ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA

Vol. XXXIII, 1 Sectio AAA 1978

Instytut Fizyki UMCS
Zakład Fizyki Teoretycznej
Kierownik: prof. dr hab. Stanisław Szpikowski

Andrzej GÓŹDŹ, Stanisław SZPIKOWSKI

Interacting Octupole Bosons and Its Group-Theory Background

Teoriogrupowe podstawy oddziałujących bozonów oktupolowych

Теориогрупповые основы взаимодействующих октупольных бозонов

INTRODUCTION

Recently the Interacting Boson Approximation Model (IBA) to deal with the collective states for even nuclei is rapidly developed [1-5]. However, only quadrupole states can be interpreted within the given formalism with the eventual of one-phonon states of the higher multipolarity. According to known suggestions [6,5] the important role is played by other collective motions especially by the octupole degrees of freedom. Calculations based upon the Bohr-Mottelson geometrical model showed [7] that in some cases almost pure two-phonon octupole states ought to appear. The extension of theoretical predictions is rather important to suggest experimental search for many-phonon states.

The aim of the paper is to extend the IBA formalism as to take into account the interaction of octupole bosons.

GROUP-THEORY FORMALISM

Quadrupole bosons, with the angular momentum quantum number $l=2$, so far considered, were followed by the $SU(5)$ standard symmetry with five one-spinless boson states for the 5-dimension group -space. For octupole bosons the base is of 7-dimensions with seven one-boson states ($l=3$) which form the base of the fundamental representation of the $SU(7)$ symmetry.

Generators of the infinitesimal transformations are

$$\left(f_3^+ \tilde{f}_3\right)^L_M \qquad L=1,2,\ldots,6 \tag{1}$$

where $f^+_{3m}|0\rangle$ is the one-phonon octupole state and

$$\tilde{f}_{3m} = -(-1)^m f_{3-m}$$

The commutator of two generators reads

$$\left[\left(f_3^+ \tilde{f}_3\right)^{L_1}_{M_1}, \left(f_3^+ \tilde{f}_3\right)^{L_2}_{M_2}\right] = \sqrt{(2L_1+1)(2L_2+1)} \sum_{LM}\left[(-1)^L-(-1)^{L_1+L_2}\right] \times (L_1M_1L_2M_2|LM)\begin{Bmatrix} L_1 & L_2 & L \\ 3 & 3 & 3 \end{Bmatrix}\left(f_3^+ \tilde{f}_3\right)^L_M \tag{2}$$

The operators (1) for odd L ($L = 1,3,5$) form the closed set of generators of the $O(7)$ transformations, too.

Following the traditional way of atomic spectroscopy [8], we introduce here also the G_2 boson group generated by the operators (1) restricted to $L = 1,5$

The last symmetry, as usual, is the rotational symmetry with angular momentum operators as generators of $O(3)$ transformations

$$L_+ = -2\sqrt{14}\left(f_3^+ \tilde{f}_3\right)^1_1 \qquad L_- = +2\sqrt{14}\left(f_3^+ \tilde{f}_3\right)^1_{-1} \qquad L_0 = +2\sqrt{7}\left(f_3^+ \tilde{f}_3\right)^1_0 \tag{3}$$

In that way the group-symmetry chain for further considerations is

$$SU(7) \supset O(7) \supset G_2 \supset O(3) \tag{4}$$

To adopt the above symmetry chain means, among others, consideration of the interaction between the octupole bosons only.

Conclusions following this model will be consequently applied to such collective states of atomic nuclei which are pure, or almost pure, many-octupole phonon states.

Classification of many-octupole phonon states is simplified as the considered states are completely symmetrized ones. For such states the irreducible representations of the $SU(7)$ symmetry group are very restricted and are denoted by one number, namely the number of bosons involved: $[N]$.

Moreover, the irreducible representation $[N]$ of the SU (7) group splits into the irreducible representations of the orthogonal group O (7) in the simple way

$$[N] = \sum \oplus (v,0,0) \tag{5}$$

where v=N, N-2,..., 1 or 0 is the boson seniority number and $(v,0,0)$ means the fully symmetrized irreducible representation of the O (7) group. Each of allowed **representations (v,0,0)** appears only once in the decomposition (5). Even more simple is the next step in the chain (4) as the irreducible representation $(v,0,0)$ of the O (7) group remains irreducible as the representation of the G_2 group. The last one is factorized by $(v,0)$.

Nontrivial problem arises, however, in the decomposition of the irreducible representations $(v,0)$ into the irreducible representations (L) of the rotational group O (3). For a given $(v,0)$ the same (L) may appear more than once and the additional non-specified quan**tum numbers χ must be introduced to distinguish the same (L) within a given (v,0).**

The many-boson states in the chain (4) are then

$$|N v \chi L M\rangle \tag{6}$$

DECOMPOSITION OF THE IRREDUCIBLE REPRESENTATION (v,0) OF THE G_2 SYMMETRY GROUP INTO THE IRREDUCIBLE REPRESENTATIONS (L) OF THE O (3) ROTATIONAL GROUP

For the effective decomposition we make use of the paper of Shi-Sheng-Ming [9] where the formula for the multiplicity of the representation (L) in an irreducible representation of the G_2 group was given. The results of that work are here extended to a handful form in applications.

In the two-dimensional root space (Fig. 1) with the same angle of 30° between roots, we choose the non-orthogonal base

$$\vec{h}_1 = (1,0) \qquad \vec{h}_2 = (0,1) \tag{7}$$

Scalar product of two vectors in this base is

$$(\vec{a}|\vec{b}) = (a_1 a_2 | b_1 b_2) = a_1 b_1 + a_2 b_2 + \frac{1}{2}(a_1 b_2 + a_2 b_1) \tag{8}$$

and the roots are

$$\vec{\alpha}_1 \equiv h_2 = (0,1) \qquad \vec{\alpha}_2 = (1,-2) \tag{9}$$

$$\vec{\alpha}_3 = h_1 = (1,0) \qquad \vec{\alpha}_4 = (1,-1)$$

$$\vec{\alpha}_5 = \ (2,-1) \qquad \vec{\alpha}_6 = (1,1)$$

with

$$\vec{\alpha}_{-i} = -\vec{\alpha}_i$$

The roots $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are called the prime roots.

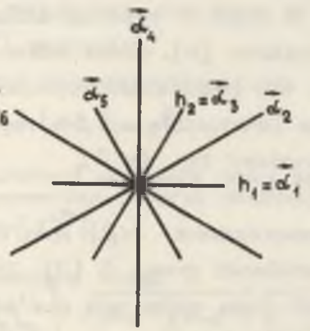

Fig. 1. Roots for the G_2 group: $\frac{|\vec{\alpha}_2|}{|\vec{\alpha}_1|} = \sqrt{3}$ and $\sphericalangle(\vec{\alpha}_1, \vec{\alpha}_2) = 30°$

We label as $\vec{\Lambda}$, $\vec{\phi}$, $\vec{\xi}$ the following vectors in the root-space

$$\vec{\Lambda} = (v,0)$$
$$\vec{\phi} = \frac{1}{2}\sum_{i=1}^{6}\vec{\alpha}_i = (3,-1) \tag{10}$$
$$\vec{\xi} = \vec{\Lambda} + \vec{\phi} = (v+3,-1)$$

Let us introduce the operators σ_i

$$\sigma_i \vec{\xi} = \vec{\xi} - 2\frac{(\vec{\xi}|\vec{\alpha}_i)}{(\vec{\alpha}_i|\vec{\alpha}_i)}\vec{\alpha}_i \qquad i = 1,2,\dots,6 \tag{11}$$

which mean the reflections of the vector $\vec{\xi}$ with respect to the straight lines passing through the origin perpendicular to the root vectors $\vec{\alpha}_i$. If we apply the transformations σ_i several times we get in addition only six further transformations σ_i, $i = 7,8,\dots,12$. These transformations form the Weyl's group with the known property

$$\det \sigma_i = -1 \qquad \text{for} \quad i \leqslant 6$$
$$\det \sigma_i = +1 \qquad \text{for} \quad i > 6 \tag{12}$$

Let us define the vector $\vec{f}$ from conditions

$$(\vec{f}|\vec{\alpha}_1)=1 \quad (\vec{f}|\vec{\alpha}_2)=1 \tag{13}$$

which give

$$\vec{f}=\left(\frac{1}{3},-\frac{2}{3}\right) \tag{14}$$

Then we introduce the c-number $O(\vec{\xi})$:

$$O(\vec{\xi})=(\vec{\xi}|\vec{f})=3\xi_1+\xi_2 \tag{15}$$

Hence

$$O(\vec{\phi})=8 \tag{16}$$

The multiplicity of a given L in the irreducible representation $(v,0)$ of the G_2 group is given by the formula [9]

$$C_L=M_L-M_{L+1} \tag{17}$$

where M_L can be calculated from recurrence relation

$$M_L=\sum_{\delta\in W}\det\delta-\sum_{\delta\in W}\det\delta\, M_{L+O(\vec{\phi}-\delta\vec{\phi})} \tag{18}$$
$$O(\delta\vec{\xi})-O(\vec{\phi})=L \quad \delta\neq 1$$

where W is the Weyl's group and 1 is its identity transformation.

We can also express the C_L from (17) and (18) by the recurrence formula

$$\begin{aligned}C_L=&\delta_{L,v-3}+\delta_{L,2v-5}+\delta_{L,3v}-\delta_{L,v-8}-\delta_{L,2v-1}-\delta_{L,3v-1}\\&+C_{L+1}+C_{L+2}+C_{L+8}+C_{L+9}+C_{L+10}+C_{L+15}\\&-C_{L+5}-C_{L+6}-C_{L+7}-C_{L+13}-C_{L+14}\end{aligned} \tag{19}$$

with **conditions**

$C_{3v}=1$ and $C_K=0$ for $k>3v$

We have obtained the more compact relation for the multiplicity C_L in the form

$$C_{3v-k} = \begin{cases} \sum\limits_{n=0} \gamma_{k-n}\, g(n-v) & \text{for } k \leqslant 3v \\ 0 & \text{for } k > 3v \end{cases} \tag{20}$$

where the coefficients γ_{k-n} are independent of seniority v and can be obtained from the relation

$$\gamma_i = \gamma_{i-1} + \gamma_{i-2} + \gamma_{i-3} + \gamma_{i-8} + \gamma_{i-10} + \gamma_{i-15} - \gamma_{i-5} - \gamma_{i-6} - \gamma_{i-7} - \gamma_{i-13} - \gamma_{i-14} \tag{21}$$

with the condition

$$\gamma_0 = 1 \quad , \gamma_i = 0 \qquad i < 0$$

and where

$$g(n,v) = \delta_{n,0} + \delta_{n,v+5} + \delta_{n,2v+3} - \delta_{n,1} - \delta_{n,v+1} - \delta_{n,2v+8} \tag{22}$$

In such a way we have completed the decomposition of the fully symmetric irreducible representation of the $SU(7)$ group according to the chain (4). In the table 1 we give the values of the γ_i coefficients

Tab. 1. The coefficients γ_i in the multiplicity formula (20)

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
γ_i	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84

up to i = 15 and in the table 2 we have gathered the results of multiplicity calculations C_{3v-k} for $k \leqslant 15$.

MATRIX ELEMENTS OF THE OCTUPOLE CREATION OPERATOR f^+_{3m} IN THE $O(3) \otimes SU(1,1)$ BASIS

We restrict calculations to the states $|Nv\chi LM\rangle$ in which the additional quantum numbers χ are not needed i.e. χ take on only one value for each $(NvLM)$. Such states are supposed to be the most important physical states.

Tab. 2. The values of the multiplicity C_{3v-k}

k \ v	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
0	1																1	for $v \geq 0$
1	0																0	0
2	0	0	1														1	2
3	0	0	0	1													1	3
4	0	0	1	1	2												2	4
5	0	0	0	1	1	2											2	5
6	0	0	0	1	2	2	3										3	6
7	0	0	0	0	1	2	2	3									3	7
8	0	0	0	1	2	3	4	4	5								5	8
9	0	0	0	0	1	2	3	4	4	5							5	9
10	0	0	0	0	1	3	4	5	6	6	7						7	10
11	0	0	0	0	0	1	3	4	5	6	6	7					7	11
12	0	0	0	0	1	2	4	6	7	8	9	9	10				10	12
13	0	0	0	0	0	1	2	4	6	7	8	9	9	10			10	13
14	0	0	0	0	0	1	3	5	7	9	10	11	12	12	13		13	14
15	0	0	0	0	0	0	2	4	6	8	10	11	12	13	13	14	14	15

It can be seen, by construction, that in these states $L = 3v$; $3v-2$; $3v-3$ only. We will keep, in what follows, the order of quantum number $|NvLM\rangle$. The construction of states $|NvLM\rangle$ are done in several steps

$$\begin{aligned} |v,v,3v,3v\rangle &= N_1(v)\left(f^+_{33}\right)^v|0\rangle \\ |v,v,3v,M\rangle &= N_2(v,M)\left(L_-\right)^{3v-M}|v,v,3v,3v\rangle \\ |v,v,L,M\rangle &= N_3(v,L)\left\{f^+_3\,|v-1,v-1,3v-3\rangle\right\}^L_M \\ |N,v,L,M\rangle &= N_4(N,v)\,S_+^{\frac{1}{2}(N-v)}|v,v,L,M\rangle \end{aligned} \tag{23}$$

where

$$S_+ = \frac{\sqrt{7}}{2}\left(f^+_3 f^+_3\right)^0 \tag{24}$$

is the pair $L = 0$ creation operator on the $f_{7/2}$ level. Normalizing coefficients $N_1(v)$, $N_2(v,M)$ and $N_4(N,v)$ are given immediately:

$$\begin{aligned} N_1(v) &= \left(\frac{1}{v!}\right)^{\frac{1}{2}} \\ N_2(v,M) &= \left\{\frac{(3v-3+M)!}{(6v-6)!\,(3v-3-M)!}\right\}^{\frac{1}{2}} \\ N_4(N,v) &= \left\{\frac{2^{\frac{1}{2}(N-v)}(2v+5)!!}{\left(\frac{N-v}{2}\right)!\,\left(N+v+5\right)!!}\right\}^{\frac{1}{2}} \end{aligned} \tag{25}$$

Non-trivial case is the calculation of the $N_3(v,L)$. At first, from the matrix element

$$\langle v+1, v+1, 3v+3, 3v+3 | f^{+}_{33} | v, v, 3v, 3v \rangle = \sqrt{v+1} \tag{26}$$

we get the reduced matrix element (in O(3))

$$\langle v+1, v+1, 3v+3 \| f^{+}_{3} \| v, v, 3v \rangle = \sqrt{(v+1)(6v+7)} \tag{27}$$

and then

$$\langle v+1, v+1, 3v+3, M' | f^{+}_{3m} | v, v, 3v, M \rangle = (3v\, M\, 3m | 3v+3\, M') \sqrt{v+1} \tag{28}$$

By streightforward calculations we get

$$\langle v, v, LM | v, v, LM \rangle = 1 = N_3^2(v,L) \left[1 + (v-1)(6v-5) \begin{Bmatrix} 3 & 3v-3 & 3v-6 \\ 3 & 3v-3 & L \end{Bmatrix} \right]$$

Hence

$$N_3(v,L) = \left[1 + (v-1)(6v-5) \begin{Bmatrix} 3 & 3v-3 & 3v-6 \\ 3 & 3v-3 & L \end{Bmatrix} \right]^{-\frac{1}{2}} \tag{29}$$

Similar method while applied to matrix element calculations gives

$$\langle v+1, v+1, L' \| f^{+}_{3} \| v, v, L \rangle_{O(3)} = N_3(v,L) N_3(v+1,L') (-1)^{v+L'+1} \sqrt{v(2L'+1)} \times \left\{ \delta_{L,3v} + \sqrt{(2L+1)(6v+1)} \begin{Bmatrix} 3 & 3v & 3v-3 \\ 3 & L & L' \end{Bmatrix} N_3^{-2}(v,L) \right\} \tag{30}$$

The general formula gives six considered here matrix elements which are given in the table 3.

Standard quasi-spin calculations have been performed to get the double reduced matrix element of the f^{+}_{3m} creation operator in the O(3) ⊗ SU (1,1). The operator f^{+}_{3m} has a well defined tensor property with respect to the SU (1,1) namely

$$f^{+}_{3m} = F^{(\frac{1}{2})(3)}_{+\frac{1}{2}\, m} \qquad \tilde{f}_{3m} = F^{(\frac{1}{2})(3)}_{-\frac{1}{2}\, m} \tag{31}$$

where

$$\tilde{f}_{3m} = (-1)^{3-m} f_{3-m}$$

Hence, by definition

$$\langle N'v'L'M' | F^{(\frac{1}{2})(3)}_{q\ m} | NvLM \rangle = \frac{(LM3m|L'M')}{\sqrt{2L'+1}} \left[SS_0 \tfrac{1}{2} q | S'S'_0 \right] \times \tag{32}$$

$$\times \left\langle v'L' \right\| F^{(\frac{1}{2})(3)} \left\| v, L \right\rangle_{0(3) \bullet SU(1,1)}$$

where

$$S = \frac{1}{2}\left(v + \frac{7}{2}\right) \qquad S_0 = \frac{1}{2}\left(N + \frac{7}{2}\right) \tag{33}$$

and $[S S_0 \frac{1}{2} q | S' S_0']$ is the Clebsch-Gordan coefficient for the $SU(1,1)$ non compact group [10].

The quasi-spin calculation gives

$$\left\langle v+1, L' \right\| F^{(\frac{1}{2})(3)} \left\| v L \right\rangle_{0(3) \bullet SU(1,1)} = -\sqrt{\frac{2v+5}{2v+7}} \left\langle v+1, v+1, L' \right\| f_3^+ \left\| v, v, L \right\rangle_{0(3)}$$

$$\left\langle v-1, L' \right\| F^{(\frac{1}{2})(3)} \left\| v L \right\rangle_{0(3) \bullet SU(1,1)} = (-1)^{L+L'+1} \left\langle v, v, L \right\| f_3^+ \left\| v-1, v-1, L' \right\rangle_{0(3)} \tag{34}$$

Tab. 3. The complete set of the reduced matrix elements of a boson creation operator f_3^+ for the states uniquely labelled by N=v,L

L	L	$\langle v+1\ v+1\ L' \Vert f_3^+ \Vert v\ v\ L \rangle_{0(3)}$
3v+3	3v	$\sqrt{(v+1)(6v+7)}$
3v+1	3v	$\sqrt{\frac{6(2v+1)(3v+2)}{6v-1}}$
3v+1	3v-2	$\sqrt{\frac{3(v-1)(2v+1)(6v+5)}{6v-1}}$
3v	3v	$-\sqrt{\frac{(v-1)(2v+1)(6v+1)}{(3v-1)(6v-1)}}$
3v	3v 2	$2\sqrt{\frac{3(3v+2)(6v-1)}{(3v-2)(6v-1)}}$
3v	3v-3	$\sqrt{\frac{(v-2)(3v+1)(3v+2)(6v+1)}{(3v-2)(3v-1)}}$

Hence we get the full set of one-particle matrix element in the $O(3) \bullet SU(1,1)$ reduction restricted to the states $N v L M$ with $L = 3v, 3v-2$ $3v-3$. We are now in position to obtain after a simple extension of calculations the matrix elements of one-body and two-body physical operators under consideration, especially, the energy and transition operators.

The results obtained in the paper will be followed by applications to nuclear calculations in the frame of the Interacting Octupole Boson Approximation.

REFERENCES

1. I a c h e l l o F., A r i m a A.: Phys. Lett. 53B, 309 (1974).
2. I a c h e l l o F., A r i m a A.: Phys. Lett. 57B, 39, (1975).
3. A r i m a A., I a c h e l l o F.: Phys. Rev. Lett. 35, 1069 (1975).
4. A r i m a A., I a c h e l l o F.: Annales of Phys. (N.Y). 99, 253 (1976).
5. A r i m a A., I a c h e l l o F.: Interacting Boson Model of Collective Nuclear States, preprint Kernfysisch Versneller Instituut, University of Groningen, I: The Vibrational Limit, 1975, II: The Rotational Limit, 1977.
6. B o h r A., M o t t e l s o n B. R.: Nuclear Structure, Vol. 2, Benjamin Inc., New York 1975.
7. I v a n o v a S. P., K o m o v A. L., K y r c h e v G., S o l o v i e v V. G., S t o y a n o v Ch.: JINR Report E4-9070, Dubna 1975.
8. R a c a h G.: Phys. Rev. 1942-1949.
9. S h i S h e n g-M i n g: Chinese Mathematics 6, 610 (1965).
10. H a r u o U i: Annales of Physics 49, 69 (1968).

S T R E S Z C Z E N I E

W pracy zostały podane podstawy klasyfikacji bozonowych stanów oktupolowych według łańcucha grupowego $SU(7) \supset O(7) \supset G_2 \supset O(3)$ mającego bezpośrednie zastosowanie w modelu oddziałujących bozonów (IBA) wzbudzeń kolektywnych jąder atomowych.

Р Е З Ю М Е

В работе представлены основы классификации октупольных бозонных состояний по групповой цепочке $SU(7) \supset O(7) \supset G_2 \supset O(3)$ имеющей непосредственное применение в модели взаимодействующих бозонов коллективных возбуждений атомных ядер.

Złożono w Redakcji 5 XII 1978 roku.