{Tr}\, G(x,x';i\tau)}. \tag{7}$$

3. *The evaluation of the Green function.*

In order to evaluate the Green functions needed for calculation of the propagation constant β_0 of the fundamental mode and the intensity distribution function I_0 one can use the *short time approximation* [3,5–7] and exploit the group property of the evolution operator $\exp(-\mathbb{H}z/\lambda)$. In the case of light propagation equation the *short time* corresponds to *short distance*. For the short distance slice 2ϵ one has

$$G(x,x';2i\epsilon) = \int_{-\infty}^{\infty} G(x,x'';i\epsilon)\, G(x'',x';i\epsilon)\, dx'' \tag{8}$$

If we chose the final distance equal to τ then the successive iterations of this equation give the Green function corresponding to τ. In the following we take $\tau = 2^N \epsilon$, where 2^N is the convenient number of distance slices in the distance interval τ, each equal to ϵ. Thus, we have made the distance discretized for our numerical purposes. If N (or 2^N) is large then this approximation converges to the "*smooth*" Green function $G(.,.;i\tau)$.

The short distance propagator $G(x,x';i\epsilon)$ is given by [3,5,6]

$$G(x,x';i\epsilon) = \langle x|e^{-2\pi\epsilon H/\lambda)}|x'\rangle \approx$$
$$\approx \exp\{-\frac{\pi\epsilon}{\lambda}[n_1(x)+n_2(x')]\} \exp[-\frac{\pi n_0}{\epsilon\lambda}|x-x'|^2]. \quad (9)$$

The integral in eq.1 can be calculated by using rectangular quadrature approximation of the form

$$G(x_k,x_l;2i\epsilon) \approx \Delta \sum_{j=0}^{M} G(x_k,x_j;i\epsilon)\, G(x_j,x_l;i\epsilon), \quad (10)$$

where $x_j \in (-C,C)$ and $j=1,\ldots,M$. The parameter Δ is the griding distance of x coordinate and is small enough to make the quadrature approximation as good as possible. The cut off parameter C is finite in calculations. However its value is larger then the lateral measure of the considered waveguide. As it was pointed out in [3] the convergence of the iterating procedure is obtained when both parameters Δ and ϵ fulfill the condition $n_0\Delta^2/\lambda\epsilon < 0.1$. This last criterion was also tested in our calculations.

The iteration procedure described above can be make very fast if one exploits the properties (reality and symmetry) of the Green matrices $G(i\epsilon)$ for the infinitesimal distance ϵ and $G(i\tau)$ for the finite distance τ. From equation (10) one has

$$G(i\tau) = \Delta^{2^N-1}\,[\,G(i\epsilon)\,]^{2^N}. \quad (11)$$

Diagonalizing real and symmetric matrix G by a proper matrix S one finds

$$D(i\tau) = S^{-1}G(i\tau)S = S^{-1}\Delta^{2^N-1}[G(i\epsilon)]^{2^N}S = \Delta^{2^N-1}D^{2^N}(i\epsilon), \quad 12)$$

where D(...) is the diagonal matrix with the eigenvalues of G(...). These results together with the approximation (10) give for the trace operation (eq.5)

$$\mathrm{Tr}\, G(i\tau) = \Delta^{2^N} \sum_k d_{kk}^{2^N}, \quad (13)$$

where d_{kk} are the diagonal elements of $D(i\epsilon)$.

3. *Example 1: Single strip.*

Consider the simple case of a strip waveguide with the graded refractive index in which the index deviation is

$$n_1(x) = \frac{\delta n}{\cosh^2(2x/h)}, \tag{14}$$

where δn is the maximum deviation from n_0 and h is the lateral measure of the strip. The solution of the analogous quantum mechanical problem is discussed in Landau and Lifshitz textbook on quantum mechanics [8] and the expression for propagation constants $\{\beta_m\}$ is given in the Kogelnik's paper [9]. The formula reads

$$\beta_m^2 = -\frac{2}{n_0 h^2}\left(\frac{\lambda}{2\pi}\right)^2 (s-m)^2, \tag{15}$$

where

$$2s=\{[1-2n_0\,\delta n(\frac{2\pi h}{\lambda})^2]^{1/2}-1\}, \tag{16}$$

and m is the mode number. We will use the following values of the waveguide parameters: $n_0=2.2$, $\delta n=0.006$, corresponding to the Ti diffused in $LiNbO_3$ [3,9].

Table 1. Propagation constant for zero mode β_0 ($\lambda=1.3$ μm)*.

h [μm]	β_0(calc)	β_0(exact)
6	−0.01903	−0.01903
7	−0.02020	−0.02020
8	−0.02112	−0.02112

*) *This result coincide with that in Table 1. of paper [9].*

Figure 1 shows the index profile and the calculated intensity distribution $I_0(x)$ for the fundamental mode in the case of strip with lateral dimension h=6μm and the typical length of the propagating light, $\lambda=1.3$ μm. Exact and calculated values of β_0 for different widths h are shown in Table 1. Notice a very good agreement of the calculated values of propagation constants with the exact ones. This example serves also as the test of our numerical procedure.

5. Example 2: Double strip.

As a more complicated example of application of the Feynman path integration let us consider the two strip planar waveguide with completely symmetric index profile. The index profile is modelled by smoothly joint parabolic curves or by two Gauss functions. The parameters listed in Figure 2 as well as in other figures in this paper are chosen so as to be close to realistic situation like this in Chapter 3. The results of numerical calculations are shown in Figures 3 – 10. By the use of different models and various parameters one can show the basic properties of the solutions of Schrödinger equation (1).

Figures 2 to 6 show the behaviour of the intensity function for various widths and heights of the index profile and for different distance between strips. For the symmetrical index the intensity distribution is also symmetrical. In the case of well separated strips the disturbed symmetry causes the signal to disappear either in the lower strip or in the narrower one. For more overlapped strips the intensity distribution is located in one of the strips and shows a hump at the side of the other strip.

The similar situation can be observed for the refractive index built of two Gauss functions. The deviation of the refractive index is

$$n_1(x) = \frac{\delta n_1}{\sigma_1\sqrt{2\pi}} \exp[-\frac{(x-x_2)^2}{2\sigma_1^2}] + \frac{\delta n_2}{\sigma_2\sqrt{2\pi}} \exp[-\frac{(x-x_2)^2}{2\sigma_2^2}]. \tag{18}$$

Here the standard convention for parameters is used. The results are shown in Figures 7 to 10. The interesting results for this case are those for asymmetrical widths and/or asymmetrical heights of the refractive index profile. The propagation constants for fundamental mode are given in each figure.

6. Discussion.

The above examples show that the method of path integration is effective in the one dimensional case and for various kinds of refractive indices. Its application is independent on the symmetry property of the index profile. The similar procedure can be applied to determine the propagation constants for higher order modes.

The path integration can also be realized by using the Monte Carlo method [6]. However the calculation times in this case are one order of magnitude larger as compared to these of the present method.

It seems that the described procedure can be succesfully applied to two dimensional cases. However the large matrices entering the calculations will considerably slow down the effectivness of the matrix multiplication algorithm and the time of calculations will be very large.

7. References

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Figure 1.

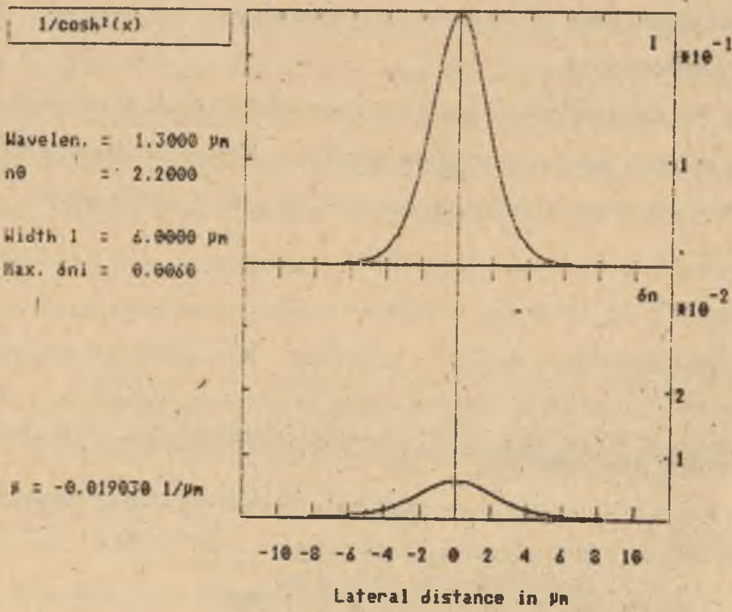
1/cosh²(x)
Wavelen. = 1.3000 µm
n0 = 2.2000
Width l = 6.0000 µm
Max. δni = 0.0060
β = -0.019030 1/µm
I
*10^-1
1
δn
*10^-2
2
1
-10 -8 -6 -4 -2 0 2 4 6 8 10
Lateral distance in µm

Figure 2.

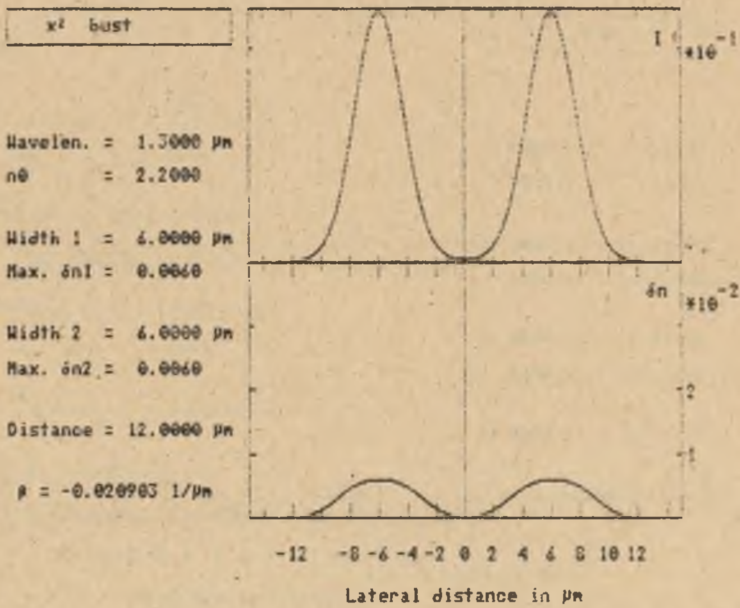

Figure 3.

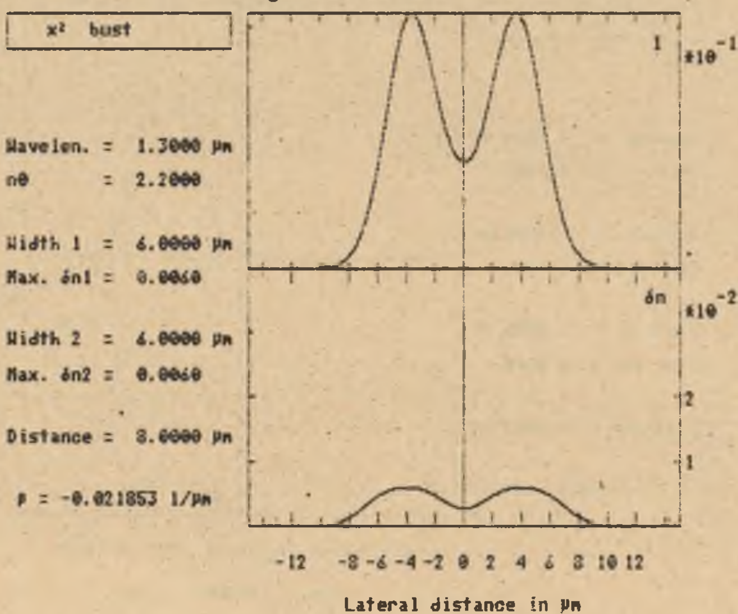

Figure 4.

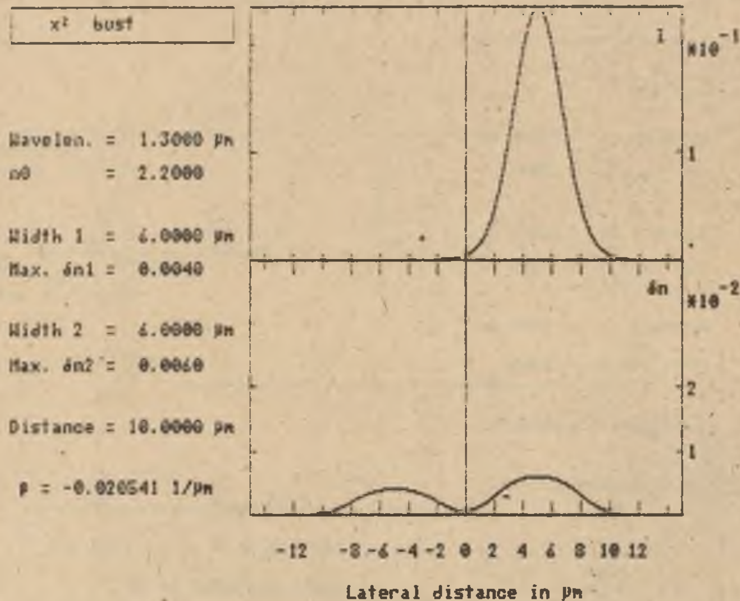

Figure 5.

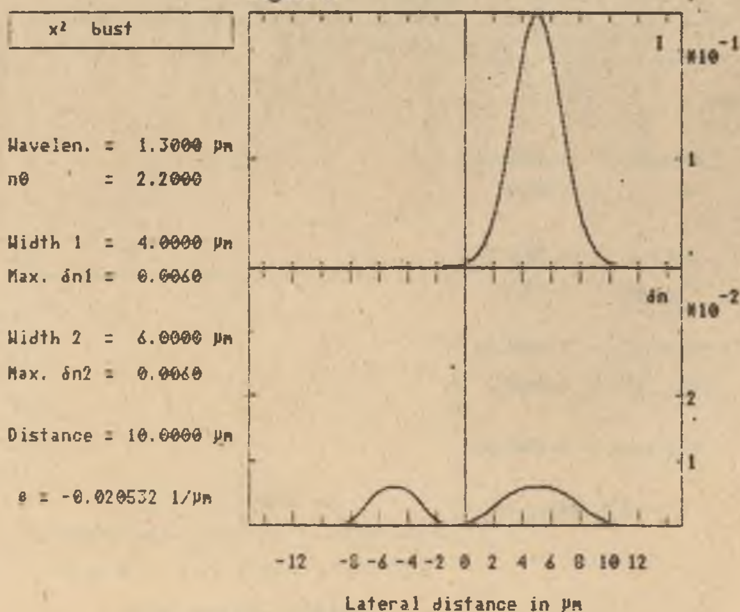

Figure 6.

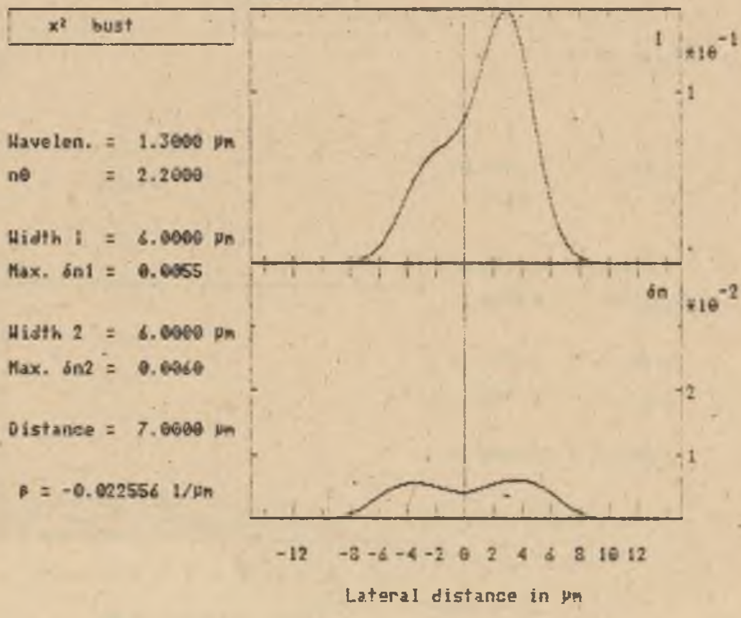
x² bust
Wavelen. = 1.3000 µm
n0 = 2.2000
Width 1 = 6.0000 µm
Max. δn1 = 0.0055
Width 2 = 6.0000 µm
Max. δn2 = 0.0060
Distance = 7.0000 µm
β = -0.022556 1/µm
I
*10⁻¹
1
δn
*10⁻²
2
1
-12 -8 -6 -4 -2 0 2 4 6 8 10 12
Lateral distance in µm

Figure 7.

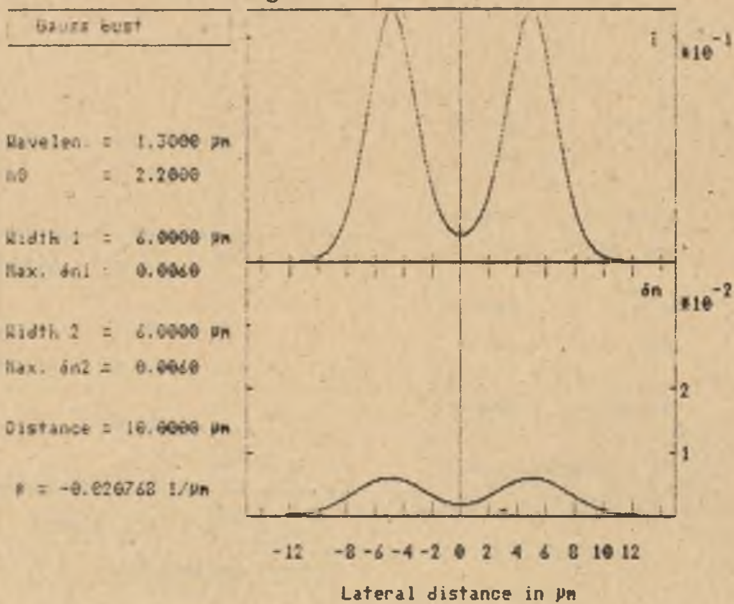

Figure 8.

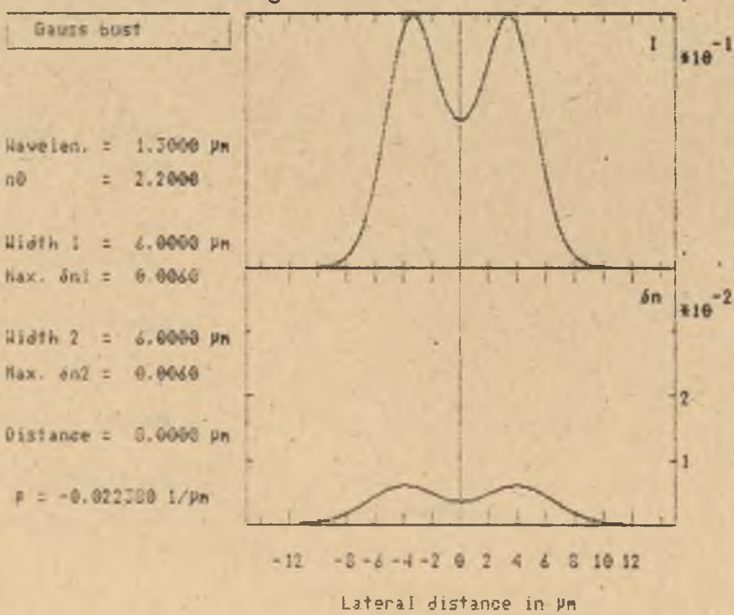

Figure 9.

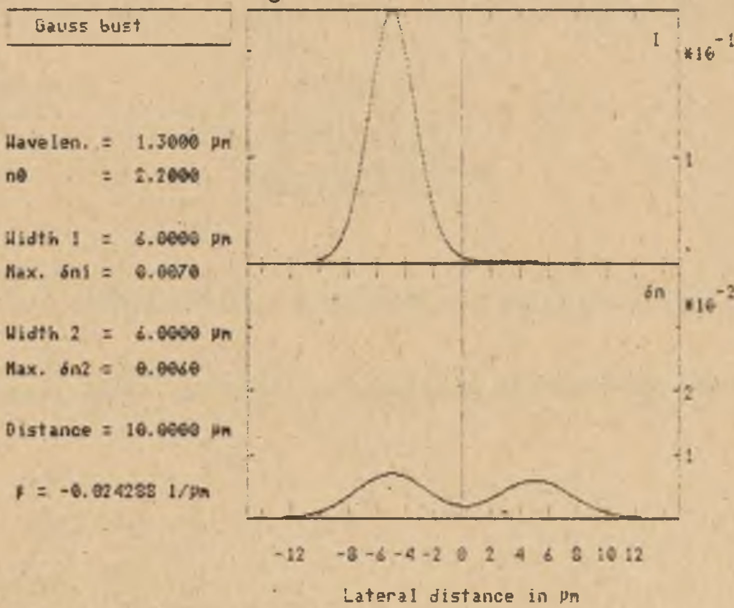

Figure 10.