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Integrals of Certain n-valent Functions

Całki pewnych funkcji n-listnych

Abstract. Some applications of Briot-Bouquet differential subordination are obtained which improve and sharpen a number of results of Libera and others.

1. Introduction. Let $H(n)$ denote the class of functions $f(z) = z^n + a_{n+1}z^{n+1} + \cdots$, n a positive integer which are regular in the unit disc $U = \{z : |z| < 1\}$. Let F and G be regular in U. Then the function F is subordinate to G, written $F \prec G$ or $F(z) \prec G(z)$, if G is univalent in U, $F(0) = G(0)$ and $F(U) \subset G(U)$. A function $f \in H(n)$ is said to be in $S_n^*(A,B)$ if

$$(1) \qquad \frac{zf'(z)}{f(z)} \prec n\left(\frac{1+Az}{1+Bz}\right), \quad (z \in U\ ;\ -1 \leq B < 1 \text{ and } B < A),$$

and is said to be in $K_n(A,B)$ if

$$(2) \qquad 1 + \frac{zf''(z)}{f'(z)} \prec n\left(\frac{1+Az}{1+Bz}\right), \quad (z \in U\ ;\ -1 \leq B < 1 \text{ and } B < A).$$

We denote by $S_1^*(A,B) = S^*(A,B)$, $S^*(1-2\alpha,-1) = S^*(\alpha)$, $S_n^*(1-2\alpha,-1) = S_n^*(\alpha)$; $K_1(A,B) = K(A,B)$, $K(1-2\alpha,-1) = K(\alpha)$ and $K_n(1-2\alpha,-1) = K_n(\alpha)$ $(\alpha < 1)$.

The function $h(z)$ regular in U, with $h'(0) \neq 0$, is convex (univalent) if and only if $\operatorname{Re}\{1 + \frac{zh''(z)}{h'(z)}\} > 0$ in U.

Let β, γ, A and B be real numbers and suppose that $\beta > 0$, $\beta n + \gamma > 0$, $-1 \leq B < 1$ and $B < A \leq 1 + \gamma(1-B)n^{-1}\beta^{-1}$. After a little manipulation — from the more general result on Briot-Bouquet differential subordination [5], it is easy to deduce that the integral operator $I_{\beta,\gamma}^n$ defined by $g = I_{\beta,\gamma}^n[f]$, where

$$(3) \qquad g(z) = \left[\frac{n\beta+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f^\beta(t)\,dt\right]^{1/\beta}, \quad z \in U\ ,\ f \in S_n^*(A,B)$$

maps $S_n^*(A,B)$ into $S_n^*(A,B)$, i.e., $I_{\beta,\gamma}^n : S_n^*(A,B) \to S_n^*(A,B)$ (Here each power takes principal value).

For given real numbers A, B with $-1 \leq B < 1$ and $B < A \leq 1+\gamma(1-B)n^{-1}\beta^{-1}$, we define the order of starlikeness of the class $I^n_{\beta,\gamma}[S^*_n(A,B)]$ by the largest number $\delta = \delta(n, A; \beta, \gamma)$ such that

$$I^n_{\beta,\gamma}[S^*_n(A,B)] \subset S^*_n(\delta) .$$

Recently many of the classical results in univalent function theory have been improved and sharpened by the powerful technique of differential subordination, e.g. [1], [5], [8], [9], etc. Recall that a function $p(z)$ regular in U is said to satisfy Briot–Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) , \quad z \in U , \ (p(0) = h(0) = n) \tag{4}$$

for β and γ complex constants and $h(z)$, a convex (univalent) in U with Re $[\beta h(z) + \gamma] > 0$ in U. The univalent function $q(z)$ is said to be a dominant of the Briot–Bouquet differential subordination (4) if $p(z) \prec q(z)$ for all p(z) satisfying (4). If $\tilde{q}(z)$ is a dominant of (4) and $\tilde{q}(z) \prec q(z)$ for all other dominants $q(z)$ of (4), then $\tilde{q}(z)$ is said to be the best dominant.

In this paper we find $\delta(n, A, B; \beta, \gamma)$ for appropriate choices of A, B, β and γ, by using the sharp subordination result recently obtained in [5]. Our general result includes some particular ones obtained by several authors [4,6,7,8]. Our result gives improve and sharp form of the recent result obtained in [2,10].

2. Preliminaries.

Lemma 1. *Let $n \in N = \{1, 2, 3, \ldots\}$, $A, B, \beta, \gamma \in \mathbb{R}$ with $\beta > 0$ and $n\beta + \gamma > 0$. Suppose that these constants satisfy*

$$-1 \leq B < 1 \quad and \quad B < A \leq 1 + \gamma(1-B)n^{-1}\beta^{-1} . \tag{5}$$

Then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = n\left(\frac{1+Az}{1+Bz}\right) \tag{6}$$

has a univalent solution given by

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} , \tag{7}$$

where

$$Q(z) = \begin{cases} \int_0^1 \left(\frac{1+Bzt}{1+Bz}\right)^{n\beta((A-B)/B)} t^{n\beta+\gamma-1}\, dt & if\ B \neq 0 \\ \int_0^1 t^{n\beta+\gamma-1} \exp(n\beta Az(t-1))\, dt & if\ B = 0 , \end{cases} \tag{8}$$

and

$$q(z) = \frac{n\beta - \gamma Bz}{\beta(1+Bz)} \quad when\ A = -\frac{(\gamma+1)B}{n\beta} , \quad B \neq 0 . \tag{9}$$

If $p(z)$ is regular in U and satisfies

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec n\left(\frac{1+Az}{1+Bz}\right), \tag{10}$$

then $p(z)\prec q(z)\prec n(\frac{1+Az}{1+Bz})$ and $q(z)$ is the best dominant.

More general form of this lemma may be found in [5].

Lemma 2. *Let $\mu(t)$ be a positive measure on the unit interval $I=[0,1]$. Let $g(t,z)$ be a complex valued function defined on $U\times[0,1]$, and integrable in t for each $z\in U$ and for almost all $t\in[0,1]$, and suppose that* Re $\{g(t,z)\}>0$ *on U and $g(z)=\int_I g(t,z)\,d\mu(t)$. If, for fixed λ $(0\le\lambda<2\pi)$, $g(t,re^{i\lambda})$ is real for r real and* Re $\{\frac{1}{g(t,z)}\}\ge\frac{1}{g(t,re^{i\lambda})}$ *, for $|z|\le r$ and $t\in[0,1]$ then* Re $\{\frac{1}{g(z)}\}\ge\frac{1}{g(re^{i\lambda})}$ *for $|z|\le r$ and $0\le\lambda<2\pi$.*

The Lemma 2 can be proved in a similar manner as that of Lemma 2 of Wilken and Feng [12]. So we omit its proof.

3. Main result.

Theorem . *Let $\beta>0$, $n\beta+\gamma>0$ and consider the integral operator defined by* (3).
(a) *If $-1\le B<1$ and $B<A\le 1+\gamma(1-B)n^{-1}\beta^{-1}$, then the order of n-valent starlikeness of the class $I^n_{\beta,\gamma}[S^*_n(A,B)]$ is given by*

$$\delta(n,A,B;\beta,\gamma)=\frac{1}{n}\,[\inf_{|z|<1}\text{Re}\, q(z)]\,. \tag{11}$$

(b) *Moreover if $-1\le B<0$, $B<A\le\min\{1+\gamma(1-B)n^{-1}\beta^{-1},-(\gamma+1)Bn^{-1}\beta^{-1}\}$ then for $f\in S^*_n(A,B)$, we have*

$$\delta(n,A,B;\beta,\gamma)=\frac{1}{n}\,[q(-1)]=\frac{1}{n\beta}\left[\frac{n\beta+\gamma}{F(1,n\beta(\frac{B-A}{B});n\beta+\gamma+1;\frac{-B}{1-B})}-\gamma\right]. \tag{12}$$

(c) *Furthermore if $0<B<1$, $B<A\le\min\{1+\gamma(1-B)n^{-1}\beta^{-1}$, $(2n\beta+\gamma+1)Bn^{-1}\beta^{-1}\}$ then for $f\in S^*_n(A,B)$, we have*

$$\delta(n,A,B;\beta,\gamma)=\tfrac{1}{n}[q(1)]=\tfrac{1}{n\beta}\left[\frac{n\beta+\gamma}{F(1,n\beta(\frac{A-B}{B});n\beta+\gamma+1;\frac{B}{1+B})}-\gamma\right] \tag{13}$$

where q is given by (7) *and $F(a,b;c;z)$ is the hyper geometric function. The results are all sharp.*

Proof. Proceeding as in [5], we see that the condition $f\in S^*_n(A,B)$ $(-1\le B<1$, $B<A\le 1+\gamma(1-B)n^{-1}\beta^{-1})$ together with $f(0)=0$ (n-times) implies that $f(z)\ne 0$

in $0 < |z| < 1$. Now the function p, defined by $p(z) = zg'(z)/g(z)$ is regular in U and from (3) it can be easily shown that

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}, \quad z \in U . \tag{14}$$

Since $f \in S_n^*(A,B)$ is equivalent to $\frac{zf'(z)}{f(z)} \prec n\left(\frac{1+Az}{1+Bz}\right)$, $z \in U$, we deduce that $p(z)$ satisfies the differential subordination (10) and hence, by Lemma 1, $p(z) \prec q(z)$ which implies (11).

Next we shall use the following well known formulae for the proof of (12), (13) and (14).

For a, b, c real numbers other than $0, -1, -2$ and $c > b > 0$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z) , \tag{15}$$

$$F(a,b;c;z) = F(b,a;c;z) , \tag{16}$$

$$F(a,b;c;z) = (1-z)^{-1}F(a,c-b;c;z/(1-z)) . \tag{17}$$

where $F(a,b;c;z) = 1 + \frac{a\cdot b}{1\cdot c} z + \frac{a(a+1)\cdot b(b+1)}{1\cdot 2\cdot c(c+1)} z^2 + \cdots$ is the hypergeometric series, which holds for $z \in \mathbb{C} \setminus (1,\infty)$.

Suppose $-1 \le B < 0$, and $B < A < \min\{1 + \frac{\gamma(1-B)}{n\beta}, \frac{-(\gamma+1)B}{n\beta}\}$ and denote $a = n\beta(\frac{B-A}{B})$, $b = n\beta + \gamma$ and $c = n\beta + \gamma + 1 = b + 1$. Since $c > b > 0$, from (8), by using (15), (16) and (17) we deduce

$$\begin{aligned} Q(z) &= (1+Bz)^a \int_0^1 (1+Btz)^{-a} t^{b-1}\, dt = \\ &= (1+Bz)^a \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \left[(1-Bz)^{-a}F(a,c-b;c;Bz/(1+Bz))\right] = \\ &= \frac{1}{n\beta+\gamma}\left[F(1,a;c;Bz/(1+Bz))\right] . \end{aligned} \tag{18}$$

Since $A < -\frac{(\gamma+1)B}{n\beta}$ implies $c > a$, by using (15), (18) yields,

$$Q(z) = \int_0^1 g(t,z)\, d\mu(t) , \tag{19}$$

where

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz} , \tag{20}$$

and

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\,dt \; . \tag{21}$$

For $-1 \leq B < 0$, it may be noted that Re $\{g(t,z)\} > 0$, for $|z| \leq r < 1$, $g(t,-r)$ is real for $0 \leq r < 1$, $t \in [0,1]$ and

$$\text{Re}\left\{\frac{1}{g(z,t)}\right\} \geq \text{Re}\left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(t,-r)}$$

for $|z| \leq r < 1$ and $t \in [0,1]$. Therefore, by using Lemma 2 we deduce that $\text{Re}\{\frac{1}{Q(z)}\} \geq \frac{1}{Q(-r)}$, $|z| \leq r < 1$, and by letting $r \to 1^-$ we obtain $\text{Re}\{\frac{1}{Q(z)}\} \geq \frac{1}{Q(-1)}$, $z \in U$. Thus by letting $A \to (-\frac{(\gamma+1)B}{n\beta})^+$ for the case $A = -\frac{(\gamma+1)B}{n\beta}$, and using (7) we obtain (12).

To prove the third part we proceed as follows :

Suppose that $0 < B < 1$ with $B < A < \min\{1 + \frac{\gamma(1-B)}{n\beta}, (\frac{2n\beta+\gamma+1}{n\beta})B\}$ and if we set $a = n\beta\frac{(A-B)}{B}$, $b = n\beta+\gamma$ and $c = n\beta+\gamma+1$, then $c > b > 0$ as well as $c > a > 0$. As in the second part

$$Q(z) = \int_0^1 g(t,z)\,d\mu(t) \; ,$$

where $g(t,z)$ and $d\mu(t)$ are respectively given by (20) and (21).

For $0 < B < 1$, it may be noted that Re $\{g(t,z)\} > 0$ in U, $g(t,r)$ is real for $0 \leq r < 1$, $t \in [0,1]$ and

$$\text{Re}\left\{\frac{1}{g(t,z)}\right\} \geq \frac{1+(1-t)Br}{1+Br} = \frac{1}{g(t,r)}$$

for $|z| \leq r < 1$ and $t \in [0,1]$. Therefore by using Lemma 2 (with $\lambda = 0$), we deduce that Re $\{\frac{1}{Q(z)}\} \geq \frac{1}{Q(r)}$, $|z| \leq r < 1$ and by letting $r \to 1^-$ we obtain Re $\{\frac{1}{Q(z)}\} \geq \frac{1}{Q(1)}$, $z \in U$. In the case $A = \left(\frac{2n\beta+\gamma+1}{n\beta}\right)B$, we obtain (13) by letting $A \to \left[\frac{(2n\beta+\gamma+1)B}{n\beta}\right]^+$. This by (7) leads to (13). The sharpness follows from the best dominant property.

Remark. In the case of $\beta = 1$, we see that the method of proof yields the same differential equation namely

$$p(z) + \frac{zp'(z)}{p(z)+\gamma} = 1 + \frac{zf''(z)}{f(z)} \; , \quad z \in U$$

where $p(z) = 1 + \frac{zg''(z)}{g'(z)}$. So an analogue problem for functions in $K_n(A,B)$ can be proved in a manner similar to that of the above theorem and the results are the same. We next give some particular cases of our results.

4. Particular cases.

i. Taking $\beta - 1$, consider the integral transform

$$I_{1,\gamma}^n[f(z)] = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\,dt\,, \quad \gamma > -n\,,$$

then from the above theorem and the above remark, we have the following : if $-1 \le B < 0$ with $B < A \le \min\{1 + \frac{\gamma(1-B)}{n}, \frac{-(\gamma+1)}{n} B\}$, we have

$$(22) \qquad I_{1,\gamma}^n[S_n^*(A,B)] \subset S_n^*(\rho_1) \quad ; \quad I_{1,\gamma}^n[K_n(A,B)] \subset K_n(\rho_1)\,.$$

Furthermore if $0 < B < 1$ with $B < A \le \min\{1 + \frac{\gamma(1-B)}{n}, (\frac{2n+\gamma+1}{n})B\}$, we have

$$(23) \qquad I_{1,\gamma}^n[S_n^*(A,B)] \subset S_n^*(\rho_2) \quad ; \quad I_{1,\gamma}^n[K_n(A,B)] \subset K_n(\rho_2)\,.$$

where $\rho_1 = \delta(n, A, B; 1, \gamma)$ and $\rho_2 = \delta(n, A, B; 1, \gamma)$ obtained respectively from (12) and (13).

ii. For $-1 \le B < 0$, $A = -B$, $\beta = 1$, $\gamma = 1$, with $n = 1$ or 2, we see that

$$I_{1,1}^n[S_n^*(-B,B)] \subset S_n^*(\delta(n,-B,B;1,1))\;;$$
$$I_{1,1}^n[K_n(-B,B)] \subset K_n(\delta(n,-B,B;1,1))\,,$$

where

$$\delta(n,-B,B;1,1) = \frac{1}{n}\left[\frac{n+1}{F(1,2n;n+2;/(1-B))} - 1\right].$$

For instance $n = 1$, it follows that if $f \in S^*(-B,B)$ or $K(-B,B)$ with $-1 \le B < 0$, then the Libera operator [3]

$$(24) \qquad I_{1,1}^1[f(z)] = \frac{2}{z}\int_0^z f(t)\,dt$$

is in $S^*(\delta(1,-B,B;1,1))$ or $K(\delta(1,-B,B;1,1)$ respectively, where

$$\delta(1,-B,B;1,1) = \frac{2}{F(1,2;3;-B/(1-B))} - 1\,.$$

Here $\delta(1,1,-1;1,1) \equiv \frac{1}{2(2\ln 2 - 1)} - 1$ is the order of starlikeness (or convexity resp.) of the class $I_{1,1}^1[S^*(0)]$ (or $I_{1,1}^1(K(0))$ resp.).

Similary for $n = 2$, it follows that if $f \in S_2^*(-B, B)$ (or $K_2(-B, B)$ resp.) with $-1 \le B < 0$, then $I_{1,1}^2[f(z)] = \frac{2}{z}\int_0^z f(t)\,dt$ is in $S_2^*\left(\frac{1+(B/2)}{1-B}\right)$ (or $K_2\left(\frac{1+(B/2)}{1-B}\right)$ resp.). In particular

$$I_{1,1}^2[S_2^*(0)] \subset S_2^*(\tfrac{1}{4}) \quad ; \quad I_{1,1}^2[K_2(0)] \subset K_2(\tfrac{1}{4}) .$$

For $0 < B < 1$ with $B < A \le \min\{2-B, 4B\}$, it follows that if $f \in S^*(A, B)$ (or $K(A, B)$ resp.) then the Libera operator $I_{1,1}^1[f(z)]$ defined by (24) is in $S^*(\delta(1, A, B; 1, 1))$ (or $K(\delta(1, A, B; 1, 1))$ resp.), where $\delta(1, A, B; 1, 1)$ is obtained from (13).

iii. Taking $B = -1$ and $A = 1-2\alpha$ with $\alpha \in [\alpha_0, 1)$, and $\alpha_0 = \max\left\{\frac{-\gamma}{n\beta}, \frac{n\beta-\gamma-1}{2n\beta}\right\}$ then we have

$$I_{\beta,\gamma}^n[S_n^*(\alpha)] \subset S_n^*(\delta(n, 1-2\alpha, -1; \beta; \gamma)) , \tag{25}$$

where $\delta(n, 1-2\alpha, -1; \beta, \gamma)$ is obtained from (12). For $n = 1$ this is due to Mocanu et al. [8].

Substituting $\beta = 1$ and $\gamma = n-1$ in (25) and using the remark, we see that if $f \in S_n^*(\alpha)$ or $K_n(\alpha)$ $(0 \le \alpha < 1)$, then

$$I_{1,n-1}^n[f(z)] = \frac{2n-1}{z^{n-1}} \int_0^z t^{n-2} f(t)\,dt$$

belongs to $S_n^*(\delta(n, 1-2\alpha, -1; 1, n-1))$ or $K_n(\delta(n, 1-2\alpha, -1; 1, n-1))$ respectively. Here

$$\delta(n, 1-2\alpha, -1; 1, n-1) = \frac{1}{n}\left[\frac{2n-1}{F(1, n(1-\alpha), 2n, 1/2)} - (n-1)\right] .$$

This for $n = 1$ reduces to a result of MacGregor [4].

iv. Let $\beta > 0$, $n\beta + \gamma + B \ge 0$ and $-1 \le B < 0$. In this case

$$\min\left\{1 + \frac{\gamma(1-B)}{n\beta}, \frac{-(\gamma+1)B}{n\beta}\right\} = \frac{-(\gamma+1)B}{n\beta} ,$$

and if we take $B < A = \dfrac{-(\gamma+1)B}{n\beta}$, we get

$$q(z) = \frac{n\beta - \gamma B z}{\beta(1+Bz)} , \quad z \in U .$$

Therefore for $B = -1$, $\operatorname{Re} q(z) > \dfrac{n\beta - \gamma}{2\beta}$. This shows that if $\beta n \ge \gamma \ge 1 - n\beta$, the integral operator $I_{\beta,\gamma}^n$ maps $S_n^*\left(\frac{n\beta-\gamma-1}{2n\beta}\right)$ into $S_n^*\left(\frac{n\beta-\gamma}{2\beta}\right)$.

For instance

$$I_{\frac{1}{n},1}^n[S_n^*(-\tfrac{1}{2})] \subset S_n^*(0) ;$$

v. Let $\beta > 0$ and $n\beta + \gamma + B \le 0$ and $-1 \le B < 0$. In this case $\min\{1 + \frac{\gamma(1-B)}{n\beta}, \frac{-(\gamma+1)B}{n\beta}\} = 1 + \frac{\gamma(1-B)}{n\beta}$ and so we have for $B = -1$ (with $n\beta + \gamma \le 1$ and $\beta > 0$),

$$\delta(n, 1 - \frac{2\gamma}{n\beta}, -1; \beta, \gamma) = \frac{1}{n\beta}\Big[\frac{n\beta+\gamma}{F(1, 2(n\beta+\gamma); n\beta+\gamma+1; \frac{1}{2})} - \gamma\Big].$$

Using the wellknown identity [12]

$$F(a, b; \tfrac{a+b+1}{2}; \tfrac{1}{2}) = \frac{\pi^{1/2}\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})},$$

we find that

$$\delta(n, 1 - \tfrac{2\gamma}{n\beta}, -1; \beta, \gamma) = \frac{1}{n\beta}\Big[\frac{\Gamma(n\beta+\gamma+\frac{1}{2})}{\pi^{1/2}\Gamma(n\beta+\gamma)} - \gamma\Big].$$

If in the last formula, taking $\gamma = 0$ and $\beta = \frac{1}{\alpha} \le \frac{1}{n}$, we have

$$\delta(n, 1, -1; \tfrac{1}{\alpha}, 0) = \frac{\Gamma(\frac{n}{\alpha} + \frac{1}{2})}{\pi^{1/2}\Gamma(\frac{n}{\alpha}+1)}$$

for $\alpha \ge n$. This shows that if $g \in H(n)$ satisfies

$$\mathrm{Re}\left\{(1-\alpha)\frac{zg'(z)}{g(z)} + \alpha\Big(1 + \frac{zg''(z)}{g'(z)}\Big)\right\} > 0, \quad z \in U$$

for $\alpha \ge n$, then $g \in S_n^*(\delta(n, 1, -1; 1/\alpha, 0))$.

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STRESZCZENIE

Przy pomocy podporządkowania różniczkowego Briota–Bouqueta poprawia się i zaostrza liczne wyniki Libery i innych.