ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN-POLONIA

VOL. XLII, 3 SECTIO A 1988

Department of Mathematics
Yerevan State University

M. D. GRIGORIAN, S. A. AJUNZ

Generalization of Legendre Polynomials

Uogólnienie wielomianów Legendre'a

Abstract. The authors give formulas for the determination of a generalized polynomial associated with a differential operator whose L_2-norm is a minimum. Problems of optimal extrapolation and interpolation within the class of generalized polynomials are also treated.

We give the formula determining the polynomial of the minimum deviation from zero in L_2-metric for a large class of the generalized polynomials induced by a differential operator. In the class of generalized polynomials the problems of the optimal extrapolation and interpolation are solved too.

Now, we give the exact formulations of the above problems.

Let

$$Dx(\cdot) = x^{(r)}(\cdot) + a_1(\cdot)x^{(r-1)}(\cdot) + \cdots + a_{r-1}(\cdot)\dot{x}(\cdot) + a_r(\cdot)$$

be a differential operator of the order r $(r \in N)$ with the variable coefficients $a_i(\cdot) \in C^{r-1}([0,1])$, $1 \le i \le r$, such that there exist r linear independent solutions $u_1(\cdot), \ldots, u_r(\cdot)$ of the equation $Dx = 0$ satisfying the condition

$$W_k(t) = W_k(u_1, \ldots, u_k) := \det\Big(u_i^{(j)}(t)\Big)_{i=1\ j=0}^{k\ k-1} > 0\,, \quad 1 \le k \le r\,. \tag{1}$$

If the operator D satisfies the condition (1), then it can be represented in the form

$$Dx(\cdot) = \frac{1}{\omega_r(\cdot)}\frac{d}{dt}\frac{1}{\omega_{r-1}(\cdot)}\frac{d}{dt}\cdots\frac{1}{\omega_1(\cdot)}\frac{d}{dt}\frac{1}{\omega_0(\cdot)}x(\cdot)$$

where $\omega_0(t) = u_1(t)$

$$\omega_i(t) = \frac{W_{i-1}(t)W_{i+1}(t)}{W_i^2(t)}\,, \quad 1 \le i \le r\,, \quad (W_0 = 1\,,\ W_{r+1} = W_r)$$

(see [1]).

By the expression of $\omega_i(\cdot)$ and from (1) it follows that $\omega_i(\cdot) > 0$, $t \in [0,1]$ and $\omega_i(\cdot) \in C^{r-1}([0,1])$, $0 \le i \le r$.

Let's consider the following extremal problems

(I) $\|x(\cdot)\|_{L_2[0,1]} \longrightarrow \inf \; ; \; Dx(\cdot) = 1$

(II) $|x^{(m)}(\tau)| \longrightarrow \sup \; ; \; \|x(\cdot)\|_{L_2([0,1])} \leq 1 \, , \quad Dx(\cdot) = 0 \, .$

If $D = \left(\frac{d}{dt}\right)^r$ then the solutions of the problem (I) are the Legendre polynomials (see [2]).

Definition 1. The solutions of the problem (I) for $r = 1, 2, \ldots$ we call the generalized Legendre polynomials.

The problem (II) is called the problem of the optimal extrapolation ($\tau \notin [0, 1]$) and interpolation ($\tau \in [0, 1]$) in L_2–metric for the generalized polynomials.

1. The generalized polynomials of the least deviation from zero in L_2–metric. We solve the problem (I). Let's introduce the following notations

$$D_i x(\cdot) := \frac{d}{dt} \frac{1}{\omega_i(\cdot)} x(\cdot) \, , \quad D_0 x(\cdot) := x(\cdot) \, .$$

$$L_i x(\cdot) := D_i D_{i-1} \ldots D_0 x(\cdot) \, , \quad 1 \leq i \leq r$$

It is known (see [3]) that the general solution of the differential equation $Dx(\cdot) = 1$ can be written in the form

$$x(t) = u_{r+1}(t) + \sum_{i=1}^{r} x_i u_i(t) \, ,$$

where

$$u_1(t) = \omega_0(t)$$

$$u_2(t) = \omega_0(t) \int_0^t \omega_1(t_1) \, dt_1$$

(2) $$u_3(t) = \omega_0(t) \int_0^t \omega_1(t_1) \int_0^{t_1} \omega_2(t_2) \, dt_2 \, dt_1$$

$$\vdots$$

$$u_{r+1}(t) = \omega_0(t) \int_0^t \omega_1(t_1) \int_0^{t_1} \omega_2(t_2) \cdots \int_0^{t_{r-1}} \omega_r(t_r) \, dt_r \ldots dt_1$$

Using the above representation of $x(\cdot)$ one can write the problem (I) in the form

(3) $$f(x) \to \inf \; ; \; x = (x_1, \ldots, x_r) \in \mathbb{R}_r$$

where

$$f(x) = \left\| u_{r+1}(\cdot) + \sum_{i=1}^{r} x_i u_i(\cdot) \right\|_{L_2([0,1])}$$

It is the convex finited dimensional problem without restrictions with the function $f(\cdot)$ which is strictly convex and continuous. Then there exists a unique solution of the problem. The existence follows from the Weierstrass theorem, as $f(\cdot) \in C(\mathbb{R}^r, \mathbb{R})$ and $f(x) \to \infty$ with $|x| \to +\infty$ and the uniqueness follows from the strict convexity of $f(x)$.

The solution of the problem (I) is denoted by

$$\hat{x}(t) = u_{r+1}(t) + \sum_{i=1}^{r} \hat{x}_i u_i(t)$$

In what follows we write the necessary condition for extremum in the problem (3) (in our case it is also sufficient).

By the Fermat theorem $f'(\hat{x}) = 0$ i.e.

$$\int_0^1 \hat{x}(t) u_i(t)\, dt = 0\,, \quad 1 \le i \le r \tag{4}$$

Since $u_1(\cdot), \ldots, u_r(\cdot)$ form the generalized Tchebycheff system (see [3], page 30), and the condition (4) is satisfied, following the proposition 1,4 (see [4], page 41) $\hat{x}(t)$ has exact r sign changes on $(0,1)$.

Let's denote by $\{T_{2D_r}(\cdot)\}_{r=1}^{\infty}$ the system of functions which are the solutions of the problem (I) for $r = 1, 2, \ldots$. It follows from (4) that it is an orthogonal system.

Now we obtain the formulas for the determination of the explicit form of the solution $\hat{x}(\cdot)$.

For this purpose we use the Lagrange principle (see [2]).

Introduce the following notations:

$$x_1 = \frac{x}{\omega_0}\,, \quad x_i = \frac{1}{\omega_i} \frac{d}{dt} x_{i-1}\,, \quad 2 \le i \le r\,,$$

and reduce the problem (I) to the following one

$$\int_0^1 x_1^2(t)\, dt \to \inf\,;\ \dot{x}_1 = \omega_1 x_2\,,\ \dot{x}_2 = \omega_2 x_3\,, \ldots,\ \dot{x}_{r-1} = \omega_{r-1} x_r\,,\ \dot{x}_r = \omega_r$$

The Lagrange function can be written as follows

$$\mathcal{L} = \lambda_0 \int_0^1 x_1^2(t)\, dt + \int_0^1 \left[\sum_{i=1}^{r-1} p_i(\dot{x}_i - \omega_i x_{i+1}) + p_r(\dot{x}_r \omega_r) \right] dt$$

a) The Euler equation

$$-\dot{p}_1 + \lambda_0 \hat{x}_1 = 0$$
$$-\dot{p}_i - \omega_{i-1} p_{i-1} = 0\,, \quad 2 \le i \le r$$

b) The transversality condition:

$$p_i(0) = p_i(1) = 0\,, \quad 1 \le i \le r\,.$$

If $\lambda_0 = 0$ then by the conditions a) and b) we have $p_i(t) \equiv 0$, $1 \le i \le r$, and this contradicts the Lagrange principle. Thus $\lambda_0 \neq 0$ and we can put $\lambda_0 = (-1)^{r+1}/2$. Now from the Euler equation we easily obtain the following equation

$$\frac{d}{dt}\frac{1}{\omega_1(\cdot)}\frac{d}{dt}\frac{1}{\omega_2(\cdot)}\frac{d}{dt}\cdots\frac{1}{\omega_{r-1}(\cdot)}\frac{d}{dt}p_r(\cdot) = \frac{\hat{x}(\cdot)}{\omega_0(\cdot)}$$

On denoting the operator on the left–hand side of the last equation by $\widetilde{D}$ and introducing the notation

$$\widetilde{L}_0 x(\cdot) = x(\cdot)\,, \quad \widetilde{L}_i x(\cdot) = \frac{1}{\omega_{r-1}(\cdot)}\widetilde{L}_{i-1}x(\cdot)\,, \quad 1 \le i \le r-1\,,$$

the conditions a) and b) can be transformed as follows

$$\widetilde{D}p_r(\cdot) = \frac{\hat{x}(\cdot)}{\omega_0(\cdot)} \tag{5}$$

$$\widetilde{L}_j p_r(0) = \widetilde{L}_j p_r(1) = 0\,, \quad 0 \le j \le r-1 \tag{6}$$

Now, from the expansion of $\hat{x}(\cdot)$ and from (3) it follows that

$$\widetilde{D}p_r(t) = \frac{u_{r+1}(t)}{\omega_0(t)} + \sum_{i=1}^{r}\hat{x}_i\frac{u_i(t)}{\omega_0(t)}\,, \tag{7}$$

where $u_i(t)$ is determined by (2).

By integrating both sides of the equation (7) and using the boundary condition $\widetilde{L}_{r-1}p_r(0) = 0$ we obtain

$$\widetilde{L}_{r-1}p_r(t) = \int_0^t \frac{u_{r+1}(\xi)}{\omega_0(\xi)}\,d\xi + \sum_{i=1}^{r}\hat{x}_i\int_0^t\frac{u_i(\xi)}{\omega_0(\xi)}\,d\xi\,.$$

Then by multiplying the last equation by $\omega_1(\cdot)$ and integrating it from 0 to t we get

$$\widetilde{L}_{r-2}p_r(t) = \int_0^t \omega_1(t_1)\int_0^{t_1}\frac{u_{r+1}(\xi)}{\omega_0(\xi)}\,d\xi\,dt_1 + \sum_{i=1}^{r}x_i\int_0^t\omega_2(t_1)\int_0^{t_1}\frac{u_i(\xi)}{\omega_0(\xi)}\,d\xi\,dt_1$$

By induction we see that

$$p_r(t) = v_{r+1}(t) + \sum_{i=1}^{r} \hat{x}_i v_i(t)$$

where

$$v_i(t) = \int_0^t \omega_{r-1}(t_{r-1}) \int_0^{t_{r-1}} \omega_{r-2}(t_{r-2}) \cdots \int_0^{t_2} \omega_1(t_1) \int_0^{t_1} \frac{u_i(\xi)}{\omega_0(\xi)}\, d\xi\, dt_1 \ldots dt_{r-1}$$

Using the boundary conditions $\widetilde{L}_j p_r(1) = 0$ we get the following linear system

$$\sum_{i=1}^{r} \hat{x}_i \int_0^1 \frac{u_i(\xi)}{\omega_0(\xi)}\, d\xi = -\int_0^1 \frac{u_{r+1}(\xi)}{\omega_0(\xi)}\, d\xi$$

(8)
$$\sum_{i=1}^{r} \hat{x}_i \int_0^1 \omega_1(t_1) \int_0^{t_1} \frac{u_2(\xi)}{\omega_0(\xi)}\, d\xi = -\int_0^1 \omega_1(t_1) \int_0^{t_1} \frac{u_{r+1}(\xi)}{\omega_0(\xi)}\, d\xi\, dt_1$$

$$\vdots$$

$$\sum_{i=1}^{r} \hat{x}_i v_1(1) = v_{r+1}(1)$$

Let's denote the coefficients in the system (8) by a_{ik}, $1 \le i \le r$, $1 \le k \le r$. Since

$$\det\left(v_i^{(j)}(t)\right)_{i=1,\, j=0}^{r\;\; r-1} > 0\,, \quad t \in [0,1]$$

and

$$\det\left(a_{ik}\right)_{i,k=1}^{r} = \frac{1}{\omega_1(1)\ldots\omega_{r-1}(1)} \det\left(v_i^{(j)}(1)\right)_{i=1,\, j=0}^{r\;\; r-1} > 0$$

we conclude that the system (8) has a unique solution.

Thus we have proved the following

Theorem 1. *The solution of the problem* (I) *is unique and it can be written as follows*

$$\hat{x}(t) = \omega_0(t)\widetilde{D}p_r(t) = u_{r+1}(t) + \sum_{i=1}^{r} \hat{x}_i u_i(t)$$

where $\hat{x}_i$ are determined from the system (8).

2. The optimal extrapolation and interpolation of generalized polynomials in L_2-metric. Consider now the problem (II) which is equivalent to the following one

(II)
$$x^{(m)}(\tau) \longrightarrow \inf\,;\ \|x(\cdot)\|^2_{L_2([0,1])} \le 1\,, \quad Dx(\cdot) = 0\,,$$
$$0 \le m \le r-1\,, \quad \tau \in \mathbb{R}$$

Since the general solution of the equation $Dx(\cdot) = 0$ can be written in the form $x(t) = \sum_{i=1}^{r} x_i u_i(t)$, the problem (II) is a problem in the convex programming which has the solution under the assumptions of compactness and continuity of the functional $f(x(\cdot)) = x^{(m)}(\tau)$. We solve the problem by Kyhu–Tucker theorem (see [2]).

Since the Slater condition is satisfied, the Lagrange function has the form

$$\mathcal{L}(x,\lambda) = x^{(m)}(\tau) + \frac{\lambda}{2}\int_0 x^2(t)\,dt\,, \quad x = (x_1,\dots,x_r)$$

Denote the solution of the problem (II) by $\hat{x}(\tau)$ and write:

1. The minimum principle

$$\min_{x\in\mathbb{R}^r}\left(x^{(m)}(\tau) + \frac{\lambda}{2}\int_0^1 x^2(t)\,dt\right) = \hat{x}(m)(\tau) + \frac{\lambda}{2}\int_0^1 \hat{x}^2(t)\,dt$$

2. The condition of the supplement nonrigid

$$\lambda\left(\int_0^1 \hat{x}^2(t)\,dt - 1\right) = 0$$

3. The condition of the nonnegativity

$$\lambda \geq 0$$

It is obvious that $\lambda > 0$. Then by the minimum principle and the Fermat theorem we get

$$x^{(m)}(\tau) + \lambda\int_0^1 \hat{x}(t)x(t)\,dt = 0\,, \quad x(\cdot) \in P_r^D \tag{9}$$

where $P_r^D = \left\{x(\cdot)\,|\,x(t) = \sum_{i=1}^{r} x_i u_i(t),\ x_i \in \mathbb{R}\right\}$.

Putting $\hat{x}(\cdot)$ instead of $x(\cdot)$ into the equation (9) and using the condition of the supplement nonrigid we see that $\lambda = -\hat{x}^{(m)}(\tau)$.

Let $e_k(\cdot) = T_{2D_k}(\cdot)/\|T_{2L_k}(\cdot)\|_{L_2(I)}$ be the orthonormal system of the generalized Legendre polynomials. Then $\hat{x}(\cdot)$ can be represented in the form

$$\hat{x}(t) = \sum_{i=1}^{r} \hat{x}_i e_i(t)\,. \tag{10}$$

Putting $e_k(\cdot)$ instead of $x(\cdot)$ in the equation (9) and using the representation (10) we obtain

$$e_k^{(m)}(\tau) + \lambda\int_0^1 \sum_{i=1}^{r} \hat{x}_i e_i(t)e_k(t)\,dt = e_k^{(m)}(\tau) + \lambda\hat{x}_k = 0\,.$$

Hence $\hat{x}_k = -\dfrac{e_k^{(m)}(\tau)}{\lambda}$ and as a consequence of (10) we get

$$\hat{x}(t) = \sum_{k=1}^{r} e_k^{(m)}(\tau) e_k(t) / \hat{x}^{(m)}(\tau) .$$

Differentiating m times both sides of that equality at the point τ, we obtain

$$\left[\hat{x}^{(m)}(\tau)\right]^2 = \sum_{k=1}^{r} \left(e_k^{(m)}(\tau)\right)^2 .$$

Hence

$$\hat{x}^{(m)}(\tau) = -\left(\sum_{k=1}^{r} \left(e_k^{(m)}(\tau)\right)^2\right)^{1/2}$$

Therefore

$$\hat{x}(t) = -\sum_{k=1}^{r} e_k^{(m)}(\tau) e_k(t) / \left(\sum_{k=1}^{r} \left(e_k^{(m)}(\tau)\right)^2\right)^{1/2}$$

As a corollary of that result we get the following inequality for the generalized polynomials.

$$|x^{(m)}(\tau)| \le \left(\sum_{k=1}^{r} e_k^{(m)}(\tau)^2\right)^{1/2} \|x(\cdot)\|_{L_2([0,1])} .$$

This inequality can be proved using the Cauchy–Bunyakowsky inequality.

Thus we are lead to the following result.

Theorem 2. *The solution of the problem* (II) *is unique and has the following form*

$$\hat{x}(t) = -\sum_{k=1}^{r} e_k^{(m)}(\tau) e_k(t) / \left(\sum_{k=1}^{r} \left(e_k^{(m)}(\tau)\right)^2\right)^{1/2}$$

where $e_k(\cdot)$, $k = 1, \ldots, r$ *is the orthonormal system of the generalized Legendre polynomials.*

REFERENCES

[1] Pólya, G., Szegö, G., *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Springer, Berlin 1925.

[2] Tichomirov, V. M., *Some Problems in the Approximation Theory* (Russian), Publications of Moscow State University 1976.

[3] Karlin, S., Studden, W., *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience Publ., New York–London–Sydney 1966.

[4] Pinkus, A., *n-widths in Approximation Theory*, Springer, Berlin 1985.

STRESZCZENIE

Autorzy podają wzory pozwalające wyznaczyć w klasie uogólnionych wielomianów związanych z pewnym operatorem różnicakowym uogólniony wielomian o minimalnej L_2-normie. Zostały również rozwiązane dla klasy uogólnionych wielomianów problemy optymalnej ekstrapolacji i interpolacji.