ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XL/XLI, 24 SECTIO AAA 1985/1986

Department of Electrical Engineering and Computer Science
University of California at San Diego

J. LEWAK

Analytical Results in Magnetic Saturation

Analityczne rezultaty dotyczące wysycenia magnetycznego

Аналитические решения проблемы магнитного насыщения

1. INTRODUCTION

Saturation of magnetization is of interest in many applications and in particular in application to magnetic recording heads. Current work on calculation of this nonlinear effect uses the direct numerical integration of the nonlinear equations [1]. The purpose of this paper is to explore analytical methods in the hope that they will be useful in uderstanding this and other similar problems.

We analyze the solution, in two dimensions, to the equation $\nabla\cdot\mathbf{B}=0$, where $\mathbf{B} = -\mu\nabla\phi$, and μ is a function of $H = |\nabla\phi|$. It is therefore also applicable to the case of dielectric saturation, where $\mathbf{B}$ is replaced by $\mathbf{E}$ and μ by ϵ, as well as steady two dimensional flows of an ideal compressible fluid where μ is replaced by ρ, the fluid density. We shall however focus on the case of magnetic saturation effects with the eventual possible application to recording heads.

Since the linear problem is solved by the real or imaginary part of any analytic function, it is natural to ask if any such harmonic solutions are possible for the nonlinear case. The answer to that is by no means obvious. We show however that such nontrivial harmonic solutions do not exist. Although there is no reason to expect such solutions to be possible, it is perhaps surprising that even the conjugate potential cannot be harmonic for any but the trivial case.

We explore the possibility of solutions which are parallel to harmonic functions. Since even in the nonlinear problem the potential and field lines (or conjugate potential) are orthogonal, it might be possible to have nonharmonic solutions which are parallel to harmonic functions. We find that this is possible but for only two special solutions. Thus we find two exact nonlinear solutions and suggest an approximate method based on these.

Exploring further a complex function formulation, we use the complex potential of the linear problem to conformally map the x-y plane to the α-β plane. The equipotentials of the linear problem are α = constant, and the field lines are β = constant and are thus coordinate lines. Saturation effects will distort these straight lines. If we assume that the distortions are such that the nonlinear equipotentials in the α-β plane have small slopes, we can find a linear equation for a certain "ν-potential". Once this equation is solved for ν, the first derivatives of the potential ϕ are related to ν through an algebraic nonlinear equation which can easily be solved.

In section 2 we set up the equations for the model, and in section 3 introduce the conjugate potential. Section 4 interprets the meaning of possible harmonic solutions. Formulation in terms of the complex variable is developed in section 5 where the linear and nonlinear cases are contrasted. In section 6 we show that the conjugate potential cannot be harmonic. Section 7 develops the exact solutions which are parallel to harmonic functions and concludes with the deduction that the potential cannot be harmonic. Section 8 suggests an approximation method using weak dependencies. Another approximation method is introduced in section 9 where liner approximate equations for the ν-potential are derived.

2. THE MODEL

The magnetic permeability μ, of ferromagnetic materials is generally not a constant but depends on the field strength H. This dependence is such that the nonlinear effect is called "saturation". Thus as the field strength increases, μ decreases. A simple, commonly used model for this dependence is Monson's model [2] which is given by: $\mu = \mu_0[1 + \chi/(1 + H/H_s)]$, where μ_0, χ, and H_s are constants.

Typically for magnetic cores of recording heads $\chi \sim 10^3$, and $H_s \sim 1$ Oerstead. Saturation is strongest near the corners of the recording heads where H is largest and according to numerical calculations [1] reaches orders of $500H_s$. Therefore even at and above saturation fields the χ term in μ is 10 times as big as the 1. Thus neglecting the 1 term and using H_s as the unit of the field we can write approximately: $\mu = \mu_0\chi/(1 + H)$. These approximations are not necessary, they are made for simplicity. The methods developed here will also apply to other models of $\mu(H)$.

The equations to solve inside the magnetic medium are therefore:

$$\mathbf{H} = -\nabla\phi \text{ and } \nabla\cdot(\mu\nabla\phi) = 0 \text{ or } \nabla\cdot\{\nabla\phi/(1 + H)\} = 0 \qquad (1)$$

which is the same as $\nabla\cdot(\mu\nabla\phi) = 0$ with μ replaced by $1/(1 + H)$.

3. REAL FUNCTION FORMULATION

We confine ourselves to two dimensions. Define $\nabla^* = (-\partial/\partial y, \partial/\partial x)$. This operator is "orthogonal" to the normal "unstarred" gradient. That is $\nabla\cdot\nabla^* = \nabla^*\cdot\nabla = 0$, and is equivalent to the operator $-\mathbf{k}\times\nabla$, where $\mathbf{k}$ is the unit vector perpendicular to the x-y plane.

The following properties of this operator are evident:

$$\nabla^{**} = -\nabla \;;\; (\nabla^*\psi)\cdot\nabla\psi = -(\nabla\psi)\cdot\nabla^*\psi = 0; \;\; \nabla^*\cdot\nabla^* = \nabla^2; \;\; |\nabla^*\psi| = |\nabla\psi|.$$

Therefore it follows that when $\nabla\cdot\mathbf{B} = 0$, we can find a scaler ψ such that $\mathbf{B} = -\nabla^*\psi$. (The negative sign is not significant, but is picked for later convenience.) Using this we get from (1):

$$\mu\nabla\phi = -\nabla^*\psi \qquad (2)$$

It follows from (2) that the curves ϕ = constant and ψ = constant are everywhere orthogonal. This in turn means that ψ is constant along field lines, which are orthogonal to the equipotentials. In analogy with the linear problem we call ψ the "Conjugate Potential".

4. PHYSICAL REQUIREMENTS FOR HARMONIC SOLUTIONS

Consider the general case where ψ is not harmonic. We obtain a physical condition for a harmonic potential by evaluating $\nabla^2\psi$. Operate with $\nabla^*\cdot$ on both sides of equation (2) to get:

$$\nabla^*\cdot(\mu\nabla\phi) = -\nabla^2\psi = -(\nabla^*\mu)\cdot\nabla\phi = (\nabla^*\mu)\cdot\nabla^*\psi/\mu = (\nabla\psi)\cdot\nabla\zeta$$

where we have used (2) to replace $\nabla\phi$ with $-\nabla^*\psi/\mu$. We have also substituted $\zeta = \ln(\mu)$. Therefore

$$\nabla^2\psi = -(\nabla\zeta)\cdot\nabla\psi \qquad (3)$$

Similarly using the starred version of equation (2) and operating with $\nabla\cdot$ on both sides, then making similar replacements we get

$$\nabla^2\phi = (\nabla\zeta)\cdot\nabla\phi \qquad (4)$$

Consider the condition necessary for harmonic solutions. The RHS of equation (3) will be zero only if ζ = constant (i.e. μ = constant) or $\nabla\zeta$ and $\nabla\psi$ are orthogonal. Since we presume that $\mu \neq$ constant, the only way to have a harmonic solution for ψ is to have ζ = constant and ψ = constant be orthogonal. Since $\zeta = \zeta(H)$, it follows that this condition requires that the magnitude of the H field change only along the field lines and not perpendicularly.

Similarly from equation (4) we see that if ϕ is to be harmonic, then the H field magnitude should only change in the perpendicular direction to the field lines and not parallel to them.

It will be shown that neither of these cases is possible except for the trivial case of a uniform field.

5. FORMULATION USING FUNCTIONS OF A COMPLEX VARIABLE.

Using $z = x + iy$, with $z^* = x - iy$, and treating z and z^* as formally independent variables, we can deduce that $\partial/\partial z^* = \frac{1}{2}(\partial/x + i\partial/\partial y)$. Therefore since μ, ϕ and ψ are all real functions, the two components of equation (2) can be written as the real and imaginary parts of

$$\mu\partial\phi/\partial z^* + i\,\partial\psi/\partial z^* = 0 \qquad (5)$$

If μ = constant, then equation (5) can be written more simply as $\partial(\mu\phi + i\psi)/\partial z^* = 0$, which implies that

$$\mu\phi + i\psi = f(z) \qquad (6)$$

where $f(z)$ is an analytic function.

This is the linear case in which any analytic function of z is a solution of the equations, where the real part is $\mu\phi$ and the imaginary part is ψ, the conjugate potential.

When μ is not constant, however, we can always find a function η such that

$$\mu\partial\phi/\partial z^* = \partial\eta/\partial z^* = -i\partial\psi/\partial z^*. \qquad (7)$$

From which it follows that

$$\partial(\eta + i\psi)/\partial z^* = 0 \text{ and thus } \eta + i\psi = F(z) \qquad (8)$$

Note however that whereas if μ is constant, $\eta = \mu\phi$, and hence is <u>real</u>, when μ is not constant, η is not real. In fact all we can say at this point is that it <u>need not</u> be real. The fact that it <u>cannot</u> be real except for the trivial special case of a uniform field, follows from the following.

Suppose η is real then it is harmonic because by (8) it is the real part of $F(z)$ and ψ is its imaginary part. Transform (7) from the pair z, z^* to the pair F, F^* as independent variables to get

$$\mu\partial\phi/\partial F^* = \partial\eta/\partial F^*$$

or in real terms

$$\mu\partial\phi/\partial\eta = 1, \text{ and } \mu\partial\phi/\partial\psi = 0; \qquad (9)$$

Hence $\phi = \phi(\eta)$ and from the first of equations (9) we conclude that $\mu = \mu(\eta)$. But $\mu = (1 + |F'|\phi')^{-1}$, where prime denotes differentiation. Therefore $|F'| = f(\eta)$. Taking this to its logical conclusion we deduce that since $\ln(f)$ is harmonic $\partial^2\ln(f)/\partial\eta^2 = 0$, and so $f = |F'| = Ae^{a\eta}$. This leads to the result that $F' = Ae^{aF'}$, and so F' must be a constant, hence the trivial solution. This proves that η cannot be real for nontrivial solutions.

Proceeding with the general formulation we note that (8) relates the nonlinear problem to an analytic function F(z). However only the real and imaginary parts of F(z) are harmonic. Since η and ψ are not the real and imaginary parts of F, they are, in general, neither orthogonal nor harmonic.

Comparing the linear and nonlinear cases we note that in the linear case given an analytic function we immediately have the potentials. In the nonlinear case on the other hand, given the analytic function F(z) we still do not know the potentials.

6. ψ NOT HARMONIC.

Consider the possibility of the **special case** $\nabla^2\psi = 0$, i.e. ψ is harmonic. We show that η can be made real.

Since by (8) $\eta = F(z) - i\psi$ it is also harmonic. We can always add an arbitrary analytic function of z to η and not change the determining equation (7). Therefore without loss of generality, by adding analytic functions to η we can arrange to make it real at which point $\eta = \mathrm{Re}\{F(z)\}$ and $\psi = \mathrm{Im}\{F(z)\}$ Since we have shown however, that η cannot be real for nontrivial solutions, it follows that ψ cannot be harmonic.

7. EXACT SOLUTIONS.

Equations 3 and 4 indicate a great deal of symmetry between the potential ϕ, and the conjugate potential ψ. It is not surprising therefore that ϕ cannot be harmonic either. Rather than prove this at this time we shall derive two exact solutions, which would include harmonic ones if they existed.

Since we have ruled out the possibility of harmonic solutions for ψ of the nonlinear problem, it is natural to ask if there are any solutions which are functions of the harmonic linear ones.

Consider the fact that if α,β are the real and imaginary parts of f(z) and so harmonic, nonlinear functions of α and β (say $f_1(\alpha)$ and $f_2(\beta)$), are not harmonic. We look for exact solutions to the nonlinear potential ϕ in the form of a function of the real part of f(z).

These exact solutions may be useful as starting points for more general solutions. In particular we suggest how the use of "slowly varying coefficients" can generalize these exact solutions to approximate, but more general, solutions where the equipotentials α = constant locally approximate the actual equipotentials.

Let α and β be two harmonic functions such that $F(z) = \alpha + i\beta$. We pick α and β such that the curves α = constant and ϕ = constant can be made parallel, and therefore β = constant and ψ = constant are also parallel. It follows that $\phi = \phi(\alpha)$ and $\psi = \psi(\beta)$. Substitute this into equation (2) and obtain:

$$\mu\phi'\nabla\alpha = -\psi'\nabla^*\beta, \text{ where } \phi' = d\phi/d\alpha \text{ etc..}$$

Writing this in complex notation we have

$$\mu(\phi'/\psi')\partial\alpha/\partial z^* + i\partial\beta/\partial z^* = 0 \qquad (10)$$

But since $\partial(\alpha + i\beta)/\partial z^* = 0$, it follows that

$$\mu(\phi'/\psi') = 1 \qquad (11)$$

Substituting $\mu = (1 + H)^{-1}$ we get:

$$H/\phi' = 1/\psi' - 1/\phi' \qquad (12)$$

Substituting for $H = |\nabla\phi| = |\phi'\nabla\alpha| = |\phi' F'(z)|$ and letting $\epsilon = \text{sign}(\phi')$ (= 1 for $\phi'>0$ and -1 for $\phi'<0$), we obtain

$$|\nabla\alpha|\epsilon = 1/\psi' - 1/\phi' = |F'(z)|\epsilon \qquad (13)$$

Since ϵ is either +1 or -1, if we solve equation 13 with $\epsilon = 1$, then $\epsilon = -1$ case is simply ψ and ϕ interchanged. This presumes that ϵ does not change sign inside our region. Clearly ϵ can only change sign when ϕ' vanishes. But ϕ' can only vanish at a singularity of F. We assume that this can only happen at isolated points in our region, and therefore no transition of signs is possible. Consider therefore $\epsilon = 1$.

We examine the conditions under which this equation has solutions. Take the logarithms of both sides of equation 13:

$$\mathrm{Ln}|F'(z)| = \mathrm{Ln}(A(\alpha) + B(\beta)) \qquad (14)$$

where $A(\alpha) = -1/\phi'$ and $B(\beta) = 1/\psi'$. Since $F'(z)$ is analytic, $\mathrm{Ln}|F'|$ is harmonic. Therefore $\nabla^2 \mathrm{Ln}|F'| = 0 = \nabla^2 \mathrm{Ln}(A + B)$. Therefore

$$(\partial^2/\partial\alpha^2 + \partial^2/\partial\beta^2)\,\mathrm{Ln}(A + B) = 0 \qquad (15)$$

Equation (15) is clearly both a necessary and sufficient condition for (13) to be satisfied. Evaluating the derivatives we have:

$$A''A + B''B + AB'' + BA'' = (A')^2 + (B')^2 \qquad (16)$$

If (16) is to be true then $\partial^2/\partial\alpha\partial\beta$ of the equation is also true. Therefore a necessary (though not sufficient) condition for (13) is:

$$A'B''' + B'A''' = 0 \qquad (17)$$

Separating variables we get: $A'''/A' = -B'''/B' = \lambda$ = constant.

We have 3 possible cases:

$\lambda = 0$

$\lambda > 0$, typify with $\lambda = 1$

$\lambda < 0$, typify with $\lambda = -1$. This case however is simply $\alpha \leftrightarrow \beta$ exchanged and so needs no separate consideration.

(Different values of λ amount to different scalings of α and β)

Consider $\lambda = 0$, then $A''' = B''' = 0$.

$$A = a_0 + a_1\alpha + a_2\alpha^2.$$

$$B = b_0 + b_1\beta + b_2\beta^2. \qquad (18)$$

This solution has to be substituted back into equation (16) in order to get the necessary and sufficient condition for a solution. When this is done we obtain

$$a_2 = b_2; \quad a_1^2 + b_1^2 = 4(a_0 + b_0)a_2. \qquad (19)$$

Considering the case $\lambda = 1$ we get $B''' = -B'$ and $A''' = A'$, therefore

$$A = a_0 + a_1e^{\alpha} + a_2e^{-\alpha}; \quad B = b_0 + b_1\cos\beta + b_2\sin\beta. \quad (20)$$

Substitution of these back into equation 16 gives the necessary and sufficient condition:

$$b_0 = -a_0; \quad b_1^2 + b_2^2 = 4a_1a_2. \qquad (21)$$

We now solve for ϕ and ψ, and then for $F(z)$.

Consider first the $\lambda = 0$ case. We can simplify the algebra considerably by noting that if we shift the origin of (α,β) we can arrange to have the quadratics in (18) without the linear terms:

$A = a_0' + a_2\alpha_1^2$. Where $\alpha_1 = \alpha + a_1/(2a_2)$; $a_0' = (4a_0a_2 - a_1^2)/(4a_2)$

similarly

$B = b_0' + b_2\beta_1^2$. Where $\beta_1 = \beta + b_1/(2b_2)$; $b_0' = (4b_0b_2 - b_1^2)/(4b_2)$

Therefore define $F_1(z) = \alpha_1 + i\beta_1 = F(z) + c$, where $c = -(a_1/(2a_2) + b_1/(2b_2))$ Then $|F'_1| = a_2(\alpha_1^2 + \beta_1^2) = a_2 \mid F_1 \mid^2$ or $F_1' = e^{i\gamma}a_2F_1^2$, where γ is a real constant. Integrating, this gives

$$F_1 = -e^{i\gamma}a_2/(z - z_0). \qquad (22)$$

Without loss of generality, picking the origin at z_0 and choosing our axes appropriately ($\gamma = 0$) we get $F_1 = -a_2/z$. where a_2 is real. Now $\alpha_1 = -a_2x/(x^2 + y^2)$, and $\beta_1 = -a_2y/(x^2 + y^2)$, and so the equipotentials α_1 = constant and the field lines β_1 = constant are circles. The field is given by

$$\mathbf{H} = \nabla\phi = (\nabla\alpha_1)\phi' = -\nabla\alpha_1/A = -\nabla\alpha_1/(a_0' + a_2\alpha_1^2) \quad (23)$$

and the ϕ and ψ potentials by

$$\phi = \phi_0 - \int d\alpha/(a_0 + a_1\alpha + a_2\alpha^2) = \phi_0 + ik^{-1}\mathrm{Ln}\{(\alpha - \alpha_+)/(\alpha - \alpha_-)\}$$

$$= \phi_0 - 2k^{-1}\tan^{-1}\{k/(2a_2\alpha + a_1)\}$$

$$\psi = \psi_0 + \int d\beta/(b_0 + b_1\beta + b_2\beta^2) = \psi_0 + k^{-1}\mathrm{Ln}\{(\beta - \beta_+)/(\beta - \beta_-)\}$$

where $2\alpha_\pm = -a_1/a_2 \pm ik/a_2$, $2\beta_\pm = -b_1/a_2 \pm k/a_2$,
and $k = \sqrt{(4a_0a_2 - a_1^2)}$.

This solution requires $a_2 \neq 0$ and represents field lines with the same shape as those of a line dipole in the linear case.

For the special case $a_2 = 0$, equation 19 requires $a_1 = 0 = b_1$ also and the result is $\phi = \phi_0 + \alpha/a_0$ and $\psi = \psi_0 + \beta/b_0$, i.e. the harmonic solution. Substituting this into the equation for F(z) we find in that case

$$F(z) = (a_0 + b_0)z + C.$$

or the trivial case of a constant field.

The case $\lambda = 1$ is similarly simple. Rewrite A and B from equations (20) as:

$$A = a_0 + 2\sqrt{(a_1a_2)}\cosh(\alpha + \gamma)$$

$$\text{and } B = -a_0 + 2\sqrt{(a_1a_2)}\cos(\beta + \gamma_1), \quad (24)$$

where $\tanh\gamma = (a_1 - a_2)/(a_1 + a_2)$ and $\tan\gamma_1 = -b_2/b_1$.

This gives the result:

$$|F'_1| = 2\sqrt{(a_1a_2)}(\cosh(\alpha_1) + \cos(\beta_1)) \qquad (25)$$

where we have substituted

$$\alpha_1 = \alpha + \gamma \text{ and } \beta_1 = \beta + \gamma_1 \text{ and } F_1 = F + (\gamma + i\gamma_1)$$

Equation 24 may also be written as

$$|F'_1| = 2\sqrt{(a_1a_2)}\left|\cosh(\tfrac{1}{2}F_1)\right|^2 \qquad (26)$$

This is solved by: $F'_1 = 2\sqrt{(a_1a_2)}e^{i\theta}\cosh^2(\frac{1}{2}F_1)$, where θ is a real constant. The solution to this in turn gives:

$$F_1 = \ln(1 + zR) - \ln(1 - zR) \qquad (27)$$

where we have replaced $\sqrt{(a_1a_2)}e^{i\theta}$ by R, allowing R to be complex. z may be replaced by $z - z_0$, where z_0 is an arbitrary complex constant.

The potentials are given by:

$$\phi = \phi_0 - k_1^{-1}\ln((e^{\alpha+\gamma} - a_+)/(e^{\alpha+\gamma} - a_-))$$

where $a_\pm = -a_0/r \pm k_1/r$; $r = 2\sqrt{(a_1a_2)}$; $k_1 = \sqrt{(a_0^2 - r^2)}$

The equipotentials for this solution are also circles. For the special case of $a_2 = 0$, equation 19 again requires $a_1 = 0 = b_1$ which gives the trivial result of F = constant, and thus zero field.

We have deduced all the solutions to the nonlinear problem whose potential can be written as a function of the real part of an analytic function F. This includes the possibility of harmonic solutions for the potential ϕ.

Consider now the possibility of a harmonic ϕ solution in the light of our results. Since $\phi = \phi(\alpha)$ then $\nabla^2\phi = \phi'\nabla^2\alpha + (\nabla\alpha)^2\phi'' = (\nabla\alpha)^2\phi'' = 0$ for a harmonic solution. Therefore $\phi'' = 0$ is the only possibility. The only such solutions we have found (both $\lambda = 0$ and $\lambda = 1$) are the trivial constant field solutions.

In conclusion therefore we have shown that the only harmonic solution to the nonlinear problem is the trivial one of a constant field.

8. THE ADIABATIC APPROXIMATION.

Suppose we now relax the condition that ϕ = constant be exactly parallel to α = constant everywhere, and replace it with the condition that it be approximately parallel everywhere.. This implies that $\phi = \phi(\alpha,\beta)$ where the β dependence is weak in some sense. This implies that the derivatives (using $\partial\phi/\partial\alpha = \phi_\alpha$ etc.) $|\phi_\alpha| \gg |\phi_\beta|$.

For a graphical picture of this condition imagine the curves ϕ = constant drawn in the α, β plane. The condition then requires the curves ϕ = constant are to have small slopes, i.e. to be approximately parallel to the β axis. Similarly we can require $\psi = \psi(\beta,\alpha)$ where the α dependence is weak. Since we must still require orthogonality between the ϕ = constant and ψ = constant curves, it can easily be shown that our derivation from equation 10 through 27 is unchanged, provided we drop all second order derivatives with respect to the weak dependencies. This allows us to proceed with more general problems.

9. DERIVATION OF APPROXIMATE EQUATIONS.

Suppose we use the linear solution as the starting point for the nonlinear one. We can then use the potentials (both the real potential and the conjugate potential) as our "grid" or curvilinear set of coordinates for the problem. This will allow us to assume that the actual potential curves have small slopes in terms of our coordinates. This in turn, as we shall see below, allows us to derive linear equations for a "ν-potential", even though the equivalent equation for the potential ϕ is nonlinear.

It is important to realize that a curve with small slopes can deviate considerably from a straight line. Thus in this approximation the nonlinear potential can differ considerably from the linear one.

We start with equation (5). Using $w = F(z) = \alpha + i\beta$ (and with an obvious notation, the implied relation $w^* = F^*(z) = \overline{F}(z^*)$) we transform equation (5) from the independent variable pair z,z^* to the pair w,w^*.

$$\mu\partial\phi/\partial w^* = -i\partial\psi/\partial w^* = \partial\eta/\partial w^*. \qquad (28)$$

In (28) we substitute $\eta = \alpha + i\nu$, (or the equivalent $\psi = \beta - \nu$) and equate the real and imaginary parts separately to obtain

$$\mu\phi_{\alpha} = 1 - \nu_{\beta};\ \mu\phi_{\beta} = \nu_{\alpha}; \qquad (29)$$

where the subscripts indicate partial derivatives with respect to the independent variables α and β. To complete the set we add to (29) the expression for μ:

$$\mu = \{1 + |F'|(\phi_{\alpha}^2 + \phi_{\beta}^2)^{1/2}\}^{-1}, \text{ where } F' = dF/dz.$$

The following procedure will yield the exact equation for ν: The pair of equations (29) is solved for ϕ_{α} and ϕ_{β} in terms of ν_{α} and ν_{β}. Then the "Consistency Condition" $\phi_{\alpha\beta} = \phi_{\beta\alpha}$ is applied giving the equation for ν. Of course the alternative procedure of applying the Consistency Condition $\nu_{\alpha\beta} = \nu_{\beta\alpha}$ to equations (29) will yield the original equation for ϕ with α,β as independent variables. At this point there is nothing to choose between these two approaches, and thus no simplification apparent. The following discussion will make clear how a judicious choice of F(z) will allow us to make reasonable approximations and dictate the use of the equation for ν rather than ϕ.

Let us choose F(z) to be the complex potential of the linear problem. Thus the linear solution, indicated by the suffix "L", is given by $\phi_L = \alpha = \eta_L$ and $\psi_L = \beta$, with $\nu_L = 0$. In the α-β plane the equipotentials are parallel to the β axis, and the field lines are parallel to the α axis. Since the nonlinear solution will in most of the z plane follow closely the linear one, except in regions like corners, where saturation is important, it is reasonable to suppose that in the α-β plane the approximate nonlinear solution has a small ν. From equations (29), and the fact that $\phi_{L\alpha} = 1$ and $\phi_{L\beta} = 0$, it follows that $|\phi_{\alpha}| \gg |\phi_{\beta}|$ is a reasonable approximation. Therefore as a first approximation we can deduce a linearized equation for ν, or an equivalent one for ϕ. However that the equivalent approximate equation for ϕ is nonlinear and rather involved, whereas the equation for ν is linear and considerably simpler.

Proceeding as outlined above, and neglecting nonlinear terms in ν, we obtain the following equation:

$$\nu_{\alpha\alpha}(1 - f) + \nu_{\beta\beta}(1 - 2f) + \nu_{\alpha} f_{\alpha} + 2\,\nu_{\beta} f_{\beta} = f_{\beta}. \quad (30)$$

where $f = |F'|$. If we further assume that f is small (which limits the approximation to weak nonlinearities) we get the much simpler equation:

$$\nu_{\alpha\alpha} + \nu_{\beta\beta} = - f_{\beta}, \quad (31).$$

Equation (31) is particularly easy to solve, since the solution can be written in terms of the Green's function G(x,y.x',y') for the linear problem. To exhibit this we re-write equation (31) in the z-plane as

$$\nabla^2 \nu = - f_{\beta}/f^2 = \partial\{f^{-1}\}/\partial\beta$$

which has the formal solution

$$\nu(x,y) = \iint G(x,y,x',y')\partial\{f^{-1}(x',y')\}/\partial\beta dx'dy'$$

The linear solution is assumed to satisfy the correct boundary conditions and so we have used the fact that $\nu = 0$ on the boundary.

Since the Green's function G is easily determined from analytic function theory (see for example Hildebrand 1976), this part of the problem is standard.

Once we have the solution for ν, the equations relating ϕ_{α} and ϕ_{β} to ν_{α} and ν_{β} (equations 29) are algebraic, though nonlinear, and can be easily solved.

10. CONCLUSION.

We have shown that there are no nontrivial harmonic solutions to the nonlinear problem. Exact solutions have been found and suggested as the starting point for a general approximation method.

A new potential function (ν) is defined which approximately satisfies a linear equation (equation 30).

In a future publication these equations will be solved for some realistic geometries.

The methods developed in this paper should also be applicable to other physical nonlinear problems. Electrostatic saturation is an obvious case. The two dimensional ideal compressible fuid flow equations though somewhat different in detail, should be amenable to a simmilar treatement.

REFERENCES

1. Bertram Niel H. and Steele Charles W. IEEE Trans. Mag. 1976, MAG-12, 702-206

2. Monson J. E. IEEE Trans. Mag. MAG-8, 536-538.

STRESZCZENIE

Metodami analitycznymi badano problem nasycenia magnetyzacji w dwu wymiarach używając modelu Morisona dla μ. Zdefiniowano potencjał magnetyczny i potencjał sprzężony i pokazano, że nie istnieją dla nich nietrywialne rozwiązania harmoniczne. Wykazano, że istnieją tylko dwa dokładne rozwiązania o liniach ekwipotencjalnych równoległych do potencjałów harmonicznych. Rozwiązania te mogą stanowić punkt wyjścia dla przybliżonych ogólniejszych rozwiązań.

Zaproponowano także metodę przybliżeń wykorzystującą funkcje zespolone i prowadzącą do liniowego równania dla "potencjałów ν" nawet wtedy gdy odpowiednie równania dla rzeczywistego potencjału są nieliniowe.

Р Е З Ю М Е

С помощью аналитического метода, при использовании модели Морисона для μ рассматривалась проблема насыщения магнетизации в двух размерах. Определены магнитный и сопряженный потенциалы и показано, что для них не существуют нетривиальные гармонические решения. Доказывается, что существуют всего два точные решения с эквипотенциальными линиями параллельными гармоническим потенциалам. Эти решения могут быть исходными для приближенных, более общих решений.

Предложен тоже метод приближений использующий комплексные функции, ведущий к линейному уравнению для "ν-потенциалов" даже тогда, когда соответствующие уравнения для действительного потенциала - нелинейные.