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An Alternative Proof of a Result Due to Douady and Earle

Nowy dowód pewnego twierdzenia Douady – Earle'a

Abstract. In this paper we give an alternative simple proof of a Theorem due to Douady and Earle concerning homeomorphic extension of automorphisms of the unit circle $\mathbf{T}$. Taking into account a result of J. Krzyż we investigate this extension in case of quasisymmetric automorphisms.

0. Introduction. In this paper we associate with any automorphism γ of the unit circle $\mathbf{T}$ a mapping F_γ of the unit disc Δ onto itself. We show that the mapping F_γ is a homeomorphism of Δ onto itself which has a continuous extension to the automorphism γ^{-1} of $\mathbf{T}$ and satisfies the identity (1.3). In the special case when γ is a k-quasisymmetric automorphism of $\mathbf{T}$ (see the Definition 2.3 in [5]) F_γ is a K-quasiconformal automorphism of Δ and the constant K depends on k only. In fact $F_\gamma^{-1} = E(\gamma)$, where $E(\gamma)$ is the mapping introduced by A. Douady and C. J. Earle in [2]. However our definition of F_γ is formally different and simpler that of $E(\gamma)$. This way we get alternative proofs of Theorems 1 and 2 established in [2].

1. We denote by Δ the unit disc. For each $z \in \Delta$ the Möbius transformation h_z of the closed disc $\overline{\Delta}$ is given by the following formula

$$h_z(\xi) = \frac{\xi - z}{1 - \bar{z}\xi}, \quad \xi \in \overline{\Delta}.$$

We also consider the class $\mathcal{M}$ of all Möbius transformations of $\overline{\Delta}$ and this class $\mathrm{Aut}_{\mathbf{T}}$ of all automorphisms (i.e. sense-preserving homeomorphic self-mappings) of the unit circle $\mathbf{T} = \partial\Delta$. Evidently

$$\mathcal{M} = \{e^{i\varphi} h_z : \varphi \in \mathbf{R}\,,\ z \in \Delta\}\,.$$

For any automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$ we define

$$\phi(z, w; \gamma) = \frac{1}{2\pi} \int\limits_{\mathbf{T}} (h_z \circ \gamma)(\xi)\ \mathrm{Re}\, \frac{\xi + w}{\xi - w}\, |d\xi|\,, \quad z, w \in \Delta\,. \tag{1.1}$$

As shown by Choquet [1] the mapping

$$\Delta \ni w \to \phi(z, w; \gamma) \in \Delta$$

is an automorphism of Δ for any fixed $z \in \Delta$ and consequently there exists the function $w = F_\gamma(z)$ defined implicitly by the equation

$$\phi(z, w; \gamma) = 0 . \tag{1.2}$$

This way we obtain the mapping F_γ of Δ into itself. Moreover, the following theorem holds :

Theorem 1.1. *For any automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$ the mapping F_γ is an automorphism of Δ which has a continuous extension to the automorphism γ^{-1} of $\mathbf{T}$. Moreover*

$$F_{\eta_1 \circ \gamma \circ \eta_2} = \eta_2^{-1} \circ F_\gamma \circ \eta_1^{-1} \tag{1.3}$$

for all Möbius transformation $\eta_1, \eta_2 \in \mathcal{M}$.

Proof. Let $\gamma \in \mathrm{Aut}_{\mathbf{T}}$. We first prove that F_γ is a cotinuous extension of the automorphism γ^{-1} of $\mathbf{T}$ on Δ. Let $z_n \in \Delta$, $n = 1, 2, \dots$ be a sequence which converges to the point $z \in \overline{\Delta}$ and let $F_\gamma(z_{n_k}) \in \Delta$, $k = 1, 2, \dots$ be an arbitrary subsequence of the sequence $F_\gamma(z_n) \in \Delta$, $n = 1, 2, \dots$. There exists a subsequence $F_\gamma(z_{n_{k_l}})$, $l = 1, 2, \dots$ which converges to a certain point $w \in \overline{\Delta}$. Assume that $z \in \Delta$. Then

$$\max_{\xi \in T} |h_{z_n} \circ \gamma(\xi) - h_z \circ \gamma(\xi)| \to 0 \quad \text{as} \quad n \to \infty . \tag{1.4}$$

If $w \in \mathbf{T}$ then by (1.1), (1.4) and the properties of Poisson integral we have

$$\begin{aligned} &|\phi(z_{n_{k_l}}, F_\gamma(z_{n_{k_l}}); \gamma) - h_z \circ \gamma(w)| \le \\ &\le \frac{1}{2\pi} \int_{\mathbf{T}} |h_{z_{n_{k_l}}} \circ \gamma(\xi) - h_z \circ \gamma(\xi)| \,\mathrm{Re}\, \frac{\xi + F_\gamma(z_{n_{k_l}})}{\xi - F_\gamma(z_{n_{k_l}})} |d\xi| + \\ &+ |\phi(z, F_\gamma(z_{n_{k_l}}); \gamma| - h_z \circ \gamma(w)| \to 0 \quad \text{as } l \to \infty . \end{aligned}$$

Therefore by (1.2) we get $h_z \circ \gamma(w) = 0$ which is impossible in view of $h_z \circ \gamma(w) \in \mathbf{T}$. Thus $w \in \Delta$ and in the limiting case we obtain

$$0 = \lim_{i \to \infty} \phi(z_{n_{k_l}}, F_\gamma(z_{n_{k_l}}); \gamma) = \phi(z, w; \gamma) .$$

Hence $w = F_\gamma(z)$ and this means that

$$\lim_{n \to \infty} F_\gamma(z_n) = F_\gamma(z) .$$

Now we assume that $z \in T$. Then

$$\max_{\xi \in T \setminus I_\epsilon(z)} |h_{z_n} \circ \gamma(\xi) + z| \to 0 \quad \text{as} \quad n \to \infty \tag{1.5}$$

where $I_\varepsilon(z) = \{\xi \in \mathbf{T} : |\xi - \gamma^{-1}(z)| < \varepsilon\}$ for all ε, $0 < \varepsilon < 2$. If $w \in \Delta$ then by the bounded convergence theorem and the properties of Poisson integral we get

$$0 = \lim_{l\to\infty} \phi(z_{n_{k_l}}, F_\gamma(z_{n_{k_l}}); \gamma) = -z$$

because of (1.5). This is a contradiction if $z \in \mathbf{T}$, whence $w \in \mathbf{T}$. If $w \neq \gamma^{-1}(z)$ then setting $\varepsilon = \dfrac{|w - \gamma^{-1}(z)|}{2}$ we obtain analogously by (1.1) and (1.5)

$$\begin{aligned}
&\phi(z_{n_{k_l}}, F_\gamma(z_{n_{k_l}}); \gamma) = \\
&= \frac{1}{2\pi} \int_{I_\varepsilon(z)} (h_{z_{n_{k_l}}} \circ \gamma(\xi) + z) \operatorname{Re} \frac{\xi + F_\gamma(z_{n_{k_l}})}{\xi - F_\gamma(z_{n_{k_l}})} |d\xi| + \\
&+ \frac{1}{2\pi} \int_{\mathbf{T} \setminus I_\varepsilon(z)} (h_{z_{n_{k_l}}} \circ \gamma(\xi) + z) \operatorname{Re} \frac{\xi + F_\gamma(z_{n_{k_l}})}{\xi - F_\gamma(z_{n_{k_l}})} |d\xi| - \\
&- z \frac{1}{2\pi} \int_{\mathbf{T}} \operatorname{Re} \frac{\xi + F_\gamma(z_{n_{k_l}})}{\xi - F_\gamma(z_{n_{k_l}})} |d\xi| \to -z \quad \text{as} \quad l \to \infty.
\end{aligned}$$

On the other hand due to (1.2) we have

$$\lim_{l\to\infty} \phi(z_{n_{k_l}}, F_\gamma(z_{n_{k_l}}); \gamma) = 0$$

which is impossible because of $z \in \mathbf{T}$. This means that $w = \gamma^{-1}(z)$ and

$$\lim_{n\to\infty} F_\gamma(z_n) = \gamma^{-1}(z) .$$

Now we show that (1.3) holds. Let $\eta \in \mathcal{M}$ be any Möbius transformation and $z \in \Delta$ be fixed. By (1.1) the functions

$$\Delta \ni w \to \phi(z, w; \gamma \circ \eta) \in \Delta$$

and with respect to the conformal invariance

$$\Delta \ni w \to \phi(z, \eta(w); \gamma) \in \Delta$$

are the solution of Dirichlet problem for Δ with the boundary values $h_z \circ \gamma \circ \eta$ on $\mathbf{T}$. Hence and from (1.2) it follows that

$$0 = \phi(z, F_{\gamma\circ\eta}(z); \gamma \circ \eta) = \phi(z, \eta \circ F_{\gamma\circ\eta}(z); \gamma)$$

and

$$0 = \phi(z, F_\gamma(z); \gamma) .$$

This implies due to the Choquet theorem $F_\gamma(z) = \eta \circ F_{\gamma\circ\eta(z)}$. Therefore

$$F_{\gamma\circ\eta} = \eta^{-1} \circ F_\gamma . \tag{1.6}$$

Since $h_z \circ \eta \in \mathcal{M}$, there exist $\varphi \in \mathbb{R}$ and $z' \in \Delta$ such that

$$h_z \circ \eta = e^{i\varphi} h_{z'} . \tag{1.7}$$

From (1.1) and (1.7) it follows that for any $w \in \Delta$

$$\phi(z, w; \eta \circ \gamma) = e^{i\varphi} \phi(z', w; \gamma) . \tag{1.8}$$

Setting $w = F_{\eta\circ\gamma}(z)$ in (1.8) we obtain

$$0 = \phi(z, F_{\eta\circ\gamma}(z); \eta \circ \gamma) = e^{i\varphi} \phi(z', F_{\eta\circ\gamma}(z); \gamma)$$

and by (1.2)

$$0 = \phi(z', F_\gamma(z'); \gamma) .$$

This gives by virtue of the Choquet theorem that

$$F_{\eta\circ\gamma}(z) = F_\gamma(z') . \tag{1.9}$$

From (1.7) we have

$$h_z \circ \eta(z') = e^{i\varphi} h_{z'}(z') = 0 .$$

Hence $\eta(z') = z$ and by (1.9)

$$F_{\eta\circ\gamma}(z) = F_\gamma \circ \eta^{-1}(z) .$$

Thus

$$F_{\eta\circ\gamma} = F_\gamma \circ \eta^{-1}$$

and this together with (1.6) implies (1.3).

Now we will show that $F_\gamma : \Delta \to \Delta$ is a sense-preserving local diffeomorphism. Let us fix $w \in \Delta$. We set $\gamma_w = h_w \circ \gamma \circ h^{-1}_{F_\gamma(w)} \in \mathrm{Aut}_{\mathbb{T}}$. By (1.3) we get

$$F_{\gamma_w}(0) = h_{F_\gamma(w)} \circ F_\gamma \circ h_w^{-1}(0) = 0 . \tag{1.10}$$

A simple calculation gives

$$\partial_u \phi(z, u; \gamma_w) = \frac{1}{4\pi} \partial_u \Big(\int_{\mathbb{T}} \frac{\gamma_w(\xi) - z}{1 - \bar{z}\gamma_w(\xi)} \Big(\frac{\xi + u}{\xi - u} + \frac{\bar{\xi} + \bar{u}}{\bar{\xi} - \bar{u}} \Big) |d\xi| \Big)$$

Hence

$$\partial_u \phi(0, 0; \gamma_w) = \frac{1}{2\pi} \int_{\mathbb{T}} \bar{\xi} \gamma_w(\xi) |d\xi| = a$$

and similarly

$$\partial_{\bar{u}} \phi(0, 0; \gamma_w) = \frac{1}{2\pi} \int_{\mathbb{T}} \xi \gamma_w(\xi) |d\xi| = b .$$

It has been shown (see [4], [1] and also [2]) that

$$|a|^2 - |b|^2 > 0 . \tag{1.11}$$

The implicit function theorem, (1.2) and (1.10) imply that there exists a neighbourhood U of 0 and exactly one continuously differentiable function $U \ni z \to u(z) \in \Delta$ such that $\phi(z, u(z); \gamma_w) = 0$, for $z \in U$ and $u(0) = 0$. From Choquet theorem, (1.2) and (1.10) it follows that $F_{\gamma_w}(z) = u(z)$ for $z \in U$. Thus the mapping F_{γ_w} is continuously differentiable in U and differentiating with respect z and $\bar{z}$ at the point $z = 0$ both sides of the equation

$$\phi(z, F_{\gamma_w}(z); \gamma_w) = 0$$

we obtain

$$a\partial_z(F_{\gamma_w})(0) + b\overline{\partial_{\bar{z}}(F_{\gamma_w})(0)} = 1 ,$$

$$a\partial_{\bar{z}}(F_{\gamma_w})(0) + b\overline{\partial_z(F_{\gamma_w})(0)} = -\frac{1}{2\pi}\int_{\mathbf{T}} \gamma^2(\xi)|d\xi| = c$$

whence

$$\partial_z(F_{\gamma_w})(0) = \frac{\bar{a} - \bar{c}b}{|a|^2 - |b|^2} \quad , \quad \partial_{\bar{z}}(F_{\gamma_w})(0) = \frac{\bar{a}c - b}{|a|^2 - |b|^2} . \tag{1.12}$$

If $\varphi \in \mathbf{R}$ satisfies $c = |c|e^{i\varphi}$ then $\mathrm{Re}\left(-e^{-i\varphi}\gamma_w^2(\xi)\right) \le 1$ for any $\xi \in \mathbf{T}$ and we have

$$|c| = \frac{1}{2\pi}\int_{\mathbf{T}} \mathrm{Re}\left(-e^{-i\varphi}\gamma_w^2(\xi)\right)|d\xi| \le \frac{1}{2\pi}\int_{\mathbf{T}} |d\xi| = 1 .$$

If $|c| = 1$ then $\gamma_w(\xi) = ie^{i\varphi/2}$ for every $\xi \in \mathbf{T}$, but this is impossible. Therefore $|c| < 1$ and from (1.11) and (1.12) it follows that the Jacobian of the mapping F_{γ_w} at $z = 0$ is positive, i.e.

$$|\partial_z(F_{\gamma_w})(0)|^2 - |\partial_{\bar{z}}(F_{\gamma_w})(0)|^2 = \frac{1 - |c|^2}{|a|^2 - |b|^2} > 0 . \tag{1.13}$$

By (1.3), (1.10) and (1.13) we see that the mapping $F_\gamma = h^{-1}_{F_\gamma(w)} \circ F_{\gamma_w} \circ h_w$ is a sense-preserving diffeomorphism in the neighbourhood $h_w^{-1}(U)$ of w. Furthermore, as proved earlier, the mapping F_γ has a continuous extension on the circle $\mathbf{T}$ to the automorphism $\gamma^{-1} \in \mathrm{Aut}_{\mathbf{T}}$. Applying the argument principle we state that F_γ is a diffeomorphism of Δ onto itself. In fact F_γ is real analytic because of regularity of the function $\Delta \times \Delta \ni (z, w) \to \phi(z, w; \gamma) \in \Delta$ and this ends the proof.

Corollary 1.2. *For any automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$ the mapping F_γ^{-1} is a real-analytic diffeomorphism of the unit disc Δ onto itself which is a continuous extension of γ on Δ and for any Möbius transformations $\eta_1, \eta_2 \in \mathcal{M}$ the following equality holds*

$$F^{-1}_{\eta_1\circ\gamma\circ\eta_2} = \eta_1 \circ F_\gamma^{-1} \circ \eta_2 .$$

Remark. As a metter of fact the mapping F_γ^{-1} coincides with the mapping $E(\gamma)$ found by Douady and Earle in [2]. In such a way we get an alternative proof of the Theorem 1 from [2].

2. Lemma 2.1. *If an automorphism $\gamma \in Aut_{\mathbf{T}}$ is normalized by the equality*

$$\frac{1}{2\pi}\int_{\mathbf{T}} \gamma(\xi)|d\xi| = \phi(0,0;\gamma) = 0 \tag{2.1}$$

then for every open arc $I \subset \mathbf{T}$ of length $|I| \leq \frac{2}{3}\pi$ we have

$$|\gamma(I)| \leq \tfrac{4}{3}\pi\,. \tag{2.2}$$

Proof. Let $I \subset \mathbf{T}$ be an arbitrary open arc of length $|I| = \frac{2}{3}\pi$. Without loss of generality we may assume that $-1 \in \mathbf{T} \setminus \gamma(I)$ and the arc $\gamma(I)$ is symmetric with respect to the real axis. Suppose that $|\gamma(I)| > \frac{4}{3}\pi$. Then

$$\left|\frac{1}{2\pi}\int_{\mathbf{T}\setminus I} \gamma(\xi)|d\xi|\right| \geq \frac{1}{2\pi}\int_{\mathbf{T}\setminus I} |\mathrm{Re}\,\gamma(\xi)|\,|d\xi| \geq -\left(1-\frac{|I|}{2\pi}\right)\cos\frac{|\gamma(I)|}{2} >$$

$$> \frac{1}{3} \geq \left|\frac{1}{2\pi}\int_I \gamma(\xi)|d\xi|\right|$$

and this contradicts (2.1). This proves the inequality (2.2).

Lemma 2.2. *If an automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$ normalized by (2.1) has a K-quasiconformal ($K-qc$) extension φ on the unit disc Δ, $1 \leq K < \infty$, then*

$$|\varphi(0)| \leq \delta(K) = \frac{1}{2} + \frac{\sqrt{3}}{2}\cot\left(\frac{\pi}{3} + \arccos\Phi_K\left(\frac{\sqrt{3}}{2}\right)\right) \tag{2.3}$$

where $\Phi_K = \mu^{-1}\left(\frac{1}{K}\mu\right)$ and $\mu(r)$, $0 < r < 1$, is the module of the ring domain $\Delta \setminus [0,r]$, (see [7]).

Proof. Let $\gamma \in \mathrm{Aut}_{\mathbf{T}}$ be an arbitrary automorphism satisfying the assumption of the lemma. Without loss of generality we may assume that $\varphi(0) = -a$, where $0 \leq a < 1$. By the Darboux principle there exists an open arc $I \subset \mathbf{T}$ of length $|I| = \frac{2}{3}\pi$ such that the arc $\gamma(I)$ is symmetric with respect to the real axis and contains the point 1. Denoting by $\omega(z, I)$ the harmonic measure in Δ we have

$$\frac{1}{K}\,\mu(\cos\tfrac{\pi}{2}\omega(0,I)) \leq \mu(\cos\tfrac{\pi}{2}\omega(\varphi(0),\gamma(I))) \tag{2.4}$$

because of the quasi-invariance of the harmonic measure, (see [3]). Putting in (2.4) $|I| = \frac{2}{3}\pi$ we get

$$\cos\frac{\pi}{2}\omega(\varphi(0),\gamma(I)) \leq \Phi_K\left(\frac{\sqrt{3}}{2}\right). \tag{2.5}$$

Since

$$\omega(-a,\gamma(I)) = \omega(0,h_{-a}(\gamma(I))) = \frac{|h_{-a}(\gamma(I))|}{2\pi}$$

then applying (2.5) we obtain

(2.6) $$|h_{-a}(\gamma(I))| \geq 4\arccos \Phi_K\left(\frac{\sqrt{3}}{2}\right).$$

From (2.2) it follows that for $a > \frac{1}{2}$

$$|h_{-a}(\gamma(I))| \leq 2\arg\left(\frac{e^{i\frac{2}{3}\pi}+a}{1+ae^{i\frac{2}{3}\pi}}\right) = -\tfrac{4}{3}\pi + \arctan\frac{\sqrt{3}}{2a-1}$$

and the desired formula (2.3) follows in view of (2.6).

Definition 2.3. A sense-preserving automorphism γ of $\mathbf{T}$ is said to be a k-quasisymmetric (k-qs) automorphism of $\mathbf{T}$ if and only if for any pair $I_1, I_2 \subset \mathbf{T}$ of disjoint adjacent open subarc of $\mathbf{T}$ of equal lengths $|I_1| = |I_2|$ the inequality

(2.7) $$|\gamma(I_1)| \leq k|\gamma(I_2)|$$

holds, (see [5])

Theorem 2.4. *If an automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$ is k-qs, $1 \leq k < \infty$, then F_γ is K^*-qc mapping of Δ onto itself where the constant K^* depends only on k.*

Proof. To start with we shall show that for any k-qs automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$, $1 \leq k < \infty$ the following inequalities hold :

(2.8) $$|c| \leq \cos\frac{2\pi}{(1+k)^2},$$

(2.9) $$|a|^2 - |b|^2 \geq \frac{\sqrt{2}}{\pi^2}\left(\sin\frac{\pi}{1+k}\right)^3 \sin\frac{\pi}{(1+k)^2}.$$

For any points $z_1, z_2 \in \mathbf{T}$, $z_1 \neq z_2$, $I(z_1,z_2)$ stands for the subarc $\{z \in \mathbf{T} : \arg z_1 < \arg z < \arg z_2\}$ of $\mathbf{T}$. Let us fix $\xi \in \mathbf{T}$ and let $\alpha_l = |\gamma(I(i^{l-1}\xi, i^l\xi))|$, $l = 1,2,\ldots$. By (2.7) we have for any $l = 1,2,3,4$,

$$\alpha_l + \alpha_{l+1} \leq \frac{2\pi k}{1+k} \quad \text{and} \quad \alpha_l \geq \frac{2\pi}{(1+k)^2}.$$

Consequently there exists l such that

$$\frac{2\pi}{(1+k)^2} \leq \alpha_l \leq \alpha_{l+2} \leq \frac{\pi k}{1+k}.$$

Therefore

$$\left|\sum_{n=1}^{4} \gamma^2(i^n\xi)\right| \leq 2|\cos\alpha_l| + 2|\cos\alpha_{l+2}| \leq 4\cos\frac{2\pi}{(1+k)^2}$$

for every $\xi \in \mathbf{T}$ and this leads to

$$|c| = \frac{1}{2\pi}\Big|\int_{\mathbf{T}} \gamma^2(\xi)\,|d\xi|\Big| = \frac{1}{2\pi}\Big|\int_{I(1,i)} \sum_{n=1}^{4} \gamma^2(i^n\xi)\,|d\xi|\Big| \le \cos\frac{2\pi}{(1+k)^2} .$$

Now, for any $t \in \mathbf{R}$ and $u \in [0,\pi]$ we define

$$\beta_1(t,u) = \big|\gamma\big(I(e^{it}, e^{i(t+u)})\big)\big| ,$$
$$\beta_2(t,u) = \big|\gamma\big(I(e^{i(t+u)}, -e^{it})\big)\big| ,$$
$$\beta_3(t,u) = \big|\gamma\big(I(-e^{it}, -e^{i(t+u)})\big)\big| ,$$
$$\beta_4(t,u) = \big|\gamma\big(I(-e^{i(t+u)}, e^{it})\big)\big| .$$

By (2.7) we have

$$\frac{2\pi}{1+k} \le \beta_l(t+u) + \beta_{l+1}(t+u) \le \frac{2\pi k}{1+k} , \quad l = 1, 2$$

and hence

$$(2.10) \qquad \sum_{n=1}^{4} \sin\beta_n(t,u) =$$
$$4\sin\frac{\beta_1(t,u)+\beta_2(t,u)}{2}\,\sin\frac{\beta_2(t,u)+\beta_3(t,u)}{2}\,\sin\frac{\beta_1(t,u)\beta_3(t,u)}{2} \ge$$
$$\ge 4\Big(\sin\frac{\pi}{1+k}\Big)^2 \sin\frac{\beta_1(t,u)+\beta_3(t,u)}{2} \ge 0 .$$

Applying again the inequality (2.7) we obtain for any $t \in \mathbf{R}$ and $u \in [\frac{\pi}{4}, \frac{3}{4}\pi]$ the following inequalities

$$\beta_1(t,u) + \beta_3(t,u) \ge \beta_1(t,\tfrac{\pi}{4}) + \beta_3(t,\tfrac{\pi}{4}) \ge \frac{1}{1+k}\big[\beta_1(t,\tfrac{\pi}{2}) + \beta_3(t,\tfrac{\pi}{2})\big] \ge \frac{2\pi}{(1+k)^2}$$

and similarly

$$\beta_2(t,u) + \beta_4(t,u) \ge \frac{2\pi}{(1+k)^2} .$$

Therefore

$$(2.11) \qquad \frac{2\pi}{(1+k)^2} \le \beta_1(t,u) + \beta_3(t,u) \le 2\pi - \frac{2\pi}{(1+k)^2} .$$

As shown in [1]

$$|a|^2 - |b|^2 = \Big(\frac{1}{2\pi}\Big)^2 \int_0^{\pi} \Big(\sin u \int_0^{2\pi} \sum_{n=1}^{4} \sin\beta_n(t,u)\,dt\Big)\,du .$$

Hence by (2.10), (2.11) we get the estimate (2.9). Let us consider any k–qs automorphism $\gamma \in \mathrm{Aut}_{\mathbf{T}}$, $1 \leq k < \infty$. It follows from the Theorem 1.1 that F_γ is a sense–preserving diffeomorphism of Δ onto itself. We shall estimate its complex dilatation in Δ. Setting $\gamma_w = h_w \circ \gamma \circ h^{-1}_{F_\gamma(w)} \in \mathrm{Aut}_{\mathbf{T}}$ for any $w \in \Delta$ we have by (1.3)

$$F_{\gamma_w} = h_{F_\gamma(w)} \circ F_\gamma \circ h_w^{-1}$$

and hence

$$\left|\frac{\partial_{\bar z} F_{\gamma_w}(0)}{\partial_z F_{\gamma_w}(0)}\right| = \left|\frac{\partial_{\bar z} F_\gamma(w)}{\partial_z F_\gamma(w)}\right| . \tag{2.12}$$

It follows from the Theorem 2 [5] that the automorphism γ admits K–qc extension φ on Δ and the constant K depends on k only. Then the mapping $\varphi_w = h_w \circ \varphi \circ h^{-1}_{F_\gamma(w)}$ is a K–qc extension of γ_w on Δ and

$$\frac{1}{2\pi}\int\limits_{\mathbf{T}} \gamma_w(\xi)\,|d\xi| = \phi(0,0;\gamma_w) = 0 ,$$

in view of $F_{\gamma_w}(0) = 0$ and (1.2). Thus by the Lemma 2.2 we get

$$|\varphi_w(0)| \leq \delta(K) . \tag{2.13}$$

Since $h_{\varphi_w(0)} \circ \varphi_w$ is K–qc mapping of Δ onto itself such that $h_{\varphi_w(0)} \circ \varphi_w(0) = 0$ then by virtue of the Theorem 1 [5] we obtain that $h_{\varphi_w(0)} \circ \gamma_w$ is the $\lambda(K)$–qs automorphism of $\mathbf{T}$ where

$$\lambda(K) = \left[\mu^{-1}\left(\frac{\pi K}{2}\right)\right]^{-2} - 1$$

is the distortion function [7]. With regard to (2.13) we derive that γ_w is the k_w–qs automorphism of $\mathbf{T}$ where

$$k_w = \left(\frac{1+|\varphi_w(0)|}{1-|\varphi_w(0)|}\right)^2 \lambda(K) \leq \left(\frac{1+\delta(K)}{1-\delta(K)}\right)^2 \lambda(K) .$$

Hence by (1.12), (1.13), (2.3), (2.8), (2.9) and (2.12) we get for any $w \in \Delta$

$$\begin{aligned}
1 - \left|\frac{\partial_{\bar z} F_\gamma(w)}{\partial_z F_\gamma(w)}\right|^2 &= \frac{(1-|c|^2)(|a|^2-|b|^2)}{|\bar a - \bar c b|^2} \geq \\
&\geq \frac{\sqrt{2}}{4\pi^2}\left(\sin\frac{2\pi}{(1+k_w)^2}\right)^2\left(\sin\frac{\pi}{1+k_w}\right)^2 \sin\frac{\pi}{(1+k_w)^2} \geq \\
&\geq \frac{\sqrt{2}}{4\pi^2}\,\frac{4^2}{(1+k_w)^4}\,\frac{2^2}{(1+k_w)^2}\,\frac{2\sqrt{2}}{(1+k_w)^2} = \frac{64}{\pi^2(1+k_w)^8} \geq \\
&\geq \frac{64}{\pi^2}\left(1 + \frac{3\lambda(K)}{\Phi_K^{-2}(\frac{\sqrt{2}}{2}) - 1}\right)^{-8}
\end{aligned}$$

because of

$$\left(\frac{1+\delta(K)}{1-\delta(K)}\right)^2 = 3\left(\Phi_K^{-2}\left(\frac{\sqrt{3}}{2}\right)-1\right)^{-1}.$$

Thus

$$\frac{|\partial_z F_\gamma(w)| + |\partial_{\overline{z}} F_\gamma(w)|}{|\partial_z F_\gamma(w)| - |\partial_{\overline{z}} F_\gamma(w)|} \le 2\left(2\frac{\pi^2}{64}\left(1+\frac{3\lambda(K)}{\Phi_K^{-2}(\frac{\sqrt{3}}{2})-1}\right)^8 - 1\right) =$$

$$= \frac{\pi^2}{16}\left(1+3\lambda(K)\left(\Phi_K^{-2}\left(\frac{\sqrt{3}}{2}\right)-1\right)^{-1}\right)^8 - 2 = K^*$$

for every $w \in \Delta$ so F_γ is the K^*–qc mapping of Δ onto itself. Following the proof of the Theorem 2 [5] and applying the estimate from [6] we get $K \le \min\{k^{3/2}, 2k-1\}$ and this means that the constant K^* depends only on k. This way we are done.

Remark. It follows from the proof of the above theorem and Corollary 1.2 that the mapping F_γ^{-1} is a K^*–qc extension of the automorphism $\gamma \in \mathrm{Aut}_{\mathbb{T}}$ on Δ if and only if γ admits a K–qc extension on Δ. This way we get an alternative proof of the Theorem 2 from [2].

REFERENCES

[1] Choquet, G., *Sur un type de transformation analytique generalisant la representation conforme et definie au moyen de fonctions harmoniques*, Bull. Sci. Math. (2) 69 (1945), 156–165.

[2] Douady, A., Earle, C. J., *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 (1986), 23–48.

[3] Hersch, J., *Longueurs extremales mesure harmonique et distance hyperbolique*, C. R. Acad. Sci. Paris 235 (1952), 569–571.

[4] Kneser, H., *Loesung der Aufgabe 41*, Jahresber. Deutsch. Math. Verein. 35 (1926), 123–124.

[5] Krzyż, J. G., *Quasicircles and harmonic measure*, Ann. Acad. Sci. Fenn. Vol. 12 (1987), 19–24.

[6] Lehtinen, M., *Remarks on the maximal dilatation on the Beurling–Ahlfors extension*, Ann. Acad. Sci. Fenn. Vol. 9 (1984), 133–139.

[7] Lehto, O., Virtanen, K. I., *Quasiconformal Mappings in the Plane*, Springer–Verlag, Berlin–Heidelberg–New York, 1973.

STRESZCZENIE

W pracy podany jest nowy, prosty dowód twierdzeń 1 i 2 uzyskanych przez A. Douady i C. J. Earle w pracy [2], dotyczących homeomorficznych rozszerzeń automorfizmów okręgu jednostkowego $\mathbb{T}$. Stosując wynik J. Krzyża [5] badamy te rozszerzenia w przypadku quasisymetrycznych automorfizmów okręgu $\mathbb{T}$.