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On the Bloch–Landau Constant for Möbius Transforms of Convex Mappings

Abstract. Let S denote the familiar class of functions holomorphic in the unit disk $\mathbf{D}$, normalized by : $f(0) = f'(0 - 1 = 0$.

Let $K \subset S$ be the subclass of S consisting of all f such that the image domain $f(\mathbf{D})$ is convex.

Let for an arbitrary $w \in \hat{\mathbf{C}} \setminus f(\mathbf{D})$, $F_f = \bigcup_w \tau_w \circ f$, where $\tau_w(\zeta) = w\zeta/(w - \zeta)$ and let $\hat{K} = \bigcup_{f \in K} F_f$.

Barnard and Schober asked the question to find the properties of K that are inherited by $\hat{K}$. We prove that the class $\hat{K}$ shares with K the property of linear invariance in the sense of Pommerenke. We also prove that Bloch–Landau constant within both classes $\hat{K}$ and K is equal to $\pi/4$.

1. Introduction. Let $\mathcal{H}(\mathbf{D})$ stand for the class of functions holomorphic in the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ and for $f \in \mathcal{H}(\mathbf{D})$ let $L(f)$ be the least upper bound of ρ such that $f(\mathbf{D})$ contains a disk of radius ρ.

Let S denote the familiar subclass of $\mathcal{H}(\mathbf{D})$ consisting of functions f univalent in $\mathbf{D}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$ and let S_0 be a compact subclass of S.

Put

$$U(S_0) = \inf\{L(f) : f \in S_0\}. \tag{1.1}$$

The exact value of $U(S_0)$ for $S_0 = S$ is still unknown, however Landau proved [12] that $U(S) > 0.5625$, while Beller and Hummel [5] were able to show that $U(S) < 0.65641$. As shown by Robinson [16] there exists $F \in S$ such that $L(F) = U(S)$.

Let $K \subset S$ be the subclass of S consisting of all f such that the image domain $f(\mathbf{D})$ is convex.

If $\tau_w(\varsigma) = w\varsigma/(w-\varsigma)$ then with any $f \in S$ and any $w \in \hat{\mathbf{C}} \setminus f(\mathbf{D})$ we may associate a subclass F_f of S consisting of all $\tau_w \circ f$. We define now another subclass $\hat{K}$ of S as $\bigcup_{f \in K} F_f$.

Various properties of functions $f \in \hat{K}$ were established by Barnard and Schober [3], Clunie and Sheil-Small as well as by Hall [9].

The class $\hat{K}$ has obviously the property of rotational invariance. Barnard and Schober [3] asked the question to find further properties of K that are inherited by $\hat{K}$.

In this paper we prove that the class $\hat{K}$ shares with K the property of linear invariance in the sense of Pommerenke [15]. We will also show that $U(\hat{K}) = U(K) = \pi/4$.

Moreover, we show that for any compact subclass S_0 of S the g.l.b. (1.1) is attained for some $f_0 \in S_0$. The constant defined in (1.1) is associated with Bloch [6] and Landau [12] and the research initiated by them in 1925–29 was continued by e.g. Ahlfors [1], Ahlfors and Grunsky [2], Goodman [8], Heins [10], Pommerenke [13].

2. Linear invariance of $\hat{K}$. Let K_n, $n \in \mathbf{N}$, denote the subclass of K consisting of all f such that $f(\mathbf{D})$ is a polygon, not necessarily bounded, with at most n sides. Thus K_1 consists of Möbius transformations mapping $\mathbf{D}$ onto a half-plane, while $K_2 \setminus K_1$ consists of functions mapping $\mathbf{D}$ on a domain whose boundary consists of two parallel lines, or two half-lines with common origin.

Let $\hat{K}_n = \bigcup_{f \in K_n} F_f$. Any function $F = \tau_w \circ f \in \hat{K}_n$ maps $\mathbf{D}$ on a circular polygon whose boundary consists of n arcs on circles intersecting each other at $-w$. Conversely, if all sides of a circular polygon Ω_n with interior angles $\alpha_k < \pi$ are situated on circles intersecting each other at $-w$, then the homography $\varsigma \mapsto w\varsigma/(\varsigma + w)$ maps Ω_n onto a convex polygon W_n. If $0 \in \Omega_n$ and the inner radius $R(0;\Omega_n) = 1$, then there exists $F \in \hat{K}_n$ such that $\Omega_n = F(\mathbf{D})$.

The classes K_n, $n \in \mathbf{N}$, and K are compact in the usual topology of uniform convergence on compact subsets of $\mathbf{D}$ and so are $\hat{K}_n$ and $\hat{K}$, cf. [3].

Theorem 2.1. *Suppose that $F \in \hat{K}$ and $\omega(z) = (z+a)/(1+\bar{a}z)$, $a \in \mathbf{D}$. Then*

$$F_a(z) := \left[(1-|a|^2)F'(a)\right]^{-1}\left(F \circ \omega(z) - F(a)\right) \in \hat{K}\ .$$

Proof. The set $\bigcup_{n=1}^{\infty} \hat{K}_n$ is dense in $\hat{K}$ and consequently there exists a sequence (F_n), $F_n \in \hat{K}_n$, convergent to F uniformly on compact subsets of $\mathbf{D}$. It is sufficient to prove that any $\hat{K}_n$, $n \in \mathbf{N}$, is linearly invariant. For $n \in \mathbf{N}$ put

$$F_{na}(z) := \left[(1-|a|^2)F_n'(a)\right)^{-1}\left(F_n \circ \omega(z) - F_n(a)\right)\ .$$

If $F_n(\mathbf{D}) = \Omega_n$ is a circular polygon with at most n sides, then $F_{na}(\mathbf{D})$ arises from Ω_n under a translation and similarity : $\varsigma \mapsto R^{-1}(\varsigma - F_n(a))$. Moreover, $F_{na}(0) = 0$, $F'_{na}(0) = 1$ and hence $F_{na} \in \hat{K}_n$ and we are done.

3. The existence of an extremal function.

Theorem 3.1. *Let S_0 be an arbitrary fixed compact subclass of S. Then there exists $f_0 \in S_0$ such that*

$$U(S_0) := \inf\{L(f) : f \in S_0\} = L(f_0) .$$

Proof. Put for short $U(S_0) = L_0$. There exists a sequence (f_n), $f_n \in S_0$, such that $L(f_n) \geq L_0$ and $\lim_{n\to\infty} L(f_n) = L_0$. Due to compactness of S_0 we may assume that (f_n) is convergent to $f_0 \in S_0$ uniformly on compact subsets of $\mathbb{D}$ and $(f_n(\mathbb{D}))$ is convergent to its kernel $f_0(\mathbb{D})$ w.r.t. the origin. Suppose that $L(f_0) > L_0$ and take λ such that $L_0/L(f_0) < \lambda < 1$. Then for some $s_0 \in f_0(\mathbb{D})$ the closed disk $K(s_0, \lambda L(f_0))$ is not contained in $f_n(\mathbb{D})$ for sufficiently large n. However, this contradicts $f_0(\mathbb{D})$ to be the limit of $(f_n(\mathbb{D}))$ in the sense of kernel convergence, cf. [14, p.31, Problem 3].

Since the classes $\acute{K}_n$, $n \in \mathbb{N}$, and $\acute{K}$ are compact, there exist in view of Theorem 3.1, the functions $F_n \in \acute{K}_n$, $F_0 \in \acute{K}$, such that

$$L(F_n) = \inf\{L(F) : F \in \acute{K}_n\} , \quad n \geq 2 , \tag{3.1}$$

$$L(F_0) = U(\acute{K}) = \lim_{n\to\infty} L(F_n) . \tag{3.2}$$

Theorem 3.2. *If $n > 3$ then*

$$L(F_n) = \inf\{L(F) : F \in \acute{K}_3\} .$$

Proof. Suppose that the g.l.b. (3.1) is attained for F mapping $\mathbb{D}$ onto a circular polygon Ω_n with n sides, $n > 3$, situated on circles intersecting each other at $-w$. There exists a disk $K_0 \subset \Omega_n$ of radius $L(F)$ tangent to $\partial\Omega_n$ at the points Q_k. The position of K_0 is determined either by two or by three points Q_k situated on different sides of Ω_n. The first possibility corresponds to Q_k being the end points of a diameter of K_0, the second one means that three points Q_k can be chosen so as to devide ∂K_0 into three subarcs each having angular measure less than π. Since $n > 3$, at least one side $\tilde{L}$ of Ω_n does not contain any just chosen Q_k and therefore it is possible to shift $\tilde{L}$ outside of Ω_n so that it takes the position $\tilde{L}_1$ on a circle through $-w$ and the resulting circular polygon $\tilde{\Omega}_n$ will have the inner radius $R(0;\Omega_n) > 1$, while the radii $\rho(\Omega_n)$, $\rho(\tilde{\Omega}_n)$ of inscribed circles are equal. If $\tilde{F}$ maps $\mathbb{D}$ conformally onto $\tilde{\Omega}_n$, $\tilde{F}_n(0) = 0$, then $G = \tilde{F}/R(0;\tilde{\Omega}_n)$ belongs to K_n and maps $\mathbb{D}$ onto Ω'_n, while

$$L(G) = \rho(\Omega'_n) < L(\tilde{F}) = \rho(\tilde{\Omega}_n) = \rho(\Omega_n) = L(F)$$

which is a contradiction.

4. Some lemmas.

Lemma 4.1. *Let $\psi(x) = \Gamma'(x)/\Gamma(x)$ where Γ is the gamma-Euler's function. Then*

$$d(x) = \psi(x) + \tfrac{\pi}{2} \operatorname{ctg} x\tfrac{\pi}{2} - \tfrac{1}{2} \log x(1-x) \tag{4.1}$$

is decreasing function in $x \in (0; 1/2)$.

Proof. From the well known formulae :

(4.2) $$\psi(x) = -\gamma + \sum_{n=0}^{\infty}[(n+1)^{-1} - (n+x)^{-1}] ;$$

$$\pi^2(\sin x\pi)^{-2} = \sum_{n=-\infty}^{+\infty}(n-x)^{-2}, \quad (x \neq n),$$

we obtain

(4.3) $$d'(x) = \left(\tfrac{\pi}{2}\right)^2 \cos^{-2} x\tfrac{\pi}{2} - \sum_{n=1}^{\infty}(n-x)^{-2} - \tfrac{1}{2}(1-2x)[x(1-x)]^{-1}$$

and therefore it is sufficient to show that for $0 < x < 1/2$

(4.4) $$\left(\tfrac{\pi}{2}\right)^2 \cos^{-2} x\tfrac{\pi}{2} < \sum_{n=1}^{\infty}(n-x)^{-2} + \tfrac{1}{2}(1-2x)[x(1-x)]^{-1} .$$

Let us denote by $L(x)$, $P(x)$ the left and right hand side of (4.4) resp. The functions L, P are convex on the interval $(0; 1/2)$ and $L(1/2) = P(1/2)$, $L'(1/2) = \frac{\pi^3}{2} > 15.5$, $P'(1/2) < 12.96$. Besides, L increases in $(0; 1/2)$ while P is decreasing in $(0; x_0)$, $0.3 < x_0 < 0.31$ and then increasing in $(x_0; 1/2)$. Since $L(0.3) = 3.1079$, $P(0.3) > 3.7059$, inequality (4.4) holds in $(0; 0.3)$.

Let $x_1 = 0.3$, $x_2 = 0.4$, $x_3 = 0.44$, $x_4 = 0.5$, $I_k = [x_k; x_{k+1})$, $k = 1, 2, 3$. On each interval I_k it is possible to find a linear function $y_k(x)$ such that $L(x) < y_k(x) \leq P(x)$, $x \in I_k$, $k = 1, 2, 3$. We omit the details.

Lemma 4.2. *The function*

$$l(x) = \psi(x) - \psi(2x) + \pi/2 \sin x\pi + \tfrac{1}{2}\log\frac{2(1-2x)}{1-x}$$

decreases on $(0; 1/2)$ and $l(1/3) = 0$.

Proof. From the well known identity : $\psi(x) - \psi(1-x) = -\pi \operatorname{ctg} x\pi$ as well as from (4.2) it follows that

(4.5) $$l(x) = -\pi \operatorname{ctg} x\pi + \pi/2 \sin x\pi + (1-3x)\Big[(2x(1-x))^{-1} + \sum_{n=1}^{\infty}(2x+n)^{-1}(1-x+n)^{-1}\Big] + \tfrac{1}{2}\log\frac{2(1-2x)}{1-x} .$$

Hence $l(1/3) = 0$. Besides,

$$l'(x) = -\pi^2 \sin^{-2} x\pi - \tfrac{\pi^2}{2}\cos x\pi \sin^{-2} x\pi - \tfrac{1}{2}x^{-2} + (1-x)^{-2} - \sum_{n=1}^{\infty}[2(2x+n)^{-2} + (1-x+n)^{-2}] - \tfrac{1}{2}(1-2x)^{-1}(1-x)^{-1} .$$

For any $n \in \mathbf{N}$ the function : $x \mapsto 2(2x+n)^{-2}+(1-x+n)^{-2}$ is decreasing on $[0;1/2]$ so that

$$\sum_{n=1}^{\infty} 2(2x+n)^{-2}+(1-x+n)^{-2} \geq 2(2x+1)^{-2}+(2-x)^{-2}+\sum_{n=2}^{\infty}\left[2(1+n)^{-2}+(\tfrac{1}{2}+n)^{-2}\right].$$

Hence, using the identities : $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$; $\sum_{n=1}^{\infty} 4(1+2n)^{-2} = \frac{\pi^2}{2}$ we obtain inequality

$$l'(x) \leq h_1(x) - h_2(x) \tag{4.6}$$

where

$$\begin{aligned} h_1(x) &= (\tfrac{\pi}{2})^2 \cos^{-2} x\tfrac{\pi}{2} + \tfrac{\pi^2}{2}\left(\sin^{-2} x\pi - (x\pi)^{-2}\right) - 2(1+2x)^{-2}, \\ h_2(x) &= (1-x)^{-2} + \tfrac{1}{2}(1-2x)^{-1}(1-x)^{-1} + (2-x)^{-2} + \delta, \\ &\delta = (5/6)\pi^2 - 125/18. \end{aligned}$$

The functions h_1, h_2 are increasing on $[0;1/2]$ and $h_1(t_k) < h_2(t_{k-1})$ on a sequence (t_k), $k = 0,1,\ldots,8$, $t_k = 0;\ 0.1;\ 0.15;\ 0.2;\ 0.25;\ 0.3;\ 0.33;\ 0.4;\ 0.5$. From (4.6) the function $l(x)$ is decreasing on $(0;1/2)$.

Let G be a simply connected domain of hiperbolic type in $\mathbf{C}$. Let $w_0 \in G$ and f is a conformal mapping of the unit disk $\mathbf{D}$ onto G, $f(0) = w_0$. Then $R(w_0;G) = |f'(0)|$.

Let $z_0 \in H := \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ and let the function h maps the upper half-plane H onto G, $h(z_0) = w_0$. Then the homography : $\varsigma \mapsto (z_0 - \bar{z}_0\varsigma)/(1-\varsigma)$ maps the unit disk $\mathbf{D}$ onto $H : 0 \longmapsto z_0$. Hence

$$R(w_0;G) = 2\operatorname{Im} z_0 |h'(z_0)| \tag{4.7}$$

The function

$$h(z) = \int_0^z u^{\alpha-1}(1-u)^{\beta-1}\,du \tag{4.8}$$

maps conformally the upper half-plane H onto the triangle $\tilde{T}$ of internal angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, $\alpha+\beta+\gamma = 1$. From (4.7), (4.8) we have

$$R(w;\tilde{T}) = 2y/\left[(x^2+y^2)^{1-\alpha}\left((1-x)^2+y^2\right)^{1-\beta}\right]^{1/2} \tag{4.9}$$

where $w = h(z)$, $z = x+iy \in H$. Besides, $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$ is the beta-Euler's function, while

$$\rho(\tilde{T}) = \frac{\sin\alpha\frac{\pi}{2}\sin\beta\frac{\pi}{2}}{\sin(\alpha+\beta)\frac{\pi}{2}}\,B(\alpha,\beta) \tag{4.10}$$

is the radius of the disk inscribed in the triangle $\tilde{T}$. We will consider the right hand side of (4.10) as a function defined on $I^2 = (0; 1/2) \times (0; 1/2)$.

Lemma 4.3. *Let $\Phi(x, y, \alpha, \beta) = R(w; \tilde{T})/\rho(\tilde{T})$ where $R(w; \tilde{T})$, $\rho(\tilde{T})$ are given by (4.9), (4.10) resp. Hence the function $\Phi(x, y, \alpha, \beta)$ doesn't have critical points on $H \times \big(I^2 \setminus (1/3; 1/3)\big)$.*

Proof. Suppose that (x, y, α, β) is a critical point of Φ. Then it satisfies the system of equations :

$$(4.11) \qquad \begin{cases} y^{-1} - (1-\alpha)y/(x^2+y^2) - (1-\beta)y/[(1-x)^2+y^2] = 0 \\ \quad -(1-\alpha)x/(x^2+y^2) + (1-\beta)(1-x)/[(1-x)^2+y^2] = 0 \end{cases}$$

$$(4.12) \quad \begin{cases} \frac{1}{2}\log(x^2+y^2) - \frac{\pi}{2}\big(\operatorname{ctg} \alpha\frac{\pi}{2} - \operatorname{ctg}(\alpha+\beta)\frac{\pi}{2}\big) - \psi(\alpha) + \psi(\alpha+\beta) = 0 \\ \frac{1}{2}\log\big((1-x)^2+y^2\big) - \frac{\pi}{2}\big(\operatorname{ctg} \beta\frac{\pi}{2} - \operatorname{ctg}(\alpha+\beta)\frac{\pi}{2}\big) - \psi(\beta) + \psi(\alpha+\beta) = 0 \end{cases}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

The only solution of (4.11) is the pair (x_0, y_0) where

$$(4.13) \qquad x_0 = \alpha/(\alpha+\beta) \quad ; \quad y_0 = \sqrt{\alpha\beta}/\big[(\alpha+\beta)\sqrt{1-\alpha-\beta}\big] \, .$$

Putting (4.13) into (4.12) we get

(4.14)

$$\frac{1}{2}\log\frac{\alpha(1-\alpha)}{(\alpha+\beta)(1-\alpha-\beta)} - \psi(\alpha) + \psi(\alpha+\beta) - \frac{\pi}{2}\big(\operatorname{ctg} \alpha\frac{\pi}{2} - \operatorname{ctg}(\alpha+\beta)\frac{\pi}{2}\big) = 0$$

$$\frac{1}{2}\log\frac{\beta(1-\beta)}{(\alpha+\beta)(1-\alpha-\beta)} - \psi(\beta) + \psi(\alpha+\beta) - \frac{\pi}{2}\big(\operatorname{ctg} \beta\frac{\pi}{2} - \operatorname{ctg}(\alpha+\beta)\frac{\pi}{2}\big) = 0 \, .$$

Subtracting the both sides of (4.14) we get

$$(4.15) \qquad d(\alpha) - d(\beta) = 0$$

where d is the function of Lemma 4.1. Since d is decreasing (4.15) may has the solution only if $\alpha = \beta$. Putting $\alpha = \beta$ into one of the equation of (4.14) we obtain the equation

$$(4.16) \qquad l(\alpha) = 0 \, .$$

According to Lemma 4.2 equation (4.16) has the only solution $\alpha = 1/3$ so that the pair $(1/3; 1/3)$ is the only solution of (4.14). This implies the conclusion of Lemma 4.3.

Lemma 4.4. *Suppose that $P = \{\varsigma \in \mathbb{C} : |\operatorname{Im} \varsigma| < \frac{\pi}{4}\}$, $\tau_{iv}(\varsigma) = iv\varsigma/(iv - \varsigma)$, $v > \pi/4$, $\Omega = \tau_{iv}(P)$. Then for any $\tau \in \Omega$*

$$(4.17) \qquad R(\tau; \Omega) \le R_{\max} \, ,$$

where

$$R_{\max} = \frac{4y^2}{1-y^4}\,v^2\,, \tag{4.18}$$

and $y = y(v)$ is the unique solution of the equation

$$\tfrac{1}{2}(y^{-1} - y) + \tan^{-1} y = v \tag{4.19}$$

contained in the interval $(0;1)$.

Proof. If $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ then $P = f(\mathrm{D})$, $\Omega = F(\mathrm{D})$ with $F = \tau_{iv}\circ f$ and $R(\tau;\Omega) = (1-|z|^2)|F'(z)|$, where $\tau = F(z)$. Suppose $t + is = \varsigma \in P$, $|s| < \frac{\pi}{4}$. Then for any $\tau \in \Omega$

$$R(\tau;\Omega) = \frac{v^2\cos 2s}{t^2 + (v-s)^2} \le \frac{v^2\cos 2s}{(v-s)^2} = R(\tau_1;\Omega)\,. \tag{4.20}$$

Thus for s fixed $R(\tau;\Omega)$ attains its maximum at $\tau_1 = \tau_{iv}(is)$.

The function f maps the unit disk D onto the strip P such that $\varsigma = is$, $|s| < \pi/4$ corresponds to $z = iy$, $|y| < 1$. Hence $\varsigma = \frac{1}{2}\log\frac{1+iy}{1-iy} = i\tan^{-1} y$ and

$$R(\tau_1;\Omega) = \frac{1-y^2}{1+y^2}\left(1 - \tfrac{1}{v}\tan^{-1} y\right)^{-2}\,. \tag{4.21}$$

The right hand side of (4.21) attains its maximum if $y = y(v)$ is the unique root of the equation (4.19) contained in the interval $(0;1)$. Inequality in (4.20) as well as (4.21) and (4.19) gives (4.17) and (4.18). The proof is complete.

Lemma 4.5. *Suppose that the triangle $T = T(\alpha_0,\beta_0)$ containing the origin, with internal angles $\alpha_0\pi$, $\beta_0\pi$, $\gamma_0\pi$, $\alpha_0+\beta_0+\gamma_0 = 1$, has its inner radius $R(0,T) = 1$. Let $K = K(\varsigma_0, r)$ be the disk inscribed in T. Then there exists the triangle $T' = T'(\alpha,\beta)$ with the same disk K inscribed in T' such that $R(0,T') > 1$.*

Proof. The function h given by (4.8) maps the upper half-plane H onto the triangle $\dot{T} = \dot{T}(\alpha,\beta)$ similar to $T'(\alpha,\beta)$. Let $\rho(\dot{T})$ be the radius of the disk inscribed in $\dot{T}$. Then for some $w_0 \in \dot{T}$ and for some η, $|\eta| = 1$ the mapping: $w \mapsto \eta r(w-w_0)/\rho(\dot{T})$ transforms the triangle $\dot{T}$ onto T' such that $w_0 \longrightarrow 0$. Hence,
$R(0;T') = rR(w_0;\dot{T})/\rho(\dot{T}) = r\Phi(x,y,\alpha,\beta)$ where Φ is the function of Lemma 4.3.

Case (i) $(\alpha_0,\beta_0) \ne (1/3,1/3)$. If $R(0;T') \le 1$ for any admissible triangle T' then Φ would have a critical point contrary to the conclusion of Lemma 4.3.

Case (ii) $\alpha_0 = \beta_0 = 1/3$. Then it easily follows that the only critical point of Φ corresponds to the minimum of Φ.

5. Main results.

Theorem 5.1. $\inf\{L(F) : F \in K_3\} = \inf\{L(F) : F \in K_2\}$.

Proof. Suppose that the g.l.b. of $L(F)$ is attained for F which maps D onto the circular triangle Ω_3 with three sides situated on circles intersecting each other at the point $-w$. There exists a disk $\hat{K}$ of radius $L(F)$ tangent to $\partial\Omega_3$ at two or three points.

If the position of $\hat{K}$ is determined by two points situated on two different sides, then – similarly like in the proof of Theorem 3.2 we may shift the third free side outside of Ω_3 so that it takes the position on a circle through $-w$ and the resulting circular triangle $\tilde{\Omega}_3$ will have the inner radius $R(0;\tilde{\Omega}_3) > 1$, while $\rho(\tilde{\Omega}_3) = \rho(\Omega_3)$. Consequently, there is a function $G \in \hat{K}_3$ such that $L(G) < L(F)$ contrary to that F gives the g.l.b. of $L(F)$, $F \in \hat{K}_3$.

Suppose now, that the position of $\hat{K}$ is determined by three points Q_k. This means that the circle $\partial\hat{K}$ is devided by these Q_k onto three subarcs each having angular measure less than π. Besides, there exists the function $f \in K_3$ which maps $\mathbf{D}$ onto the triangle $T = f(\mathbf{D})$ and the point $w \in \mathbf{C} \setminus T$, such that $F(\mathbf{D}) = \Omega_3$, $F = \tau_w \circ f$.

The disk $\hat{K}$ is the image of the disk $K = \{\varsigma : |\varsigma - \varsigma_0| < r\} \subset T$ by the mapping $\tau_w(\varsigma) = w\varsigma/(w - \varsigma)$ and therefore the radius of the disk $\hat{K}$ is equal to $L(F) = r|w|^2/(|w - \varsigma_0|^2 - r^2)$. From Lemma 4.5 it follows that there exists the triangle T' with the same disk K inscribed in T' and conformal mapping $\tilde{f}$ which maps $\mathbf{D}$ onto T', such that $\tilde{f}(0) = 0$, $\tilde{f}'(0) = R(0;T') > 1$.

Let $\tilde{G} = \tau_w \circ \tilde{f}$. Then $L(\tilde{f}) = r$ and

$$L(\tilde{G}) = L(F)\,. \tag{5.1}$$

If $f_1 = \tilde{f}/\tilde{f}'(0)$, $F_1 = \tau_w \circ f_1$ then $f_1 \in K_3$, $F_1 \in \hat{K}_3$. Besides, $f_1(\mathbf{D}) \subset T'$ and therefore $L(F_1) < L(\tilde{G})$. From (5.1) $L(F_1) < L(F)$. The proof is complete.

From the Theorems : 3.1, 5.1 it follows that the g.l.b. of $L(F)$ on $\hat{K}$ is attained by $F \in \hat{K}_2$. It is very easy to see that the extremal image domain $\Omega_2 = F(\mathbf{D})$ can't have an internal angles greater than zero.

Let

$$F = \tau_w \circ f \tag{5.2}$$

where

$$\varsigma = f(z) = \tfrac{1}{2}\log\frac{1+z}{1-z}\quad,\quad w = u + iv\quad,\quad |v| > \tfrac{\pi}{4}\,.$$

The homography $\tau_w(\varsigma) = w\varsigma/(w-\varsigma)$ maps then any stright line : $\varsigma = x + iy_0$, y_0 being fixed, $y_0 \neq v$ onto a circle through $-w$ with the diameter $2r = \max_x|\tau_w(x+iy_0)+w|$.

Since the boundary of $f(\mathbf{D})$ consists of two lines : $l_1 : \varsigma = x + i\frac{\pi}{4}$; $l_2 : \varsigma = x - i\frac{\pi}{4}$, $-\infty < x < +\infty$, $\tau_w(l_1 \cup l_2)$ consists of two circles with the diameters $2r_1 = |w|^2/|v - \pi/4|$, $2r_2 = |w|^2/|v + \pi/4|$ resp. Hence, the radius of the disk inscribed in Ω_2 is $\rho(\Omega_2) = \frac{\pi}{4}|w|^2/(v^2 - (\frac{\pi}{4})^2)$. For fixed $v = \operatorname{Im} w$,

$$\rho_{\min}(\Omega_2) = \tfrac{\pi}{4}\, v^2/(v^2 - (\tfrac{\pi}{4})^2)\,. \tag{5.3}$$

Theorem 5.2. $\inf\{L(F), F \in \hat{K}_2\} = \frac{\pi}{4}$.

Proof. Since $\hat{K}_2$ is linearly invariant we shall consider the g.l.b. of $L(G)$ for all

$$G(z) = (F \circ \omega(z) - \tau_0)/R(\tau_0;\Omega_2)\,, \tag{5.4}$$

where $\omega(z) = (z + z_0)/(1 + \bar{z}_0 z)$, $z_0 \in \mathbf{D}$, $\tau_0 = F(z_0)$ while F is given by (5.2).

For any function (5.4) inequality (4.17) of Lemma 4.4 gives $L(G) \geq \rho_{\min}(\Omega_2)/R_{\max}$ where $\rho_{\min}(\Omega_2)$, $R_{\max}$ are given by (5.3), (4.18) resp. Hence, we will have to show that

(5.5) $$\rho_{\min}(\Omega_2)/R_{\max} > \tfrac{\pi}{4} .$$

Taking into account (4.19) as well as (5.3) we obtain inequality

(5.6) $$\left(1 - y^2 + 2y \tan^{-1} y\right)^2 < 1 - y^4 + y^2\left(\tfrac{\pi}{2}\right)^2 \quad , \quad 0 < y < 1 .$$

An elementary real analysis technique shows the truth of (5.6).

The left hand side of (5.5) can be as close to $\pi/4$ as we want, so that $\pi/4$ is indeed the g.l.b. of $L(F)$, $F \in \mathring{K}_2$.

The extremal function $F = \tau_w \circ f$ corresponds to $w = \infty$ so that $F \equiv f \in K$ and therefore $L(\mathring{K}) = L\mathring{K}_2) = L(K)$.

Hence $f(z) = \frac{1}{2} \log \frac{1+z}{1-z}$ and its rotations are the only extremal functions.

6. Univalence criteria. The Bloch–Landau constant within the class $\mathring{K}$ is connected with some geometric aspect of univalence criteria introduced by Krzyż [11]. A domain Ω in the finite plane $\mathbf{C}$ is called a univalence domain (for short : a $\mathcal{U}$–domain) if the inclusion : $\{\log g'(z) : z \in \mathbf{D}\} \subset \Omega$ for $g \in \mathcal{H}(\mathbf{D})$ and some branch of $\log g'$ implies the univalence of g in $\mathbf{D}$. Each $\mathcal{U}$–domain corresponds to a particular criterion of univalence. For example, the strip $\{\varsigma : |\mathrm{Im}\, \varsigma| < \pi/2\}$ corresponds to Noshiro–Warshawski univalence criterion [7, p.47].

We will use the following results in further considerations

Theorem B [4]. *Suppose that $g \in \mathcal{H}(\mathbf{D})$, $g'(0) \neq 0$. If*

(6.1) $$\left(1 - |z|^2\right) \frac{|g''(z)|}{|g'(z)|} \leq 1 \qquad (z \in \mathbf{D}) ,$$

then g is univalent in $\mathbf{D}$.

Theorem K [11]. *Suppose $\varphi \in \mathcal{H}(\mathbf{D})$ and the values of φ are contained in a domain Ω possessing a generalized Green's function. Then for any $z \in \mathbf{D}$*

(6.2) $$(1 - |z|^2)|\varphi'(z)| \leq R\big(\varphi(z); \Omega\big)$$

The sign of equality at some point $z_0 \in \mathbf{D}$ holds only for the univalent function φ and for a simply connected domain $\Omega = \varphi(\mathbf{D})$

Theorem 6.1. *Suppose $F \in \mathring{K}$, $L(F) = \mu\frac{\pi}{4}$, $1 \leq \mu < \infty$. Let moreover for $0 < \lambda < \mu^{-1}$, $\tilde{f} := \lambda F$ and*

(6.3) $$\Omega = \tilde{f}(\mathbf{D}) .$$

Then

(6.4) $$R(w; \Omega) \leq 1 \qquad (w \in \Omega) .$$

Proof. Suppose, contrary to (6.4) that $R(w_0;\Omega) > 1$ for particular $w_0 \in \Omega$. Then $G(z) = (F \circ \omega(z) - w_0)/R(w_0; F(\mathrm{D})) \in \hat{K}$, where $\omega(z) = (z + z_0)/(1 + \bar{z}_0 z)$, $z_0 \in \mathrm{D}$, $F(z_0) = w_0$. Hence

$$L(G) = L(F)/R(w_0; F(\mathrm{D})) = L(\hat{f})/R(w_0;\Omega) < \pi/4$$

which gives a contradiction.

Inequality (6.4) as well as Theorem K allow us to apply Theorem B to the function $g \in \mathcal{H}(\mathrm{D})$ such that $\{\log g'(z) : z \in \mathrm{D}\} \subset \Omega$.

Hence we get

Theorem 6.2. *The domain $\Omega = \hat{f}(\mathrm{D})$ given by (6.3) is a $\mathcal{U}$-domain.*

In particular, if $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ and $F = \tau_{iv} \circ f \in \hat{K}$, then

$$L(F) = \rho(\Omega_2) = \frac{\pi}{4}v^2/(v^2 - (\pi/4)^2) .$$

If we take $0 < \lambda < 1 - (\pi/4v)^2$ then $\hat{f} = \lambda F$ yields a $\mathcal{U}$-domain $\Omega = \hat{f}(\mathrm{D})$. Moreover, if $|v| > 3\pi/4$ then Ω is not contained in the strip of width π so that Theorem 6.2 gives a criterion of univalence which does not follow from the Noshiro–Warshawski Theorem.

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STRESZCZENIE

Niech S oznacza klasę funkcji holomorficznych i jednolistnych w kole jednostkowym $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ i takich, że $f(0) = f'(0) - 1 = 0$.

Niech $K \subset S$ będzie podklasą tych funkcji $f \in S$, dla których zbiór $f(\mathbb{D})$ jest wypukły.

Dla dowolnej liczby $w \in \hat{\mathbb{C}} \setminus f(\mathbb{D})$ niech $F_f = \bigcup_w \tau_w \circ f$, gdzie $\tau_w(\zeta) = w\zeta/(w - \zeta)$, oraz $\hat{K} = \bigcup_{f \in K} F_f$.

Barnard i Schober postawili problem badania tych własności klasy K, które dziedziczone są od klasy $\hat{K}$. Wykażemy, że klasa $\hat{K}$ jest liniowo niezmiennicza w sensie Pommerenke, oraz że stała Blocha–Landaua zarówno w klasie $\hat{K}$ jak i w klasie K jest równa $\pi/4$.