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Quasiconformal Extension for Functions Defined in the Upper Half-plane

Quasikonforemne przedłużenie funkcji zdefiniowanych w górnej półpłaszczyźnie

Abstract. In this paper sufficient conditions for functions defined and locally univalent in the upper half-plane U to have a K-quasiconformal extension to the whole complex plane $\overline{C}$ are given. This result generalizes the results of Anderson, Hinkkanen [2] and Lewandowski, Stankiewicz [3].

Anderson and Hinkkanen in a recent paper [2] proved a certain univalence condition for functions f meromorphic in upper half-plane $U = \{z : \operatorname{Im} z > 0\}$ given in terms of the Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

This result was based on a hypothesis established by Ahlfors [1].

The Theorem proved here and based on a theorem given in [4] is an essential generalization of results due to Ahlfors [1], Anderson and Hinkkanen [2], Lewandowski and Stankiewicz [3].

Theorem . *Let f and g be meromorphic and locally univalent functions in U. If for the function g and some fixed k, $0 < k < 1$, there exists a holomorphic function $c(z)$, $z \in U$, such that*

$$(1) \qquad |c(z) - 1| \le k \ , \quad \left|c(z) - 1 + 2iy\left(\frac{c'(z)}{c(z)} - \frac{g''(z)}{g'(z)}\right)\right| \le k$$

and the function f satisfies the inequality

$$(2) \quad \left|2y^2(S_f(z) - S_g(z)) - 2iy\left(c'(z) - c(z)\frac{g''(z)}{g'(z)}\right) - c(z)(c(z) - 1)\right| \le k|c(z)| ,$$

$$y = \operatorname{Im} z \ , \quad z \in U$$

then f is univalent in U and has a quasiconformal extension to the whole complex plane $\overline{C}$. If additionally the assumptions of the theorem are satisfied in $\overline{U} = \{z : \operatorname{Im} z \geq 0\}$ then the quasiconformal extension has be form :

$$F(z) = \begin{cases} f(z) & \text{for } z \in \overline{U} \\ f(\overline{z}) + \dfrac{\dfrac{z-\overline{z}}{c(\overline{z})} f'(\overline{z})}{1 + \dfrac{z-\overline{z}}{2c(\overline{z})}\left(\dfrac{g''(\overline{z})}{g'(\overline{z})} - \dfrac{f''(\overline{z})}{f'(\overline{z})}\right)}, & z \in L = \{z,\ \operatorname{Im} z < 0\} \end{cases}$$

For $k = 1$ the inequality (2) together with (1) imply univalence of function f in U.

Proof. We may assume the non-existence of poles of the functions f and g and zeros f'' and g'' on the positive imaginary axis and also $\dfrac{f''}{f'} - \dfrac{g''}{g'} \neq 0$.

If the above holds then the proof is trivial. In the case $\dfrac{f''}{f'} - \dfrac{g''}{g'} = 0$ in U, $S_f = S_g$ and further part of the proof with some changes remains true.

For $R > 0$ we denote

$$\begin{aligned} D_R &= \{z : |z - z_0| < R\}, \quad z_0 = z_0(R) = i\sqrt{1+R^2} \\ D_R^* &= \{z \in \overline{C} :\ |z - z_0| > R\} = \overline{C} - \overline{D}_R \\ C_R &= \{z : |z - z_0| = R\} = \partial D_R = \partial D_R^* \end{aligned} \tag{3}$$

Let $z = h_R(w) = z_0 + \dfrac{R^2}{\overline{w} - \overline{z}_0} = \dfrac{\overline{w}z_0 - 1}{\overline{w} - \overline{z}_0}$, $h_R(w)$ is an anticonformal reflection with respect to the circumference C_R and

$$\lim_{R \to \infty} h_R(w) = \overline{w}.$$

We observe that if $0 < R_1 < R_2 < \cdots < R_n \to \infty$ as $n \to \infty$ then

$$U = \bigcup_{n=1}^{\infty} D_{R_n}. \tag{4}$$

Moreover

$$\frac{\partial z}{\partial w} = 0, \quad \frac{\partial z}{\partial \overline{w}} = \frac{-R^2}{(\overline{w} - \overline{z}_0)^2}, \quad z = h_R(z) \text{ and } \lim_{R \to \infty} \frac{\partial z}{\partial \overline{w}} = 1 \tag{5}$$

and

$$\left|\frac{\partial z}{\partial \overline{w}}\right| \leq 1 \quad \text{for} \quad w \in \overline{D}_R \quad , \quad \left|\frac{\partial z}{\partial \overline{w}}\right| = 1 \quad \text{for} \quad z \in C_R. \tag{6}$$

And finally we put

$$v(z) = \sqrt{\frac{g'(z)}{f'(z)}} \tag{7}$$

$$u(z) = f(z) \cdot v(z). \tag{8}$$

Hence

$$(9)\qquad \begin{cases} v' = \dfrac{1}{2}\Big(\dfrac{g''}{g'} - \dfrac{f''}{f'}\Big)\Big(\dfrac{g'}{f'}\Big)^{1/2} \\ u' = f\Big[\dfrac{f'}{f} + \dfrac{1}{2}\Big(\dfrac{g''}{g'} - \dfrac{f''}{f'}\Big)\Big]\cdot\Big(\dfrac{g'}{f'}\Big)^{1/2} \end{cases}$$

Also from (8) taking (9) we have

$$(10)\qquad \begin{cases} u'v - uv' = g' \\ u''v - uv'' = g'' \\ u''v' - u'v'' = \frac{1}{2}\, g'(S_f - S_g) \end{cases}$$

We observe that u and v are meromorphic functions in U, f, g cannot have multiple poles and f', g' cannot have zeros.

Let F_R be the function obtained from f by restricting the range to $\overline{D}_R$ and subsequently extended on the whole closed plane $\overline{C}$ in the following way :

$$(11)\qquad F_R(w) = \begin{cases} f(w) & \text{for } w \in \overline{D}_R \\ \dfrac{u(z) + \dfrac{w-z}{c(z)}\, u'(z)}{v(z) + \dfrac{w-z}{c(z)}\, v'(z)}, & \text{for } w \in D_R^* \end{cases}$$

By (9) $F_R(w)$ has the form

$$(12)\qquad F_R(w) = \begin{cases} f(w) & \text{for } w \in \overline{D}_R \\ f(z) + \dfrac{\dfrac{w-z}{c(z)}\, f'(z)}{1 - \dfrac{w-z}{2c(z)}\Big(\dfrac{g''(z)}{g'(z)} - \dfrac{f''(z)}{f'(z)}\Big)}, & z = h_R(w) \in D_R,\ w \in D_R^*. \end{cases}$$

Next it needs to be proved that the function is a local homeomorphism mapping the plane $\overline{C}$ onto $\overline{C}$ with complex dilatation bounded by k, $0 < k < 1$.

In order to demonstrate this we consider the following cases :

1°. The function $F_R = f(z)$ is a locally conformal mapping in the domain $\overline{D}_R$ and so it is a local homeomorphism with the complex dilatation equal zero in D_R.

2°. In the domain $D_R^* \setminus \{\infty\}$ the formal derivatives of the function $F_R(w)$ are expressed using (10) and (9) by the equalities :

$$(13)\qquad \frac{\partial F_R}{\partial w} = \frac{f'(z)}{c(z)\Big[1 + \dfrac{w-z}{2c(z)}\Big(\dfrac{g''(z)}{g'(z)} - \dfrac{f''(z)}{f'(z)}\Big)\Big]^2},$$

$$(14)\qquad \frac{\partial F_R}{\partial \overline{w}} = \frac{\partial F_R}{\partial w}\Big\{c(z) - 1 - (w-z)\Big(\frac{c'(z)}{c(z)} - \frac{g''(z)}{g'(z)}\Big) + \\ + \frac{1}{2}\,\frac{(w-z)^2}{c(z)}\,(S_f(z) - S_g(z))\Big\}\cdot\Big(-\frac{R^2}{(\overline{w} - \overline{z}_0)^2}\Big), \quad z = h_R(w)$$

It is known that $F_R(w)$ is a local homeomorphism preserving the orientation for $w \in D^* \setminus \{\infty\}$ if the modulus of complex dilatation

$$\mu = \mu(F_R) = \left| \frac{\partial F_R}{\partial \overline{w}} \Big/ \frac{\partial F_R}{\partial w} \right| \le k < 1 .$$

So $\mu(F_R(w))$ is bounded by $k < 1$ if

$$\frac{R^2}{|\overline{w} - \overline{z}_0|^2} \cdot \Big| c(z) - 1 - (w - z)\Big(\frac{c'(z)}{c(z)} - \frac{g''(z)}{g'(z)}\Big) + \\ + \frac{1}{2}\, \frac{(w-z)^2}{c(z)}\, (S_f(z) - S_g(z)) \Big| \le k .$$

The last inequality is equivalent to the following

$$(15) \qquad \Big| 2y^2 (S_f(z) - S_g(z)) - \frac{c'(z) - c(z)\dfrac{g''(z)}{g'(z)}}{w - z}\, 4y^2 + \\ + \frac{c(z)(c(z) - 1)}{(w-z)^2}\, 4y^2 \Big| \le k\, \frac{4y^2 |\overline{w} - \overline{z}_0|^2}{R^2 |w - z|^2} \cdot |c(z)|$$

From the assumption of the theorem and in particular from the inequality (2) it follows that the condition (15) will be satisfied if the following condition, which follows from geometrical considerations, is satisfied

$$(16) \qquad \Big| (c(z) - 1)\frac{(w-z)^2 + 4y^2}{(w-z)^2} + \Big(\frac{c'(z)}{c(z)} - \frac{g''(z)}{g'(z)}\Big)\frac{2iy(w-z) - 4y^2}{w - z} \Big| \le \\ \le k\Big(\frac{4y^2 |w - z_0|^2}{R^2 |w - z|^2} - 1\Big).$$

We observe that

$$(17) \qquad \begin{gathered} 2iy(w-z) - 4y^2 = 2iy(w - z + 2iy) \\ (w-z)^2 + 4y^2 = (w - z + 2iy)(w - z - 2iy) \end{gathered}$$

Hence the inequality (16) can be rewritten in the form :

$$(18) \qquad \Big| (c(z) - 1)\frac{w - z - 2iy}{w - z} + \Big(\frac{c'(z)}{c(z)} - \frac{g''(z)}{g'(z)}\Big) 2iy \Big| \le \\ \le k\, \frac{4y^2 |w - z_0|^2 - R^2 |w - z|^2}{R^2 |w - z| \cdot |w - z + 2iy|}$$

Now using (1) the left hand side of the inequality (18) can be estimated as follows :

$$\Big| (c(z) - 1)\frac{w - z - 2iy}{w - z} + \Big(\frac{c'(z)}{c(z)} - \frac{g''(z)}{g'(z)}\Big) 2iy \Big| \le k\Big(1 + \frac{2y}{|w - z|}\Big) .$$

Considering the above we can see that the inequality (18) will be satisfied only if one of the following inequalities is satisfied :

$$1+\frac{2y}{|w-z|}\le\frac{4y^2|w-z_0|^2-R^2|w-z|^2}{R^2|w-z|\cdot|w-z+2iy|}\,,$$

(19) $$(|w-z|+2y)|w-z+2iy|\le\frac{4y^2}{R^2}\,|w-z_0|^2-|w-z|^2\,.$$

Moreover, we observe that for $z=h_R(w)$ the expression

(20) $$\frac{w-z}{w-z_0}=1-\frac{z-z_0}{w-z_0}=1-\frac{|z-z_0|^2}{R^2}$$

is nonnegative for $w\in D_R^*$.

If we put

(21) $$\begin{cases} z-z_0=re^{i\varphi} & 0\le r\le R\,,\quad \varphi\in\langle 0,2\pi)\\ \dfrac{y}{R}=A & \\ \dfrac{r}{R}=\dfrac{|z-z_0|}{R}=t\,, & t\in\langle 0,1)\end{cases}$$

then

(22) $$\begin{cases}\dfrac{1}{w-z_0}=\dfrac{\overline{z}-\overline{z}_0}{R^2}=rR^{-2}e^{-i\varphi}\\ y=\operatorname{Im} z=\operatorname{Im}(re^{i\varphi})+\sqrt{1+R^2}=r\sin\varphi+\sqrt{1+R^2}\\ A=\dfrac{y}{R}\;\dfrac{\sqrt{1+R^2}}{R}+t\sin\varphi\end{cases}$$

Moreover,

$$0<1-t^2+2At\le 1-t^2+2A\quad\text{for}\quad t\in\langle 0,1)\,,\quad A>0\,,$$

$$2A-(1-t^2)=\frac{2\sqrt{1+R^2}}{R}+2t\sin\varphi-1+t^2=$$

$$=\frac{2}{R}\left(\sqrt{1+R^2}-1\right)+1+2t\sin\varphi+t^2>0\,.$$

Hence the inequality (19) can be rewritten by (20), (21) and (22) in the following form :

(23) $$(1-t^2+2At)\cdot|1-t^2+2Atie^{-i\varphi}|\le 4A^2-(1-t^2)^2\,,$$

which is satisfied if

$$|1-t^2+2Atie^{-i\varphi}|\le 2A-(1-t^2)\,.$$

The last inequality is equivalent to : $1+t\sin\varphi\le A$.

By (22) we have

$$1+t\sin\varphi\le\frac{\sqrt{1+R^2}}{R}+t\sin\varphi$$

This inequality is satisfied because $R < \sqrt{1+R^2}$. Hence the inequality (16) is satisfied and so is the inequality (15) which ensures that $F_R(w)$ is a local homeomorphism in $D_R^* \setminus \{\infty\}$ with the complex dilatation $\mu(F_R(w)) \le k < 1$. The above considerations also hold for $w \in D^* \setminus \{\infty\}$, as well as in those points in which $f(z(w))$, F_R, $\dfrac{\partial F_R}{\partial \overline{w}}$ are equal to infinity.

If $f(z(w)) = \infty$ and $f(z(w)+q) = \dfrac{A_{-1}}{q} + A_0 + A_{1q} + \cdots$ for sufficiently small q then it is easy to show that $F_R(w) = \infty$. Therefore the above considerations are true in this case.

If $f(z(z)) \neq 0$ and F_R, $\dfrac{\partial F_R}{\partial w}$ or $\dfrac{\partial F_R}{\partial \overline{w}}$ are infinite then

$$\frac{g''(z)}{g'(z)} - \frac{f''(z)}{f'(z)} = -\frac{2c(z)}{w-z} \quad , \quad z = h_R(w)\,.$$

In those points we can consider $\dfrac{1}{F_R}$ because $\mu\left(\dfrac{1}{F_R}\right) = \mu(F_R)$.

3°. Let $w_0 \in C_R$. Then $z(w_0) = w_0$ and let q be sufficiently small.

We put $w = w_0 + q \in \overline{D}_R$, then

$$F_R(w_0+q) = f(w_0+q) = f(w_0) + f'(w_0)\cdot q + O(q^2) = F_R(w_0) + f'(w_0)\cdot q + O(q^2)\,.$$

Hence

$$F_R(w_0+q) - F_R(w_0) = f'(w_0)\cdot q + O(q^2)\,. \tag{25}$$

Let us put $w = w_0 + q \in D_R^*$ then by (12) we have

$$F_R(w_0+q) = f(z) + \frac{\dfrac{w_0+q-z}{c(z)}\, f'(z)}{1 - \dfrac{w_0+q-z}{c(z)}\left(\dfrac{g''(z)}{g'(z)} - \dfrac{f''(z)}{f'(z)}\right)}\,,$$

where $z = z(w_0+q) = h_R(w_0+q) = h_R(w_0) + \dfrac{\partial z}{\partial w}\, q + \dfrac{\partial z}{\partial \overline{w}}\, \overline{q} + O(q^2) =$

$$= w_0 + a\overline{q} + O(q^2)\,,$$

$$a = \left.\frac{\partial z}{\partial \overline{w}}\right|_{w=w_0}$$

Using the fact that f and g are locally univalent functions, f', f'', g', g'', c are the holomorphic functions, after long calculations considering the above, we obtain

$$F_R(w_0+q) - f(w_0) = f'(w_0)\left[\frac{q}{c(w_0)} + a\left(1 - \frac{1}{c(w_0)}\right)\cdot \overline{q}\right] + O(q^2)\,. \tag{26}$$

Because $w_0 \in C_R$ we have $|a| = 1$ and (1) implies $|c(z) - 1| < 1$, hence the expression in the square brackets in (26) is different from zero.

By virtue of (25) and (26) it follows that for every sufficiently small q the expression $F_R(w_0+q)-F_R(w_0)\neq 0$ and therefore $F_R(w)$ is a local homeomorphism on C_R.

4°. Finally let $w_0=\infty$ and let w belong to a certain neighbourhood of infinity. Then

$$z=z(w)=z_0+\frac{R^2}{\overline{w}}\,\frac{1}{1-\dfrac{\overline{z}_0}{\overline{w}_0}}=z_0+\frac{R^2}{\overline{w}}+O\Big(\frac{1}{\overline{w}^2}\Big),$$

$$\lim_{w\to\infty} z(w)=h_R(\infty)=z_0$$

$$\lim_{w\to\infty} F_R(w)=f(z_0)+\frac{2f'(z_0)}{\dfrac{g''(z_0)}{g'(z_0)}-\dfrac{f''(z_0)}{f'(z_0)}}=B\neq\infty$$

because we may suppose that $\dfrac{g''(z_0)}{g'(z_0)}-\dfrac{f''(z_0)}{f'(z_0)}\neq 0$ on the imaginary axis. The function $F_R(w)$ is a local homeomorphism at ∞ if the function $g(w)=F_R\big(\frac{1}{w}\big)$ is a local homeomorphism at the origin.

Using the calculations in 2° it can be seen that G can be represented in a neighbourhood $w=0$ in the form

$$G(w)=B+B_1\overline{w}+B_2w+O(w^2)$$

where

$$B=G(0)=\lim_{w\to\infty}F_R(w)\neq\infty$$

$$B_1=\frac{\partial G}{\partial\overline{w}}(0)=-\frac{1}{\overline{w}^2}\,\frac{\partial F_R}{\partial\overline{w}}\Big(\frac{1}{w}\Big)\Big|_{w=0}=$$

$$=\lim_{w\to\infty}\frac{\overline{w}^2 f'(z)R^2}{c^2(z)\Big[1-\dfrac{w-z}{2c(z)}\Big(\dfrac{g''(z)}{g'(z)}-\dfrac{f''(z)}{f'(z)}\Big)\Big]^2(\overline{w}-\overline{z}_0)^2}\,.$$

$$\cdot\, c(z)\big(c(z)-1\big)+(w-z)\Big(c'(z)-c(z)\frac{g''(z)}{g'(z)}\Big)+\frac{1}{2}(w-z)^2(S_f-S_g)=$$

$$=\frac{2f'(z_0)\big(S_f(z_0)-S_g(z_0)\big)\cdot R^2}{\Big(\dfrac{g''(z_0)}{g'(z_0)}-\dfrac{f''(z_0)}{f'(z_0)}\Big)^2}$$

$$B_2=\frac{\partial G}{\partial w}(0)=-\frac{1}{w^2}\,\frac{\partial F_R}{\partial w}\Big(\frac{1}{w}\Big)\Big|_{w=0}=$$

$$=\lim_{w\to\infty}\frac{-w^2\dfrac{f'(z)}{c(z)}}{\Big[1+\dfrac{w-z}{2c(z)}\Big(\dfrac{g''(z)}{g'(z)}-\dfrac{f''(z)}{f'(z)}\Big)\Big]^2}=$$

$$=-\frac{4f'(z_0)\cdot c(z_0)}{\Big[\dfrac{g''(z_0)}{g'(z_0)}-\dfrac{f''(z_0)}{f'(z_0)}\Big]^2}\,.$$

The function G will be a local orientation preserving homeomorphism at $w = 0$ if

$$B_2 \neq 0 \quad \text{and} \quad \left|\frac{B_1}{B_2}\right| < 1 .$$

The first of these conditions is satisfied because $f'(z_0) \neq 0$ and $c(z_0) \neq 0$. The second is also satisfied as the calculations below demonstrate:

$$\left|\frac{B_1}{B_2}\right| = \left|\frac{R^2}{2}\,\frac{S_f(z_0) - S_g(z_0)}{c(z_0)}\right| = \left|\frac{R^2}{4(1+R^2)}\,\frac{2y_0^2(S_f(z_0) - S_g(z_0))}{c(z_0)}\right| ,$$

where $y_0 = \operatorname{Im} z_0 = \sqrt{1+R^2}$.

With the inequality (2) we obtain by (1)

$$\begin{aligned} &|2y^2(S_f(z) - S_g(z))| \leq \\ &\leq k|c(z)| + |2iy(c'(z) - c(z)\frac{g''(z)}{g'(z)}) + c(z)(c(z) - 1)| \leq 2k|c(z)| . \end{aligned}$$

Finally

$$\left|\frac{B_1}{B_2}\right| \leq \frac{R^2 k}{2(1+R^2)} \leq \frac{k}{2} \leq \frac{1}{2} .$$

For $k = 1$ we have $\left|\frac{B_1}{B_2}\right| \leq \frac{1}{2}$.

Thus $G(u)$ is a local homeomorphism at $w = 0$, that is $F_R(w)$ is a local homeomorphism at $w = \infty$.

In this way we have proved that for every fixed $R > 0$ the function F_R is a local homeomorphism mapping the complex plane $\overline{C}$ onto itself, which preserves the orientation and has the modulus of the complex dilatation $\mu(F_R(w))$ bounded by $k < 1$.

This means that F_R is a global homeomorphism of $\overline{C}$ with the modulus of the complex dilatation bounded by $k < 1$, so it is a $\frac{1+k}{1-k} = K$-quasiconformal self-mapping of the plane $\overline{C}$.

Now let us consider the sequence $\{R_n\}$

$$0 < R_1 < R_2 < \cdots < R_n \to \infty$$

For every integer n we can consider the function

$$F_n(w) = F_{R_n}(w) .$$

In this way we obtain a sequence of K-quasiconformal mappings. This sequence forms a normal family. Thus we can choose a subsequence $F_{n_\mu}(w)$ which is almost uniformly convergent to the function $F(w)$ which is K-quasiconformal mapping.

We observe that for $R_n \to \infty$ $F_n(w) \to f(w)$. It means that the limit function $F(w)$ coincides with the function $f(w)$. Therefore $F(w)$ is a well defined K-quasiconformal extension of the function $f(w)$ on the closed complex plane $\overline{C}$.

For $k = 1$ the above considerations can be repeated with some modifications. In this way we obtain a sequence $F_n(w)$ of global homeomorphisms of the plane $\overline{C}$ onto itself.

The limit function restricted to the upper halfplane U $(f = F|_U)$ is the limit of $1:1$ mappings, so is a univalent function in U. q.e.d.

Corollary . *Let f be a meromorphic and locally univalent function in U. If for any fixed k, $0 < k < 1$, there exists a function $c(z)$, $|c(z) - 1| < k$ such that*

$$\left|c(z)\big(c(z)-1\big) + 2iy\Big(c'(z) - c(z)\frac{f''(z)}{f'(z)}\Big)\right| \le k|c(z)| \ , \quad y = \operatorname{Im} z \ , \ z \in U \ ,$$

then f is univalent in U and has a quasiconformal extension on $\overline{C}$.

If further assumptions are satisfied in $\overline{U} = \{z : \operatorname{Im} z \ge 0\}$ then the quasiconformal extension can be rewritten in the form

$$F(z) = \begin{cases} f(z) & \text{for } z \in \overline{U} \\ f(\overline{z}) + \dfrac{z - \overline{z}}{c(\overline{z})} f'(\overline{z}) & \text{for } z \in L = \{z : \operatorname{Im} z < 0\} \ . \end{cases}$$

For $k = 1$ the Corollary is a sufficient condition for the univalence of the function f in U. By a suitable choice of the functions $g(z)$ and $c(z)$ from Theorem we obtain the results of Lewandowski and Stankiewicz [3], Ahlfors [1], Anderson and Hinkkanen [2], respectively.

Moreover, the result contained in the Corollary cannot be obtained from the results obtained by the authors mentioned above.

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STRESZCZENIE

W pracy tej podano warunki dostateczne na to, aby funkcje określone i lokalnie jednolistne w górnej półpłaszczyźnie U miały K-quasi-konforemne rozszerzenie na całą płaszczyznę zespoloną $\overline{C}$.

Wynik ten uogólnia rezultaty prac Andersona i Hinkkanena [2] oraz Lewandowskiego i Stankiewicza [3].

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