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On the existence of connections with a prescribed skew-symmetric Ricci tensor

Abstract. We study the so-called inverse problem. Namely, given a prescribed skew-symmetric Ricci tensor we find (locally) a respective linear connection.

1. Introduction. All manifolds and maps between manifolds considered in the paper are assumed to be smooth (i.e. of class C^∞).

The concept of a linear connection ∇ on a manifold M and its Ricci tensor S can be found in the fundamental monograph [4].

In the present paper, we study the so-called inverse problem.

More detailed, under some assumption on a tensor field r of type $(0,2)$ on M, we prove the existence of a local solution of the equation

$$S = r \tag{1}$$

with unknown linear connection ∇ on M.

In particular, we deduce that any 2-form ω on a manifold M with $\dim(M) \geq 2$ is locally the Ricci tensor S of some linear connection ∇ on M.

In the analytic situation, the inverse problem was studied in many papers, e.g. [1, 2, 3, 5]. For example, in [5], using the Cauchy–Kowalevski theorem, the authors found (locally) all analytic linear connections for a prescribed analytic Ricci tensor. In the C^∞ situation, we can not apply the Cauchy–Kowalevski theorem.

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From now on, $x^1, \dots, x^n$ denote the usual coordinates on $\mathbf{R}^n$ and $\partial_1, \dots, \partial_n$ denote the usual canonical vector fields on $\mathbf{R}^n$. Given a map $f : \mathbf{R}^n \to \mathbf{R}$ let $(f)_i := \partial_i(f) = \frac{\partial f}{\partial x^i}$ for $i = 1, \dots, n$.

2. The main result. The main result of the paper is the following

Theorem 1. *Let M be a manifold such that $\dim(M) \geq 2$ and let $x_o \in M$. Let r be a tensor field of type $(0,2)$ on M such that $r(X,X) = 0$ around x_o for some vector field $X \in \mathcal{X}(M)$ with $X_{x_o} \neq 0$. Then there is a linear connection ∇ on M such that r is the Ricci tensor S of ∇ on some neighborhood of x_o.*

Proof. We may assume that $M = \mathbf{R}^n$, $x_o = 0$ and $X = \partial_1$.

Let r be the tensor field of type $(0,2)$ on $\mathbf{R}^n$ and denote $r_{ij} = r(\partial_i, \partial_j)$ for $i,j = 1, \dots, n$. Then

$$r_{11} = 0. \tag{2}$$

The Ricci tensor S of a linear connection ∇ has the following rather well-known coordinate expression

$$S(\partial_i, \partial_j) = \sum_{k=1}^{n} [(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i] + \sum_{k,l=1}^{n} [\Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}],\ i,j = 1, \dots, n, \tag{3}$$

where Γ^i_{jk} are the Christoffel symbols of ∇, see [4].

It is sufficient to show that under assumption (2), equation (1) has a local solution (defined on some neighborhood of 0) $\nabla = (\Gamma^a_{bc})$ such that

$$\begin{aligned} &\Gamma^a_{bc} = 0 \text{ for } a = 3, \dots, n,\ b,c = 1, \dots, n, \\ &\Gamma^2_{bc} = 0 \text{ for } b,c = 2, \dots, n, \\ &\Gamma^2_{b1} = 0 \text{ for } b = 1, \dots, n, \\ &\Gamma^1_{1b} = 0 \text{ for } b = 1, \dots, n. \end{aligned} \tag{4}$$

In other words, we put $\Gamma^a_{bc} = 0$ for $a,b,c = 1, \dots, n$ except for Γ^2_{1j} with $j = 2, \dots, n$ and Γ^1_{ij} with $i = 2, \dots, n$ and $j = 1, \dots, n$.

Using (4) and the coordinate expression (3), we get

$$S(\partial_i, \partial_j) = \sum_{k=1}^{2} (\Gamma^k_{ij})_k - \sum_{\substack{k,l \in \{1,2\} \\ k \neq 1}} \Gamma^l_{kj}\Gamma^k_{il} = (\Gamma^1_{ij})_1 + (\Gamma^2_{ij})_2 - \Gamma^1_{2j}\Gamma^2_{i1} - \Gamma^2_{1j}\Gamma^1_{i2} \tag{5}$$

as $\Gamma^a_{bc} = 0$ if $a = 3, \dots, n$ and $b,c = 1, \dots, n$, and $\Gamma^a_{ac} = 0$ if $a,c = 1, \dots, n$.

Then using (5) and (4), we get

$$S(\partial_1, \partial_1) = 0, \tag{6}$$

$$S(\partial_1, \partial_j) = (\Gamma^2_{1j})_2 \text{ for } j = 2, \dots, n, \tag{7}$$

$$S(\partial_i, \partial_1) = (\Gamma^1_{i1})_1 \text{ for } i = 2, \ldots, n, \tag{8}$$

$$S(\partial_i, \partial_j) = (\Gamma^1_{ij})_1 - \Gamma^2_{1j}\Gamma^1_{i2} \text{ for } i, j = 2, \ldots, n. \tag{9}$$

More precisely, to obtain (6) we use (5) with $(i, j) = (1, 1)$ and the assumed (in (4)) conditions $\Gamma^1_{11} = \Gamma^2_{11} = 0$. To obtain (7), we use (5) with $(i, j) = (1, j)$ and the assumed (in (4)) conditions $\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{1j} = 0$. To obtain (8), we use (5) with $(i, j) = (i, 1)$ and the assumed (in (4)) conditions $\Gamma^2_{11} = \Gamma^2_{i1} = 0$. To obtain (9), we use (5) with $i, j = 2, \ldots, n$ and the assumed (in (4)) conditions $\Gamma^2_{i1} = \Gamma^2_{ij} = 0$.

Then, by (2), (4) and (6)–(9), the equation (1) with unknown ∇ satisfying (4) is equivalent to the system of systems of differential equations

$$(\Gamma^2_{1j})_2 = r_{1j} \quad \text{for } j = 2, \ldots, n, \tag{10}$$

$$(\Gamma^1_{i1})_1 = r_{i1} \quad \text{for } i = 2, \ldots, n, \tag{11}$$

$$(\Gamma^1_{ij})_1 = \Gamma^2_{1j}\Gamma^1_{i2} + r_{ij} \quad \text{for } i, j = 2, \ldots, n. \tag{12}$$

It remains to observe that the system (10)–(12) has a solution of class C^∞.

We see that the solution of (10) is

$$\Gamma^2_{1j}(x) = \int_0^{x^2} r_{1j}(x^1, t, x^3, \ldots, x^n)dt + a_j(x^1, x^3, \ldots, x^n)$$

for $j = 2, \ldots, n$, and that the solution of (11) is

$$\Gamma^1_{i1}(x) = \int_0^{x^1} r_{i1}(t, x^2, \ldots, x^n)dt + b_i(x^2, \ldots, x^n)$$

for $i = 2, \ldots, n$, where a_j, b_i are arbitrary maps in $n - 1$ variables.

Substituting the obtained Γ^2_{1j} into (12), we get the system of ordinary first order differential equations with parameters $x^2, \ldots, x^n$.

Such obtained system (12) has a solution of class C^∞ according to the well-known theory of differential equations. We can even solve it explicitly as follows.

Each of the equations

$$(\Gamma^1_{i2})_1 = \Gamma^2_{12}\Gamma^1_{i2} + r_{i2} \text{ for } i = 2, \ldots, n$$

(from the system (12)) is linear non-homogeneous with parameters. Solving them separately (using the well-known method), we obtain

$$\begin{aligned}
&\Gamma^1_{i2}(x^1, \ldots, x^n) \\
&\quad = \left(\int_0^{x^1} r_{i2}(t, x^2, \ldots, x^n)e^{-\int_0^t \Gamma^2_{12}(\tau, x^2, \ldots, x^n)d\tau}dt + c_{i2}(x^2, \ldots, x^n)\right) \\
&\quad \times e^{\int_0^{x^1} \Gamma^2_{12}(t, x^2, \ldots, x^n)dt}
\end{aligned}$$

for $i = 2, \dots, n$, where c_{i2} are arbitrary maps in $n-1$ variables. Then the other equations of (12) (with Γ^1_{i2} as above) have solutions given by

$$\begin{aligned}\Gamma^1_{ij}&(x^1, \dots, x^n)\\ &= \int_0^{x^1} (\Gamma^2_{1j}(t, x^2, \dots, x^n)\Gamma^1_{i2}(t, x^2, \dots, x^n) + r_{ij}(t, x^2, \dots, x^n))dt\\ &\quad + d_{ij}(x^2, \dots, x^n),\end{aligned}$$

where d_{ij} are arbitrary maps in $n-1$ variables.

The proof of Theorem 1 is now complete. □

We have the following interesting corollary of Theorem 1.

Corollary 1. *Let M be a manifold such that $\dim(M) \geq 2$ and let $x_o \in M$. Let ω be a 2-form on M. Then there is a linear connection ∇ on M such that ω is the Ricci tensor S of ∇ on some neighborhood of x_o.*

Proof. For any vector field X (in particular with $X_{x_o} \neq 0$) we have $\omega(X, X) = 0$. Then we apply Theorem 1 with ω playing the role of r. □

References

[1] Dušek, Z., Kowalski, O., *How many are Ricci flat affine connections with arbitrary torsion?*, Publ. Math. Debrecen **88** (3–4) (2016), 511–516.

[2] Gasqui, J., *Connexions à courbure de Ricci donnée*, Math. Z. **168** (2) (1979), 167–179.

[3] Gasqui, J., *Sur la courbure de Ricci d'une connexion linéaire*, C. R. Acad. Sci. Paris Sér A–B **281** (11) (1975), 389–391.

[4] Kobayashi, S., Nomizu, K., *Foundation of Differential Geometry*, Vol. I, J. Wiley-Interscience, New York, 1963.

[5] Opozda, B., Mikulski, W. M., *The Cauchy–Kowalevski theorem applied for counting connections with a prescribed Ricci tensor*, Turkish J. Math. **42** (2) (2018), 528–536.

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