ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLII, SECTIO A 1988

Department of Mathematics
University of South Florida, Tampa

A. G. KARTSATOS

Applications of Nonlinear Perturbation Theory to the Existence of Methods of Lines for Functional Evolutions in Reflexive Banach Spaces

Wykorzystanie nieliniowej teorii perturbacji do możliwości stosowania metody łamanych dla równań ewolucji w refleksywnych przestrzeniach Banacha

Abstract. Nonlinear perturbation theory is applied to the existence of a method of lines associated with the functional evolution problem :

$$(*)\qquad \begin{aligned} &x' + A(t)x = G(t, x_t)\,, \quad t \in [0, T]\,, \\ &x_0 = \Phi\,. \end{aligned}$$

The method satisfies an equation of the type :

$$(**)\qquad A(t_{n,j-1})z_{nj} + (z_{nj} - z_{n,j-1})/h = G(t_{n,j-1}, (\overline{z}_{nj})t_{n,j-1})\,.$$

The underlying space X is a real Banach space with uniformly convex dual space. The operators $A(t)u$ are mainly m-accretive in u while $G(t,f)$ is at least Lipschitz continuous in f. Here f lies in a suitable function space over the delay interval $[-r, 0]$.

Recent results are improved and/or extended. The results are new even in the ordinary case ($G(t, x_t) \equiv G(t, x(t))$) and can be effectively used in the numerical treatment of $(*)$. It is nowhere assumed that X is a $(\pi)_1$ space or that $G(t, f)$ can be extended to a global Lipschitzian with respect to f.

1. Introduction – preliminaries. In what follows, the symbol X denotes a real Banach space with norm $\|\cdot\|$ and dual space X^*. It is always assumed that X^* is uniformly convex. The duality mapping of X is denoted by J. This mapping maps X into X^*, it is positively homogeneous of degree 1 and such that

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2\,.$$

Here, $\langle x, f\rangle$ denotes the value of the functional $f \in X^*$ at $x \in X$. An operator $A : D(A) \subset X \to X$ is "accretive" if

$$\langle Ax - Ay, J(x-y)\rangle \geq 0\,, \quad x, y \in D(A)\,.$$

An accretive operator A is "m–accretive" if $R(A+\lambda I) = X$ for every $\lambda > 0$. For an m–accretive operator, the Yosida approximants $J_n : X \to D(A)$, $A_n : X \to X$ are defined by

$$J_n x = (I + (1/n)A)^{-1} x\ , \quad A_n x = A J_n x\ ,$$

where I denotes the identity operator. An accretive operator is called "strongly acretive" if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2\ , \quad x, y \in D(A)\ .$$

An operator $A : D(A) \subset X \to X$ is "compact" if it maps bounded subsets of its domain into relatively compact subsets of X. It is called "bounded" if it maps bounded subsets of its domain into bounded subsets of X. It is called "demicontinuous" if $\{x_n\} \subset D(A)$, $x_n \to x \in D(A)$ imply that $Ax_n \rightharpoonup Ax$. The symbol "$\to$" ("$\rightharpoonup$") denotes strong (weak) convergence and the symbols R, R_+ denote the real line and the half line $[0,\infty)$, respectively. A real Banach space X is called a "$(\pi)_1$ space" if there exists a sequence $\{P_n\}$ of linear projections, each of norm 1, with finite dimensional range $P_n X = X_n$ such that $P_n x \to x$ as $n \to \infty$ and each subspace X_n has dimension n and is contained in X_{n+1}.

We denote by $PC([-r,0],Y)$ the space of all piecewise continuous functions $f : [-r,0] \to Y$ associated with the sup–norm $\|\cdot\|_\infty$. For a function $u : [-r,T] \to X$ we set $u_t(s) = u(t+s)$, $t \in [0,T]$, $s \in [-r,0]$. In this paper we study the equation

$$(\ast)\qquad \begin{aligned} &x' + A(t)x = G(t, x_t)\ , \quad t \in [0,T]\ , \\ &x_0 = \Phi\ . \end{aligned}$$

We assume mainly that for each $t \in [0,T]$, $A(t)$ is a (possibly nonlinear) m–accreative operator with fixed domain $D \subset X$. We also assume that $G : [0,T] \times PC([-r,0],\overline{D_0}) \longrightarrow X$, where D_0 equals either the ball $B_{\bar r}(0) = \{x \in X\ ;\ \|x\| < \bar r\}$ or D, is Lipschitz continuous in its second variable :

$$\|G(t,\psi_1) - G(t,\psi_2)\| \leq b\|\psi_1 - \psi_2\|_\infty$$

for every $t \in [0,T]$, $\psi_1, \psi_2 \in PC([-r,0],\overline{D_0})$. The function Φ in $(\ast)$ is a given function in $C([-r,0],\overline{D_0})$.

Now, consider, for $n = 1,2,\ldots$, a partition $\{t_{nj}\}$ of the interval $[0,T]$ with $j = 0,1,2,\ldots,n$, $t_{n0} = 0$ and $t_{nn} = T$. Moreover, let $h = T/n$, $t_{nj} = jh$. Also, let $z_{n0} = \Phi(0) \in D$, and if χ_L denotes the characteristic function of the interval $L \subset R$, set

$$f_1(x)(t) = \chi_{[-r,0]}(t)\Phi(t) + \chi_{(0,T]}(t)x\ ,$$

$(t,x) \in [-r,T] \times X$, and

$$F_1(t)x = G\big(t, (f_1(x))_t\big)\ , \quad (t,x) \in [0,T] \times \overline{D_0}\ .$$

The function $x \to G\big(t,(f_1(x))_t\big)$ is a Lipschitzian on $\overline{D_0}$ with Lipschitz constant b. At this point, assume that, for each n, the equation

$$(1)\qquad (A(t_{n0}) - F_1(t_{n0}) + (1/h)I)x = z_{n0}/h$$

has a solution $z_{n1} \in \overline{D_0}$. Then we let

$$\bar{z}_{n1}(t) = \begin{cases} \Phi(t) & , t \in [-r, 0], \\ z_{n1} & , t \in (0, T]. \end{cases}$$

It should be noted that $\bar{z}_{n1}(t) \in \overline{D_0}$ for all $t \in [-r, T]$. Similarly, one defines for each n and each $j \in \{2, 3, \ldots, n\}$ the functions

$$f_j(x)(t) = \chi_{[-r, t_{n,j-1}]}(t)\bar{z}_{n,j-1}(\cdot) + \chi_{(t_{n,j-1}, T]}(t)x,$$
$$F_j(t)x = G(t, (f_j(x))_t),$$

respectively. The function $F_j(t)x$ is Lipschitz continuous with respect to x on $\overline{D_0}$ with Lipschitz constant b. Calling $z_{nj} \in \overline{D_0}$ a solution of the equation

$$(A(t_{n,j-1}) - F_j(t_{n,j-1}) + (1/h)I)x = z_{n,j-1}/h, \tag{2}$$

we have actually constructed the following double sequence :

$$\bar{z}_{nj}(t) = \begin{cases} \Phi(t) & , t \in [-r, 0], \\ z_{n1} & , t \in (0, t_{n1}], \\ \vdots & \\ z_{n,j-1} & , t \in (t_{n,j-2}, t_{n,j-1}], \\ z_{nj} & , t \in (t_{n,j-1}, T], \end{cases}$$

$n = 1, 2, \ldots,\ j = 1, 2, \ldots, n$. We call the sequence $\{\bar{z}_{nj}(t)\}$ or any such sequence valid for all large n, a "method of lines" for the problem $(*)$, and observe that, for all the indices n, j we have $\bar{z}_{nj}(t) \in \overline{D_0}$, $t \in [-r, T]$. Moreover,

$$A(t_{n,j-1})z_{nj} + (z_{nj} - z_{n,j-1})/h = G(t_{n,j-1}, (\bar{z}_{nj})_{t_{n,j-1}}).$$

If $\overline{D_0}$ is a convex set, then the Rothe functions

$$z^n(t) = \begin{cases} \Phi(t) & , t \in [-r, 0], \\ z_{n,j-1} + (t - t_{n,j-1})(z_{nj} - z_{n,j-1})/h & , t \in [t_{n,j-1}, t_{nj}] \end{cases}$$

are also lying inside $\overline{D_0}$. In fact, if $t \in [t_{n,j-1}, t_{nj}]$ we have

$$z^n(t) = [1 - (t - t_{n,j-1})/h]z_{n,j-1} + [(t - t_{n,j-1})/h]z_{nj},$$

which lies on the line segment joining the points $z_{n,j} \in \overline{D_0}$, $z_{n,j-1} \in \overline{D_0}$.

This method was used by the author and Parrott [8] in order to obtain solutions to problems of the type $(*)$ under the assumption that G is a global Lipschitzian in its second variable ($D_0 = X$). It was shown however in [7] that the method actually converges as in [8] if $D_0 = B_{\bar{r}}(0)$, for some $\bar{r}$, the space X is a $(\pi)_1$ space,

$$\langle A(t)x, Jx\rangle \geq a\|x\|^2, \quad t \in [0, T], \ x \in \partial B_{\bar{r}/2},$$

for some constant $a > 0$, $G(t, \bar{0}) \equiv 0$ ($\bar{0}$ is the zero function in $C([-r,0], B_{\bar{r}})$) and b (the Lipschitz constant of G with respect to its second variable) lies in $(0, a)$. Of course, we are only mentioning here the assumptions in [7], [8] on the second variable of G.

In this paper we show, among other results with $D_0 = B_{\bar{r}}(0)$, that Theorem 2 of [7] can actually be proven in spaces X that are not necessarily of $(\pi)_1$ type. We also show that under additional assumptions on A and/or G we may assume that $D_0 = D(A)$. The reader should have in mind that we are only assuming conditions on A, G that ensure the solvability of equations of the type (2), and thus guarantee the existence of a method of lines $\{z_{nj}(t)\}$. This method will converge if we assume, in addition, the rest of the conditions on A and the first variable of G in the various results of [7], [8]. We can also apply these considerations to the Kato approximants [6] and the Galerkin approximants [5].

These results can be also applied (and the methods become simpler) to ordinary evolutions where $G(t, x_t)$ is actually replaced by $G(t, x)$. For such results the reader is referred to Kartsatos and Zigler [10].

For the general theory of accretive operators, we refer the reader to the books of Barbu [1], Browder [2], Lakshmikantham and Leela [11] and Martin [12].

2. Results for the space $PC([-r,0], \overline{B_r(0)})$. The following conditions will be needed in the sequel.

(A.1) For each $t \in [0,T]$, $A(t)u$ is m-accretive in u with domain $D(A)$ independent of t.

(G.1) For each $t \in [0,T]$ the function $G : [0,T] \times PC([-r,0)], \overline{B_{\bar{r}}(0)} \to X$ satisfies the Lipschitz condition .

$$\|G(t, \psi_1) - G(t, \psi_2)\| \leq b \|\psi_1 - \psi_2\|_\infty$$

with Lipschitz constant $b > 0$ and Φ is a fixed function in $C([-r,0], \overline{B_{\bar{r}}(0)})$ with $\Phi(0) \in D(A)$.

(G.2) Condition (G.1) holds with $B_{\bar{r}}(0)$ replaced by $D(A)$.

Theorem 1. *Assume that conditions* (A.1), (G.1) *hold. Assume, further, that for every function* $\psi \in PC([-r,0], \overline{B_{\bar{r}}(0)})$ *and every* $x \in D(A)$ *with* $\|x\| > \bar{r}$ *we have*

$$\langle A(t)x - G(t, \psi), Jx \rangle \geq 0 \,, \quad t \in [0,T] \,. \tag{3}$$

Then there exists a method of lines for $(*)$.

Proof. Using the m-accretiveness of the operator $A(t_{n0})$, for a fixed n, we obtain that the equation (1) is equivalent to

$$x = [A(t_{n0}) + (1/h)I]^{-1} [F_1(t_{n0})x + z_{n0}/h] \,, \tag{4}$$

which can be written as $x = Sx$. We first note that $F_1(t_{n0})x$ is Lipschitzian in x with constant b on the set $\overline{B_{\bar{r}}(0)}$. We observe next that the mapping
$x \to [A(t_{n0}) + (1/h)I]^{-1} x$ is also Lipschitz continuous with constant h. Consequently,

$S : \overline{B_{\bar r}(0)} \to X$ with $\|Sx - Sy\| \le bh\|x - y\|$ for every $x, y \in \overline{B_{\bar r}(0)}$. In order to apply the contraction mapping principle, we assume that n is so large that $bh < 1$ and we show that S maps $\overline{B_{\bar r}(0)}$ into itself. To this end, let $x \in \overline{B_{\bar r}(0)}$ be given and let $u = Sx$. Then we have

$$A(t_{n0})u + (1/h)u - F_1(t_{n0})x - z_{n0}/h = 0 .$$

Recall that $z_{n0} = \Phi(0) \in \overline{B_{\bar r}(0)}$. Taking "inner products" above with u and assuming that $\|u\| > \bar r$, we have

$$0 = \langle A(t_{n0})u - F_1(t_{n0})x, Ju\rangle + (1/h)\langle u - z_{n0}, Ju\rangle \ge$$
$$\ge (1/h)[\langle u, Ju\rangle - \langle z_{n0}, Ju\rangle] \ge (1/h)[\, \|u\|^2 - \|z_{n0}\|\,\|u\|\,] \ge (1/h)(\|u\| - \bar r)\|u\| > 0 .$$

Here we have used the fact that $F_1(t_{n0})x = G(t_{n0}, (f_1(x))_{t_{n0}})$ with $\|(f_1(x))_{t_{n0}}\|_\infty \le \bar r$. It follows that $Sx \in \overline{B_{\bar r}(0)}$. The rest of the proof is a repetition of the above argument. It is therefore omitted.

The following homotopy result can be found in the author's paper [4].

Lemma A. *Let $U \subset X$ be open and let $H : [0,1] \times \overline{U} \to X$ be such that*
i) *For each $t \in [0,1]$, $H(t,\cdot)$ is demicontinuous and strongly accretive ;*
ii) *$H(t,x)$ is continuous in t uniformly in $x \in \overline{U}$;*
iii) *$H(t,x) \ne 0$ for $t \in (0,1)$, $x \in \partial U$;*
iv) *There exists $x_0 \in U$ such that $\|H(0,x_0)\| \le \|H(0,x)\|$ for every $x \in \partial U$.*
Then $H(1,\cdot)$ has a unique zero in $\overline{U}$.

In order to apply the above theorem to the present setting, we need the following definition : a set $U \subset X$ is called "absorbing" if $u \in U$ implies $tu \in U$ for every $t \in (0,1)$.

Theorem 2. *Let the conditions* (A.1), (G.1) *be satisfied and assume that $D(A)$ is absorbing. Then if* (3) *holds for every $x \in D(A)$ with $\|x\| = \bar r$, there exists a method of lines for the equation* (*).

Proof. We let $U = B_{\bar r}(0)$ in Lemma A and $H(t,x) = x - tSx$, where S is as in the proof of Theorem 1. It is easy to see that

$$\langle H(t,x) - H(t,y), J(x-y)\rangle \ge (1 - bh)\|x - y\|^2$$

for all $x, y \in \overline{B_{\bar r}(0)}$, where n is chosen so large that $bh < 1$. Thus i) of Lemma A is satisfied. Now, let $t_0, t_1 \in [0,1]$ be given. Also, let K be a bound for the mapping S on the ball $\overline{B_{\bar r}(0)}$. This bound exists by the fact that S is Lipschitz continuous on $\overline{B_{\bar r}(0)}$. Then we have

$$\|H(t_1,x) - H(t_0,x)\| \le K|t_1 - t_0| , \quad x \in \overline{B_{\bar r}(0)} ,$$

which shows that ii) of Lemma A holds. Since $H(0,0) = 0$ and $H(0,x) = x$, condition iv) of Lemma A holds with $x_0 = 0$. To show iii) of that lemma, assume that $H(t, x_t) = 0$ for some $t \in (0,1)$, $x_t \in \partial B_{\bar r}(0)$. Then we have

$$A(t_{n0})(x_t/t) + (1/h)(x_t/t) = F_1(t_{n0})x_t + z_{n0}/h ,$$

which, evaluating Jx_t, implies

$$(5)\quad 0 = \langle A(t_{n0})(x_t/t), Jx_t\rangle + (1/(ht))\|x_t\|^2 - \langle F_1(t_{n0})x_t, Jx_t\rangle - \langle z_{n0}/h, Jx_t\rangle > \\ > \langle A(t_{n0})(x_t/t), Jx_t\rangle - \langle F_1(t_{n0})x_t, Jx_t\rangle$$

because

$$(1/(ht))\|x_t\|^2 - \langle z_{n0}/h, Jx_t\rangle \geq (1/(ht))\|x_t\|^2 - (1/h)\|z_{n0}\|\,\|x_t\| \geq \\ \geq (1/(ht))\|x_t\|^2 - (1/h)\|x_t\|^2 > 0\,.$$

Since $x_t/t \in D(A)$ and $D(A)$ is absorbing, $t(x_t/t) = x_t \in D(A)$. Thus,

$$\langle A(t_{n0})(x_t/t), Jx_t\rangle - \langle F_1(t_{n0})x_t, Jx_t\rangle = \\ = [t/(1-t)]\langle A(t_{n0})(x_t/t) - A(t_{n0})x_t, J([(1-t)/t]x_t)\rangle + \\ + \langle A(t_{n0})x_t - F_1(t_{n0})x_t, Jx_t\rangle \geq 0\,,$$

i.e., a contradiction of (5). Consequently, $H(1,x) = x - Sx = 0$ for some $x \in \overline{B_{\bar r}(0)}$. The rest of the proof, which follows as above, is omitted.

If the operator $A(t)$ is demicontinuous for each $t \in [0,T]$ and bounded on $\overline{B_{\bar r}(0)}$, then we can actually work with the original equation (2). This is the content of Theorem 3 below.

Theorem 3. *Assume that* (G.1) *is satisfied with* $A(t)u$ *just accretive in* u *for all* $t \in [0,T]$. *Further, assume that* $D(A) = D(A(t))$ *is independent of* t, $\overline{B_{\bar r}(0)} \subset D(A)$ *and* (3) *holds for* $\|x\| = \bar r$. *Then if* $A(t)$ *is bounded and demicontinuous on* $\overline{B_{\bar r}(0)}$ *for each* $t \in [0,1]$, *there exists a method of lines for* $(*)$.

Proof. This time we look at the homotopy $H(t,x) = (1-t)x + tSx$, where $Sx = (A(t_{n0}) - F_1(t_{n0}) + (1/h)I)x - z_{n0}/h$. Assuming again that n is so large that $bh < 1$, we see easily that

$$\langle H(t,x) - H(t,y), J(x-y)\rangle \geq (1-t)\|x-y\|^2 + t((1/h) - b)\|x-y\|^2$$

for every $t \in [0,1]$ and every $x, y \in \overline{B_{\bar r}(0)}$. Thus $H(t,x)$ is coercive in x on $\overline{B_{\bar r}(0)}$. The rest of the proof follows as in the Theorem 2 except the argument concerning x_t/t which is not needed here.

This theorem is of particular importance in view of the fact that one may consider the Yosida approximants $A_m(t_{n,j-1}) = A(t_{n,j-1})(I + (1/m)A(t_{n,j-1}))^{-1}$, instead of $A(t_{n,j-1})$ itself, in the equation (2), and then obtain a solution of (2) via a limiting process. This was done in [7, Th.2] under the assumption that X is a $(\pi)_1$ space. The following result eliminates that assumption.

Theorem 4. *Let the conditions* (A.1), (G.1) *be satisfied. Assume also that there exists a constant* $a > 0$ *such that* $\langle A(t)x, Jx\rangle \geq a\|x\|^2$ *for all* $x \in D(A) \cap \overline{B_{\bar r}(0)}$, $t \in [0,T]$. *Let the Lipschitz constant* b *of* G *lie in* $(0,a)$ *and* $G(t,\bar 0) \equiv 0$, *where* $\bar 0$

denotes the zero function in $PC([-r,0],\overline{B_{\bar r}(0)})$. *Then* (*) *possesses a method of lines provided that* $0 \in D(A)$ *and* $A(t)0 \equiv 0$.

Proof. It was shown in [7, Th.2] that given $\bar b \in (b,a)$ and m sufficiently large and independent of t, we have

$$(6) \qquad \langle A_m(t)u - G(t,\psi), Ju\rangle > 0\,, \quad t \in [0,T]\,,$$

for all $u \in \partial B_{\bar r}(0)$, $\psi \in PC([-r,0],\overline{B_{\bar r}(0)})$. We sketch this proof here because $\partial B_{\bar r}(0)$ should be replaced in [7, Th.2] by $B_{\bar r}(0)$ and the assumption on $A(t)0$ was omitted there. Whenever $x \in D(A) \cap B_{\bar r}(0)$ we have

$$\langle (I + (1/m)A(t))x, Jx\rangle \geq [1 + (a/m)]\|x\|^2\,.$$

If we set $u = (I + (1/m)A(t))x$, we can see easily that

$$\|J_m(t)u\| \leq [1 + (a/m)]^{-1}\|u\| \qquad (J_m(t) = [I + (1/m)A(t)]^{-1})\,.$$

Since the operator $I + (1/m)A(t)$ is surjective, given $u \in \partial B_{\bar r}(0)$ there exists $x \in D(A)$ with $x + (1/m)A(t)x = u$. Taking the values of Jx on this equality we obtain that $\|x\| \leq \|u\|$. Thus $x \in \overline{B_{\bar r}(0)}$. It follows that (6) holds for every $u \in \partial B_{\bar r}(0)$. Following the proof in the above reference, we obtain now that

$$\langle A_m(t)u, Ju\rangle \geq m[1 - (1 + (a/m))^{-1}]\|u\|^2\,,$$

which, for $\bar b \in (b,a)$, $u \in \partial B_{\bar r}(0)$ and all large m, gives

$$(7) \qquad \langle A_m(t)u - G(t,\psi), Ju\rangle \geq \bar b r^2 - b r^2 > 0\,, \quad t \in [0,T]\,, \quad \psi \in PC([-r,0],\overline{B_{\bar r}(0)})$$

We now consider the equation

$$(8) \qquad (A_m(t_{n0}) - F_1(t_{n0}) + (1/h)I)x - z_{n0}/h = 0$$

for all m for which (7) holds and for n large enough so that $bh < 1$. Since, for each such m, the operator $A_m(t_{n0})$ is continuous and bounded on $\overline{B_{\bar r}(0)}$, an application of Lemma A to the mapping $H(t,x) = (1-t)x + tSx$ with Sx equal to the first term of (8) shows the existence of a solution x_m of (8) for all large m.

At this point the proof follows as the corresponding part of the proof of Theorem 3.6 of Barbu [1]. We let $A_m = A_m(t_{n0})$, $F = (1/h)I - F_1(t_{n0})$, $y = z_{n0}/h$. We have $A_m x_m + F x_m = y$. We note that A_m is m–accretive and F is coercive with constant $c = (1 - bh)/h$. We have

$$\begin{aligned}
0 &= \langle A_m x_m - A_q x_q, J(x_m - x_q)\rangle + \langle F x_m - F x_q, J(x_m - x_q)\rangle \geq \\
&\geq \langle A_m x_m - A_q x_q, J(x_m - x_q)\rangle + c\|x_m - x_q\|^2 = \\
&= \langle A_m x_m - A_q x_q, J(J_m x_m - J_q x_q)\rangle + \\
&+ \langle A_m x_m - A_q x_q, J(x_m - x_q) - J(J_m x_m - J_q x_q)\rangle + c\|x_m - x_q\|^2\,,
\end{aligned}$$

where $J_m = (I + (1/m)A(t_{n0}))^{-1}$. This implies

$$\|x_m - x_q\|^2 \leq -(1/c)\langle A_m x_m - A_q x_q, J(x_m - x_q) - J(J_m x_m - J_q x_q)\rangle \leq$$
$$\leq (1/c)\|A_m x_m - A_q x_q\| \, \|J(x_m - x_q) - J(J_m x_m - J_q x_q)\| .$$

Since the boundedness of $\{\|Fx_m\|\}$ implies the boundedness of $\{\|A_m x_m\|\}$ and $\|J_m x_m - x_m\| \leq (1/m)\|A_m x_m\|$, using the uniform continuity of J on bounded subsets of X we get $x_m \to x \in \overline{B_r(0)}$ as $m \to \infty$. Since $A_m x_m = y - Fx_m \to y - Fx$, Proposition 3.4 of Barbu [1] implies that $x \in D(A)$ and $Ax = y - Fx$. We have thus solved (1) with the solution lying in $D(A) \cap \overline{B_r(0)}$. The same argument applies to (2). It is therefore omitted.

3. Results for the space $PC([-r,0], \overline{D(A)})$. It is well known that if X is also uniformly convex, then $\overline{D(A)}$ is a convex set. In addition, a result of Reich [13] says that $\overline{D(A)}$ admits a nonexpansive retraction, i.e., a mapping $P : X \xrightarrow[\text{onto}]{} \overline{D(A)}$ such that $P^2 = P$ and $\|Px - Py\| \leq \|x - y\|$, $x \in X$. This result can be effectively used in problems where the perturbations are defined only on $\overline{D(A)}$.

Theorem 5. *Let conditions* (A.1), *and* (G.2) *be satisfied. Further, assume that there exist constants* $d_1 > 0$, $d_2 > 0$ *such that*

$$\langle A(t)x, Jx\rangle \geq -d_1\|x\|$$

for all $t \in [0,T]$ *and all* $x \in D(A)$ *with* $\|x\| \geq d_2$.

Moreover, let the constant $K > 0$ *be such that*

$$\|G(t,\psi)\| \leq K$$

for all $t \in [0,T]$ *and* $\psi \in PC([-r,0], \overline{D(A)})$. *Then there exists a method of lines for the equation* (*).

Proof. This time we write (2) as follows :

$$(9) \qquad u - F_1(t_{n0})[A(t_{n0}) + (1/h)I]^{-1}u - z_{n0}/h = 0 ,$$

where $u = (A(t_{n0}) + (1/h)I)x$, or, equivalently, $u - Su = 0$ for the obvious operator $S : X \to X$. It is easy to see that S is Lipschitz continuous on X with constant bh. We assume again that n is so large that $bh < 1$, and consider only such n's. In order to apply Lemma A, with $H(t,u) = u - tSu$, we show that conditions ii) and iii) are satisfied for some open set U. As far as condition ii) is concerned, we observe that

$$\|Su\| \leq \|S0\| + bh\|u\| \leq \|S0\| + \|u\|$$

implies the uniform continuity of $H(t,u)$ in t with respect to u lying in any bounded subset of X.

Now, we are going to prove that all possible solutions u_t of $H(t,u) = 0$, $t \in (0,1)$, lie inside a ball which is independent of t. In fact, let $t_m \in (0,1)$ be such that $\|u_{t_m}\| \to +\infty$ as $n \to \infty$. Set $u_m = u_{t_m}$. Then we have

$$(10) \qquad A(t_{n0})x_m + (1/h)x_m - t_m F_1(t_{n0})x_m = t_m z_{n0}/h ,$$

where $u_m = A(t_{n0})x_m + (1/h)x_m$. If $\{\|x_m\|\}$ has a bounded subsequence, then (10) implies that $A(t_{n0})x_m$ has a bounded subsequence, which implies in turn that $\{\|u_m\|\}$ has a bounded subsequence, i.e., a contradiction. It follows that $\|x_m\| \to \infty$ as $m \to \infty$. Thus,

$$\begin{aligned}0 &= \langle A(t_{n0})x_m, Jx_m\rangle + (1/h)\|x_m\|^2 - t_m\langle F_1(t_{n0})x_m, Jx_m\rangle - t_m\langle z_{n0}/h, Jx_m\rangle \geq \\ &\geq -d_1\|x_m\| + (1/h)\|x_m\|^2 - K\|x_m\| - \|z_{n0}\|\,\|x_m\|/h = \\ &= [\,(1/h)\|x_m\| - d_1 - K - \|z_{n0}\|/h\,]\,\|x_m\| \to \infty \qquad \text{as } m \to \infty\,,\end{aligned}$$

i.e., a contradiction. If we let $\|u_t\| \leq \bar{r}$ for all possible solutions u_t of $H(t,u) = 0$, then we may take $U = B_{\bar{r}_1}(0)$, for any $\bar{r}_1 > \bar{r}$, and the conditions ii), iii) of Lemma A are satisfied. Conditions i), iv) follow easily. This completes the proof.

Theorem 6. *Let X be uniformly convex. Let conditions* (A.1), (G.2) *hold. Let $0 \in D(A)$ with $A(t)0 = 0$, $t \in [0,T]$. Let $\Phi(s) \in \overline{D(A)} \cap \overline{B_{\bar{r}}(0)}$, $s \in [-r, 0]$. Assume that for every $x \in \partial B_{\bar{r}}(0) \cap D(A)$, every $\lambda \in (0,1)$ and every $\psi \in PC([-r,0], \overline{D(A)} \cap \overline{B_{\bar{r}}(0)})$ we have*

$$\langle A(t)x - \lambda G(t,\psi), Jx\rangle \geq 0\,. \tag{11}$$

Then $(*)$ *possesses a method of lines.*

Proof. Since X is also uniformly convex there exists a nonexpansive retraction P on $\overline{D(A)}$. Since $0 \in \overline{D(A)}$, $P0 = 0$. Thus, for every $x \in X$, $\|Px\| \leq \|x\|$. We now consider the homotopy

$$H(t,x) \equiv x - [A(t_{n0}) + (1/h)I]^{-1}[t(F_1(t_{n0})Px + z_{n0}/h]\,.$$

The equation $H(1,x) = 0$ is equivalent to (2). In fact, if $H(1,x) = 0$, then $x \in D(A)$ and $Px = x$. In order to apply Lemma A to this problem, we notice again that the term in $H(t,x)$ after the minus sign is a Lipschitz continuous function on X with Lipschitz constant bh. We assume again that n is so large that $bh < 1$. In fact,

$$\|F_1(t_{n0})Px - F_1(t_{n0})Py\| \leq b\|Px - Py\| \leq b\|x - y\|$$

for all $x, y \in X$. Now, let $x \in \partial B_{\bar{r}}(0)$ be given. Since $\|Px\| \leq \|x\|$, we have that $\|Px\| \leq \bar{r}$. This implies that the entire function

$$f_1(Px)(t) = \chi_{[-r,0]}(t)\Phi(t) + \chi_{(0,T]}(t)Px\,,$$

$(t,x) \in [-r,T] \times \partial B_{\bar{r}}(0)$, lies in the closed convex set $\overline{D(A)} \cap \overline{B_{\bar{r}}(0)}$. Thus $(f_1(Px))_t \in PC([-r,0], \overline{D(A)} \cap \overline{B_{\bar{r}}(0)})$ and if x is a solution of the equation $H(t,x) = 0$ lying in $\partial B_{\bar{r}}(0)$ for some $t \in (0,1)$, then we have

$$\begin{aligned}0 &= \langle A(t_{n0})x - tG(t_{n0}, (f_1(Px))_{t_{n0}}), Jx\rangle + (1/h)\langle x, Jx\rangle - (t/h)\langle z_{n0}, Jx\rangle \geq \\ &\geq (1/h)[\,\|x\|^2 - t\|z_{n0}\|\,\|x\|\,] \geq (1/h)(1-t)\bar{r}^2 > 0\,.\end{aligned}$$

It is easy to see now that all the conditions i), iv) of Lemma A are satisfied with $U = B_{\overline{r}}(0)$. This completes the proof.

Evidently, separate, and independent of $\lambda \in (0,1)$, conditions can be imposed on $A(t)$, $G(t,\psi)$ for (11) to hold.

4. General comments. Analogous results do hold in general Banach spaces whenever $A(t)$ has compact resolvents $(A(t)+(1/h)I)^{-1}$ for all $t \in [0,T]$. However, as far as the author knows, it has not been shown whether the method of lines constructed here actually converges to the solution of (•) if X^* is not uniformly convex and $G(t,\psi)$ does depend on $\psi \in PC([-r,0],\overline{D_0})$ nontrivially.

One can also use the present results to show that Theorem 3 of [7] can be proved without assuming that $G(t,\psi)$ is extendable to a global Lipschitzian on $[0,T] \times C$, where $C = C([-r,0],X)$. One can assume instead conditions like the ones of Section 2 above. The results of that section are directly applicable in this setting.

Theorem 2 replaces the assumption that X is a $(\pi)_1$ space in Theorem 1 of [7] by the boundedness of $A(t)u$ with respect to u on $\overline{B_{\overline{r}}(0)}$. Since $A(t)u$ is demicontinuous (and thus locally bounded) in u, this boundedness assumption is certainly a natural one.

REFERENCES

[1] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff Int. Publ., Leyden 1976.

[2] Browder, F.E., *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces*, Proc. Symp. Pure Math., 18, Amer. Soc. Providence, Rhode Island 1976.

[3] Kartsatos, A.G., *The existence of a method of lines for evolution equations involving maximal monotone operators and locally defined Lipschitzian perturbation*, to appear.

[4] Kartsatos, A.G., *Zeros of demicontinuous accretive operators in Banach spaces*, J. Integral Equations, 8 (1985), 175–184.

[5] Kartsatos, A.G., Parrot, M.E., *Existence of solutions and Galerkin approximations for nonlinear functional evolution equations*, Tohoku Math. J., 34 (1982), 509–523.

[6] Kartsatos, A.G., Parrot, M.E., *Convergence of the Kato approximants for evolution equations involving functional perturbations*, J. Differential Equations, 47 (1983), 358–377.

[7] Kartsatos, A.G., Parrot, M.E., *Global solutions of functional evolution equations involving locally defined Lipschitzian perturbations*, J. London Math. Soc., 27 (1983), 306–316.

[8] Kartsatos, A.G., Parrot, M.E., *A method of lines for a nonlinear abstract functional evolution equation*, Trans. Amer. Math. Soc., 286 (1984), 73–89.

[9] Kartsatos, A.G., Toro, J., *Passivity and admissibility for evolution equations in Banach spaces*, Nonlinear Anal., 6 (1982), 225–236.

[10] Kartsatos, A.G., Zigler, W., *Rothe's method and weak solutions of perturbed evolution equations in reflexive Banach spaces*, Math. Ann., 219 (1976), 159–166.

[11] Lakshmikantham, V., Leela, S., *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford 1981.

[12] Martin, R. H., jr., *Nonlinear operators and differential equations in Banach spaces*, Willey, New York 1976.

[13] Reich, S., *Extension problems for accretive sets in Banach spaces*, J. Funct. Anal., 26 (1977), 378–395.

STRESZCZENIE

Stosuje się nieliniową teorię perturbacji do rozwiązania równania ewolucji (*) metodą łamanych. Zakłada się, że kolejne przybliżenia spełniają równanie (**). Ulepszono, względnie uogólniono dotychczas otrzymane rezultaty w tym kierunku. Są one nowe nawet w przypadku zazwyczaj rozpatrywanym $G(t, x_t) \equiv G(t, x(t))$ i mogą być efektywnie stosowane przy numerycznym rozwiązywaniu równań.