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Bounded Functions with Symmetric Normalization

Funkcje ograniczone z symetryczną normalizacją

Abstract. Let $X(B)$ denote the class of functions regular and univalent in the open unit disk Δ which satisfy the conditions $f(-a) = -a$, $f(a) = a$ and $|f(z)| < B$, where $0 < a < 1$, $a < B$. The authors obtain several covering theorems for the class $X(B)$ and its subclasses.

1. Introduction. A function $f(z)$, regular and univalent in the open unit disk Δ, $\Delta = \{z : |z| < 1\}$ is in class S if

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1 \,. \tag{1.1}$$

If, on the other hand,

$$f(0) = 0 \quad \text{and} \quad f(a) = a \,, \tag{1.2}$$

for some a, $0 < a < 1$, then it is said to have Montel's normalization, [8], and is in class **M**. Furthermore, we will let $\mathbf{S}(B)$ and $\mathbf{M}(B)$ be subclasses of S and **M**, respectively, whose members satisfy the additional condition that $|f(z)| < B$ for $z \in \Delta$. This additional hypothesis makes the study of these subclasses both difficult and interesting, [1,2].

The transformation $af(z)/f(a)$ carries members of S into **M**, consequently **M** inherits some properties directly from S. However, the effectiveness of this relationship breaks down between $\mathbf{S}(B)$ and $\mathbf{M}(B)$. The normalizations for S and **M** play a significant role in the study of these classes (see [5], [6], [7], for example).

In our present work, we look at functions f(z), regular and univalent in Δ, normalized so that

$$f(-a) = -a \quad \text{and} \quad f(a) = a \,, \tag{1.3}$$

for a fixed a, $0 < a < 1$. We call this class X. $X(B)$ is the subclass of fuctions bounded by B. The class X is compact. Its normalization renders the subclasses

$X(B)$, $\mathbf{S}(B)$, and $\mathbf{M}(B)$ quite independent. Consequently, $X(B)$ has properties not shared by other classes.

We will establish covering properties for $X(B)$ and some of its subclasses. Our methods make use of circular symmetrization [10] and a lemma established by J. Krzyż and E. Złotkiewicz [5].

2. Covering properties. The Koebe constant for a subset $\mathbf{A}$ of either $\mathbf{S}$ or $\mathbf{M}$ is the radius of the largest disk centered at the origin contained in $f(\Delta)$ for each f in $\mathbf{A}$. Since members of X may omit the origin, the classical Koebe constant for X is zero. However, it is meaningful to ask for its Koebe constants relative to a or $-a$. The symmetric normalization of X guarantees that if $f(z)$ is in X, then $-f(-z)$ is also, hence the Koebe constants relative to a and $-a$ are the same.

Theorem 1. *Let $R = R(a, B)$ be given by the formula $R = |d - a|$, where*

$$d = \hat{k}\left[\frac{k(\frac{a}{B}) - qk(-\frac{a}{B})}{1-q}\right], \quad q = \left(\frac{1-a}{1+a}\right)^4 \tag{2.1}$$

and $\hat{k}$ denotes the inverse of the Koebe function $k(z) = z/(1-z)^2$. Then

$$\{w : |w - a| < R\} \cup \{w : |w + a| < R\} \subset f(\Delta) \tag{2.2}$$

for each $f(z)$ in $X(B)$. This result is the best possible.

Proof. Let $f(z)$ be in $X(B)$ and $D = f(\Delta)$. The compactness of $X(b)$ guarantees that there be a function in the class for which $\mathrm{dist}\{a, \partial D\} = R$, $R > 0$.

Let $g(z, z_0; D)$ be Green's function of D and let D^* be the domain obtained from D under circular symmetrization with respect to the ray $(-\infty, a]$. Then

$$g(a, -a; \Delta) = g(a, -a; D) \leq g(a, -a; D^*)\ , \tag{2.3}$$

as Green's function increases under circular symmetrization [4].

Denote by K_R the domain obtained from the disk $|w| < B$ slit along the segment $[B - R, B]$, then

$$g(a, -a; D^*) \leq g(a, -a; K_R)\ , \tag{2.4}$$

because $D^* \subset K_R$. Now, if K_d is a domain like K_R, but slit along $[B - d, B]$, with d chosen so that $g(a, -a; K_d) = g(a, -a; \Delta)$, then, in view of (2.3) and (2.4), $d \leq R$. To conclude, it sufficies to find the mapping of Δ onto K_d which satisfies (1.3) and (2.1). This is done by the function $W(z)$ defined by

$$k\left(\frac{W(z)}{B}\right) - k\left(\frac{d}{B}\right) = q\left[k(z) - \frac{1}{4}\right],$$

where q is a constant determined by (2.1).

Since both $f(z)$ and $-f(-z)$ are always in our class, the proof is concluded.

The Koebe set for the class $X(B)$ is the set common to all regions $f[\Delta]$, $f(z)$ in $X(B)$, hence, it is $\mathbf{K} = \bigcap_{f(z) \in X(B)} f[\Delta]$.

$\mathbf{K}$ may not be simply-connected for suitable choices of a. The function $W = W(z)$ normalized by (1.3) and defined by the equation

$$\frac{iWB}{(M-iW)^2} = c\left[\frac{iz}{(1-iz)^2} + \frac{1}{4}\right]$$

is in $X(B)$; and $W(z)$ maps Δ onto the disk given by $|W| < B$ furnished with a cut covering the segment $[0, iB]$, providing $a \geq a_0$, with

$$4 \text{ Arctan } a_0 = 2 \text{ Arctan}\left(\frac{a_0}{B}\right) + \frac{\pi}{2} .$$

Since $W(z)$ and $\overline{W(\bar{z})}$ are simultaneously in $X(B)$, the corresponding Koebe set is separated by the imaginary axis. This observation is consistent with the analogous result for the class of $\mathbf{M}$ of functions with Montel's normalization [5].

Our methods are not sufficient at this time to enable us to find the Koebe set of $X(B)$. However, we are able to give the analog of Theorem 1 for the subclass of $X(B)$ whose members map Δ onto a convex domain. We call this class $X^c(B)$.

Theorem 2. *For each $f(z)$ in $X^c(B)$,*

$$\{w \mid |w-a| < R\} \cup \{w \mid |w+a| < R\} \subset f[\Delta] , \tag{2.5}$$

if $R = B\cos\alpha - a$, $0 < \alpha < \cos^{-1}(\frac{a}{B})$, α being a solution of the equation

$$\left|\sin\frac{\beta-\gamma}{2}\right| = \frac{2a}{1+a^2}\left|\sin\left(\frac{\pi^2}{\alpha} + \frac{1}{2}(\beta+\gamma)\right)\right| , \tag{2.6}$$

with

$$\beta = \frac{2\pi}{\alpha}\tan^{-1}\frac{a\sin\alpha}{B - a\cos\alpha} \quad \text{and}$$
$$\gamma = -\frac{2\pi}{\alpha}\tan^{-1}\frac{a\sin\alpha}{B + a\cos\alpha} .$$

Proof. As symmetrization does not generally preserve convexity of domains we must modify the technique used for Theorem 1.

Suppose $D = f[\Delta]$ and $w_0 \in \partial D$ with $|w_0| < B$. Because D is convex, there is a supporting segment of D, through w_0, which together with a properly chosen arc of the circle $|w| = B$ form the boundary of a convex domain G, with $D \subset G$. Then, the conformal invariance of Green's function and the above inclusion give

$$g(a,-a;\Delta) = g(a,-a;D) \leq g(a,-a;G) . \tag{2.7}$$

The circular symmetrization of G with respect to the ray $\{x \in \mathbf{R} : x \leq a\}$ gives the convex domain G^*. Then, as in Theorem 1,

$$g(a,-a;G) \leq g(a,-a;G^*) . \tag{2.8}$$

Now, suppose

$$D_h = \{w : |w| < B \quad \text{and} \quad \operatorname{Re} w < h\}, \tag{2.9}$$

for $a < h \le B$. We know that $g(a, -a; D_h) = g(a, -a; \Delta)$, consequently

$$h \le d, \tag{2.10}$$

for $d = \operatorname{dist}\{0, \partial G^*\}$. Furthermore, equality holds in (2.10) if and only if $D_h = G^*$. This means that h is the Koebe constant for $X^c(B)$ with respect to a and $-a$.

To find the explicit form for h, $h = R$, as given in the theorem, we use the condition

$$g(a, -a; \Delta) = g(a, -a; D_h). \tag{2.11}$$

If $Be^{i\alpha}$ and $Be^{-i\alpha}$ are the end points of the segment satisfying $\operatorname{Re} w = h$ and $|w| \le B$, then

$$U(w) = \left(\frac{Be^{i\alpha} - w}{w - Be^{-i\alpha}}\right)^{\pi/\alpha}, \tag{2.12}$$

with $U(0) = e^{i\frac{\pi^2}{\alpha}}$ maps D_h onto the lower half-plane, H.

Now, $g(z, z_0; \Delta) = -\log|L(z, z_0)|$, where $L(z, z_0) = e^{i\alpha}\dfrac{z - z_0}{z\bar{z}_0 - 1}$, $z_0 \in \Delta$, and $g(z, \lambda; H) = -\log|T(z, \lambda)|$, for $T(z, \lambda) = e^{i\theta}\left(\dfrac{z - \lambda}{z - \bar{\lambda}}\right)$, $\operatorname{Im} \lambda < 0$ and suitable θ. Letting $z = U(w)$ in $T(z, \lambda)$ and evaluating constants appropriately reduces (2.11) to

$$\left|\frac{U(a) - U(-a)}{U(a) - \overline{U(-a)}}\right| = \frac{2a}{1 + a^2}. \tag{2.13}$$

Then, setting $\beta = \frac{2\pi}{\alpha} \operatorname{Arg}(B - ae^{-i\alpha})$ and $\gamma = \frac{2\pi}{\alpha} \operatorname{Arg}(B + ae^{-i\alpha})$, yields the form

$$\frac{\left|\sin \frac{\beta - \gamma}{2}\right|}{\left|\sin\left[\frac{\pi^2}{\alpha} + \frac{(\beta + \gamma)}{2}\right]\right|} = \frac{2a}{1 + a^2}, \tag{2.14}$$

which is equivalent to (2.6).

3. An extremal problem. Let $l[w_0, \phi]$ be the ray issuing from w_0 with inclination ϕ, i.e.,

$$l[w_0, \phi] = \{w : w = w_0 + re^{i\phi},\ r \ge 0\}. \tag{3.1}$$

If $f(z)$ is univalent in Δ, then let

$$E[f(z), \phi] = f[\Delta] \cap l[w_0, \phi] \tag{3.2}$$

and let $\mu(E[f(z),\phi])$ be the Lebesgue measure of (3.2), (it may be $+\infty$).

Suppose now, that **A** is a suitably defined family of functions univalent in Δ and with w_0 in $f[\Delta]$. Then one may pose the problem of finding

$$l(\phi) = \inf_{A} \mu(E[f(z),\phi]) , \tag{3.3}$$

for $0 \le \phi \le 2\pi$.

This extremal problem is the radial analog of the omitted-arc problem resolved for S by Jenkins [3]. The solution to (3.3) for starlike or convex subclasses of S, with $w_0 = 0$, gives the Koebe set for those classes. But it is not so in general.

We have no solution to (3.3) for X or $X(B)$. It seems plausible that the solution for X coincides with that for X^{*} and for $X(B)$ it coincides with $X^{*}(B)$. (X^{*} and $X^{*}(B)$ denote the subfamilies of functions starlike with respect to the origin.) It would be useful to determine (3.3) for X and its subclasses with $w_0 = a$. However, at this time, we are able to handle the problem only for $X^c(B)$ and for odd members of $X^{*}(B)$ with $w_0 = 0$; and, we resolve it by finding the Koebe set for each class.

The Koebe set for $X^c(B)$ is $\mathbf{K}^c = \bigcap_{X^{*}(B)} f[\Delta]$. It is a closed convex set containing a and $-a$. If $w = \rho e^{i\phi}$ is in $\partial\mathbf{K}^c$, then $l(\phi) = |w| = \rho$, when $w_0 = 0$ in (3.1).

Our method depends on properties of Green's function which were established by J. Krzyż and E. Złotkiewicz [5]. They found Koebe sets for functions $f(z)$ univalent in Δ for which $f(0) = a$ and $f(z_0) = b$, (a, b and z_0 are fixed). Their work depended on the following lemma which we will use here.

Lemma [5]. *Suppose* **G** *is a class of simply connected domains in* **C** *each containing the fixed, distinct points* a *and* b. *let* $\mathbf{G}_w$ *be the subclass of* **G** *whose members omit* w. *Furthermore, if*

(i) there is Ω_w *in* **G** *such that for all* Ω *in* $\mathbf{G}_w$,

$$g(a,b;\Omega) \le g(a,b;\Omega_w) \equiv G(w;\mathbf{G}) ;$$

(ii) $\{z : g(a,z;\Omega_w) > \delta\} \in \mathbf{G}$ *for all* δ ,

$$0 < \delta < g(a,b;\Omega_w) ;$$

and

(iii) $\mathbf{G}_\gamma \equiv \{\Omega \in \mathbf{G} : g(a,b;\Omega) = \gamma\}$, *for* $\gamma > 0$ *; then*

$$\bigcap_{\Omega\in\mathbf{G}_\gamma} \Omega = \{w : G(w;\mathbf{G}) < \gamma\} .$$

Now, let $\mathbf{F}(a,-a;w_0)$ be the family of all convex domains D, each contained entirely in the disk $\{w : |w| < B\}$, including a and $-a$ but omitting the value w_0, with $|w_0| < B$. Because of the convexity, each member of the set is contained in a subdomain $D(w_0)$ bounded by an arc satisfying $|w| = B$ and a segment through w_0 with end points on the arc. Consequently,

$$g(a,-a;D) \le g(a,-a;D(w_0)) . \tag{3.4}$$

Then to find the supremum of the right side of (3.4), we confine ourselves to domains of type $D(w_0)$ and apply the lemma.

If w_0 is a boundary point of K^c, it follows from the compactness of $X^c(B)$ that the corresponding domain $D(w_0)$ is the image of Δ under a function in the class and we may write

$$g(a,-a;\Delta) = g(a,-a;D(w_0)) \, . \tag{3.5}$$

Now, because of the conformal invariance of Green's function, we may restrict our search for extremal functions and extremal domains to like $D(w_0)$, in some optimal position, and to their images in the lower half-plane, (as was done in Theorem 2).

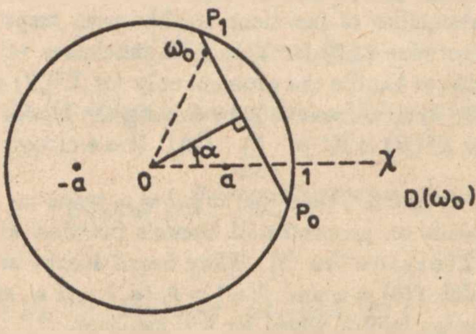

Fig. 1

Let us assume that the extremal domain appears as in Figure 1. Then a rotation through the angle $(-\alpha)$ gives a domain of the type $D(w_0)$, as shown in Figure 2 ; we call it $\tilde{D}(w_0)$.

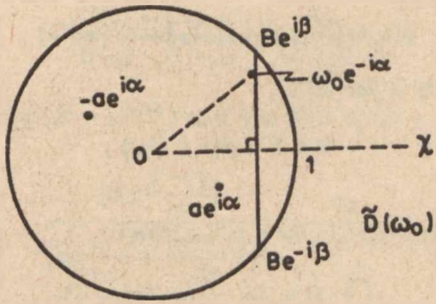

Fig. 2

From Figure 2, we can see that

$$\beta = \cos^{-1}\left(\frac{|w_0|\cos\alpha}{B}\right) .$$

Then, the function mapping $D(w_0)$ onto the lower half-plane H is

$$U(w) = \exp\left(i\frac{\pi^2}{\beta}\right)\left(\frac{B - e^{-i\beta}w}{B - e^{i\beta}w}\right)^{\pi/\beta} . \tag{3.6}$$

The invariance of Green's function guarantees that

$$\begin{aligned}(3.7)\qquad g(-a,a;D(w_0)) &= g(-a,a;\tilde{D}(w_0)) = \\ &= g\big(U(-ae^{-i\alpha}),U(ae^{-i\alpha});H\big) = \Phi(a,B,w_0,\alpha)\ ,\end{aligned}$$

where

$$(3.8)\qquad \Phi(a,B,w_0,\alpha) = \left|\frac{U(ae^{-i\alpha}) - U(-ae^{-i\alpha})}{U(ae^{-i\alpha}) - \overline{U(-ae^{-i\alpha})}}\right| .$$

We have used properties of mapping and Green's functions discussed in the proof of Theorem 2.

Finally, the extremal value for the problem corresponds to the choice α_0, of α for which

$$(3.9)\qquad \Phi(a,B,w_0,\alpha) = \frac{2a}{1+a^2} .$$

w_0 is fixed in these computations, however, α varies as the segment $[P_0,P_1]$ through w_0, (see Fig.1), is allowed to vary. We summarize our conclusion as the following theorem.

Theorem 3. *The Koebe set for the family of convex functions in $X(B)$ is*

$$(3.10)\qquad \mathbf{K}^c = \{w : \Phi(a,B,w,\alpha) \le \frac{2a}{1+a^2}\} .$$

If $w_0 \in \partial\mathbf{K}^c$, $|w_0| = \rho < B$, then the corresponding extremal function maps Δ onto a domain bounded by an arc of $|w| = B$ whose endpoints are joined by a segment through w_0.

To conclude, we look at the analogous problem for bounded, odd starlike functions in X.

Theorem 4. *The Koebe set for the class of odd functions in $X^*(B)$ is given by*

$$(3.11)\qquad \left|\frac{B^2w + a^2\overline{w}}{B^2+|w|^2} - a\right| + \left|\frac{B^2w + a^2\overline{w}}{B^2+|w|^2} + a\right| \le 1 + a^2 .$$

Furthermore, $l(\Phi) = |w|$ whenever $|w|e^{i\phi}$ gives equality in (3.11).

Proof. Let $G(a,-a;w_0)$ be the family of domains bounded by B, starlike and symmetric with respect to the origin ("odd" could be used to describe the latter), and omitting w_0 , $|w_0| < B$. If $D \in G(a,-a;w_0)$, then the ray $\{w = \rho e^{i\alpha}|\rho \ge |w_0|\}$ and its reflection in the origin, $\{w = \rho e^{i(\alpha+\pi)}|\rho \ge |w_0|\}$, $\alpha = \operatorname{Arg} w_0$, are in the complement of D. Now, if $\tilde{D}(w_0)$ is the disk $|w| \le B$ slit along these rays, then

$$(3.12)\qquad g(-a,a;D) \le g(-a,a;\tilde{D}(w_0)) .$$

To complete our proof, it suffices to find $g(-a,a;\tilde{D}(w_0))$.

First, we rotate and dilate the domain $\tilde{D}(w_0)$ by the transformation $\varsigma = \frac{e^{-i\alpha}w}{B}$. $\Delta\rho$, the image of $\tilde{D}(w_0)$ is the unit disk cut along the segments $[-1,-\rho]$ and $[\rho,1]$, $\rho = \frac{|w_0|}{B}$ and we let $b = \frac{ae^{-i\alpha}}{B}$. Then, with $U = \frac{1+\rho^2}{2\rho}\cdot\frac{\varsigma}{1+\varsigma^2}$,the transformation $Z = \frac{1-\sqrt{1-4U^2}}{2U}$ maps $\Delta\rho$ onto Δ. A computation shows that

$$(3.13) \qquad g(b,0;\Delta\rho) = \log\left|\frac{2U(b)}{1-\sqrt{1-4U^2(b)}}\right| = \log\left|\frac{1+\sqrt{1-4U^2(b)}}{2U(b)}\right| .$$

Finally, an application of the lemma, gives the Koebe set for our class as

$$(3.14) \qquad \left\{ w : \left|1+\frac{\sqrt{1-4U^2(\frac{e^{-i\alpha}w}{B})}}{2U(\frac{e^{-i\alpha}w}{B})}\right| \le \frac{1}{a} \right\},$$

which is equivalent to (3.11). The second statement of Theorem 4 follows from the special character of the domains under consideration.

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STRESZCZENIE

Niech $X(B)$ oznacza klasę funkcji regularnych i jednolistnych w kole jednostkowym Δ, spełniających warunki : $f(-a) = -a$, $f(a) = a$ oraz $|f(z)| < B$ dla $z \in \Delta$, gdzie $0 < a < 1$, $a < B$. W pracy tej autorzy otrzymują kilka twierdzeń o pokryciu dla klasy $X(B)$ i jej podklas.

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