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# Some New Inequalities for Periodic Quasisymmetric Functions

Nowe nierówności dla okresowych funkcji quasisymetrycznych

**Abstract.** As pointed out in [2], [3] the boundary correspondence under quasiconformal self-mappings of Jordan domains may be represented by $M$ - quasisymmetric functions $x \mapsto x + \sigma(x)$, $x \in \mathbf{R}$, where $\sigma$ is $2\pi$-periodic. In this paper some estimates of various norms and Fourier coefficients of $\sigma$ depending on $M$ and established in [3] are improved.

**1. Introduction. Notations. Statement of results.** Any automorphism $\varphi$ od the unit circle $T$ admitting a quasiconformal extension to the unit disc $\mathbf{D}$ also admits a quasiconformal exter..ion $\Phi$ with $\Phi(0) = 0$. By lifting the mapping $\Phi$ of $\mathbf{D} \setminus \{0\}$ under $z \mapsto -i \log z$ to the upper half-plane we obtain a quasisymmetric (abbreviated : qs) function $x \mapsto x + \sigma(x)$ , $x \in \mathbf{R}$, with $2\pi$-periodic $\sigma$. Obviously the period $2\pi$ may be replaced by any $a > 0$ and a corresponding class of functions $\sigma$ will be denoted by $E(M, a)$. The normalization

$$\sigma(0) = \sigma(a) = 0 \tag{1.1}$$

defines the subclass $E_1(M, a)$ of $E(M, a)$. Evidently $E_1(M, 1) + \mathrm{id}$ is the subclass of the familiar class $H_1(M)$ of $M$-qs functions $h$ normalized by the condition

$$h(0) = 0 \quad , \quad h(1) = 1 . \tag{1.2}$$

If $\sigma \in E(M, a)$ then $\sigma_0(x) = \sigma(x) - a^{-1} \int_0^a \sigma(t)\,dt$ satisfies :

$$\int_0^a \sigma_0(x)\,dx = 0 . \tag{1.3}$$

The subclass of $E(M, a)$ subject to the normalization (1.3) will be denoted by $E_0(M, a)$. For sake of brevity $E(M)$ will stand for $E_0(M, 2\pi)$. With some abuse of language we shall call $\sigma \in E(M, a)$ a periodic qs function.

In Section 2 we establish some basic lemmas concerning periodic qs functions which will be used further on in obtaining various estimates established in [3]. In particular we answer in the positive a conjecture posed in [3, p.232[5,6]] and obtain for $\sigma \in E(M)$ an estimate of $\sum_{n=1}^{\infty} \rho_n$ of the form $O(\sqrt{M-1})$ (Theorem 3.1). Here $\sigma(x) = \sum_{n=1}^{\infty} \rho_n \sin(nx + x_n)$ and $\rho_n \geq 0$.

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**2. Basic lemmas.**

**Lemma 2.1.** *If $\sigma \in E_0(M,1)$ then*

$$\sup\{|\sigma(x)| : x \in \mathbf{R}\} \le \frac{1}{2}\,\frac{M-1}{M+1} \tag{2.1}$$

*and*

$$\int_0^1 |\sigma(x)|^2\,dx \le \frac{1}{8}\Big(\frac{M-1}{M+1}\Big)^2 \tag{2.2}$$

**Proof.** We may assume that

$$\sigma(0) = \sigma(1) = 0\ . \tag{2.3}$$

Otherwise we could consider $\sigma_1(x) = \sigma(x + x_0)$ where $x_0 \in [0;1)$ satisfies $\sigma(x_0) = 0$. So we can take $\sigma \in E_1(M,1)$. Since $\int_0^1 \sigma(x)\,dx = 0$, there exists $x_1 \in (0;1)$ such that $\sigma(x_1) = 0$. Since $\sigma \in E_1(M,1)$ we have the estimate

$$|\sigma(x)| \le \frac{M-1}{M+1} \tag{2.4}$$

cf. [3, p.231].

If $\sigma \in E_1(M,1)$, $(\alpha;\beta) \subset (0;1)$ and $\sigma(\alpha) = \sigma(\beta) = 0$ then obviously

$$\sup\{|\sigma(x)| : x \in (\alpha,\beta)\} \le (\beta-\alpha)\frac{M-1}{M+1} \tag{2.5}$$

Note that $x \mapsto (\beta-\alpha)^{-1}\sigma((\beta-\alpha)x+\alpha) \in E_1(M,1)$ and apply (2.4).

Let $I_k$ (or $J_l$, respectively) be the system of maximal, disjoint, open intervals in $[0;1)$ such that $\sigma(x) > 0$ on $I_k$ (and $\sigma(x) < 0$ on $J_l$, resp.) and $\sigma(x) = 0$ at the end-points of $I_k$ and $J_l$. If all the intervals $I_k$ and $J_l$ are such that the length of each is at most $\frac{1}{2}$, then (2.1) immediately follows from (2.5).

Assume now that one of these intervals, say $I_0$ has the length $|I_0| > \frac{1}{2}$. Due to the normalization (1.3) we have

$$\int_{\bigcup J_l} |\sigma(x)|\,dx = \int_{\bigcup I_k} \sigma(x)\,dx\ . \tag{2.6}$$

Suppose that

$$\max\{\sigma(x) : x \in I_0\} = \sigma(x_0) > \frac{1}{2}\,\frac{M-1}{M+1}\ . \tag{2.7}$$

Since $|I_0| > \frac{1}{2}$, we have $\sum |J_l| < \frac{1}{2}$ and hence by (2.5) and [3 ; formula (2.13)]

$$\int_{\bigcup J_l} |\sigma(x)|\,dx \le \frac{1}{2}\sum |J_l|^2\,\frac{M-1}{M+1} \le \frac{1}{8}\,\frac{M-1}{M+1} \tag{2.8}$$

since $\sum |J_l|^2 < \frac{1}{4}$.

We now prove that the assumption (2.7) leads to a contradiction. To this end we examine the behaviour of $\sigma(x_0 + t)$ for $x_0 + t \in I_0$. The familiar $M$-condition for $\sigma + \mathrm{id}$ (cf. [1]):

$$\frac{1}{M} \le \frac{t + \sigma(x_0 + t) - \sigma(x_0)}{t + \sigma(x_0) - \sigma(x_0 - t)} \le M \tag{2.9}$$

implies for $t < 0$

$$\Big(\frac{1}{M} - 1\Big)t + \sigma(x_0) \le \sigma(x_0 + t)\,.$$

Note that $\sigma(x_0) - \sigma(x_0 - t) \ge 0$.

By (2.7) for $0 \le t \le \dfrac{M}{2(M+1)} = \beta$ we have

$$0 \le \Big(\frac{1}{M} - 1\Big)t + \frac{1}{2}\,\frac{M-1}{M+1} \le \sigma(x_0 + t)\,. \tag{2.10}$$

Similarly, for $t < 0$ the $M$-condition (2.9) implies

$$\sigma(x_0 + t) \ge (M-1)t + \sigma(x_0) > (M-1)t + \frac{1}{2}\,\frac{M-1}{M+1}\,. \tag{2.11}$$

The last term is positive as soon as

$$0 \ge t \ge -\frac{1}{2}\,\frac{1}{M+1} = -\alpha\,.$$

Note that the length of the interval $[-\alpha, \beta] \subset I_0$ equals $\frac{1}{2}$, i.e. $\alpha + \beta = \frac{1}{2}$. In view of (2.10) and (2.11)

$$\begin{aligned}
\int_{I_0} \sigma(x)\,dx &\ge \int_{-\alpha}^{0} \sigma(x_0 + t)\,dt + \int_0^{\beta} \sigma(x_0 + t)\,dt \\
&\ge \int_{-\alpha}^{\beta} \sigma(x_0)\,dt + \int_{-\alpha}^{0} (M-1)t\,dt + \int_0^{\beta} \Big(\frac{1}{M} - 1\Big)t\,dt \\
&> (\alpha + \beta)\frac{1}{2}\,\frac{M-1}{M+1} - (M-1)\frac{1}{2}\,\alpha^2 - \frac{1}{2}\,\frac{M-1}{M}\,\beta^2 \\
&= \frac{1}{4}\,\frac{M-1}{M+1} - \frac{1}{2}(M-1)\Big(\alpha^2 + \frac{\beta^2}{M}\Big) = \frac{1}{8}\,\frac{M-1}{M+1}\,,
\end{aligned}$$

which contradicts (2.6) and (2.8).

We have for $\sigma \in E_0(M,1)$

$$\begin{aligned}
\int_0^1 |\sigma(x)|^2\,dx &\le \max\{|\sigma(x)| : x \in R\} \cdot \int_0^1 |\sigma(x)|\,dx \\
&\le \frac{1}{2}\,\frac{M-1}{M+1} \cdot \frac{1}{4}\,\frac{M-1}{M+1}\,.
\end{aligned}$$

and (2.2) follows by (2.1) and [3 ; (2.12)].

We now prove

**Lemma 2.2.** *If* $\sigma \in E_1(M,1)$ *then*

$$\left|\sigma\left(\frac{1}{2^n}\right)\right| \le \left(\frac{M}{M+1}\right)^n - \frac{1}{2^n}\ , \quad n \in \mathbf{N}\ . \tag{2.12}$$

**Proof.** Consider the case $n = 1$. Then the inequality

$$M^{-1} \le \frac{1/2 + \sigma(1) - \sigma(1/2)}{1/2 + \sigma(1/2) - \sigma(0)} = \frac{1/2 - \sigma(1/2)}{1/2 + \sigma(1/2)} \le M$$

implies

$$\sigma(1/2) \le \frac{1}{2}\,\frac{M-1}{M+1} = \frac{M}{M+1} - \frac{1}{2} \text{ and also } -\sigma(1/2) \le \frac{M}{M+1} - \frac{1}{2}$$

and (2.12) follows for $n = 1$. Suppose now that (2.12) holds for some $n \in \mathbf{N}$. Then we have

$$\begin{aligned}\frac{1}{M} &\le \frac{2^{-n-1} + \sigma(2^{-n}) - \sigma(2^{-n-1})}{2^{-n-1} + \sigma(2^{-n-1}) - \sigma(0)} \\ &\le \frac{2^{-n-1} + \left(\frac{M}{M+1}\right)^n - 2^{-n} - \sigma(2^{-n-1})}{2^{-n-1} + \sigma(2^{-n-1})}\ .\end{aligned}$$

Hence

$$\sigma\left(\frac{1}{2^{n+1}}\right) \le \left(\frac{M}{M+1}\right)^{n+1} - \frac{1}{2^{n+1}}$$

follows after some obvious calculations. On the other hand, for $\sigma\left(\frac{1}{2^n}\right) < 0$ we have

$$M \ge \frac{2^{-n-1} + \sigma(2^{-n}) - \sigma(2^{-n-1})}{2^{-n-1} + \sigma(2^{-n-1})} \ge \frac{2^{-n-1} + 2^{-n} - \left(\frac{M}{M+1}\right)^n - \sigma(2^{-n-1})}{2^{-n-1} + \sigma(2^{-n-1})}$$

This implies

$$\begin{aligned}\sigma\left(\frac{1}{2^{n+1}}\right) &\ge -\frac{1}{2}\,\frac{M-1}{M+1}\cdot\frac{1}{2^n} - \frac{M^n}{(M+1)^{n+1}} + \frac{1}{2^n}\,\frac{1}{M+1} \\ &= \frac{1}{2^n}\left(\frac{1-M}{1+M} + \frac{1}{2}\right) - \frac{M^n}{(M+1)^{n+1}}\end{aligned}$$

It is sufficient to verify that the last term is $\ge \frac{1}{2^{n+1}} - \left(\frac{M}{M+1}\right)^{n+1}$ which is equivalent to the obvious inequality

$$M \ge 1 + \left(\frac{M+1}{2M}\right)^n (M-1)\quad ,\quad M \ge 1\ ..$$

Hence (2.12) holds for $n+1$ and we are done.

As corollaries of Lemmas 2.1 and 2.2 we obtain following inequalities :
If $\sigma \in E(M)$ and $n \in \mathbf{N}$ then

(2.13) $$\sup\{|\sigma(x)| : x \in \mathbf{R}\} \le \pi \frac{M-1}{M+1},$$

(2.14) $$\int_0^{2\pi} |\sigma(x)|^2\, dx \le \pi^2 \Big(\frac{M-1}{M+1}\Big)^2,$$

(2.15) $$\Big|\sigma\Big(\frac{\pi}{2^n}\Big)\Big| = \Big|\sigma\Big(\frac{2\pi}{2^{n+1}}\Big)\Big| \le 2\pi\Big[\Big(\frac{M}{M+1}\Big)^{n+1} - \frac{1}{2^{n+1}}\Big].$$

Moreover, for any $x \in \mathbf{R}$

(2.16) $$\Big|\sigma\Big(x+\frac{\pi}{2^n}\Big) - \sigma(x)\Big| \le 2\pi\Big[\Big(\frac{M}{M+1}\Big)^{n+1} - \frac{1}{2^{n+1}}\Big].$$

These inequalities are counterparts of inequalities (2.1), (2.2) and (2.12), resp.

**3. Main results.** We now prove a sharpened version of Theorem 2.10 in [3].

**Theorem 3.1.** *If* $\sigma \in E(M)$ *and*

(3.1) $$\sigma(x) = \sum_{n=1}^{\infty} \rho_n \sin(nx + x_n) \quad , \quad \rho_n \ge 0 \; ,$$

*then*

(3.2) $$\sum_{n=1}^{\infty} \rho_n \le \pi\Big(\frac{M-1}{M+1}\Big) + \sqrt{2}\,\pi \sum_{n=2}^{\infty} \sqrt{\Big(\frac{M}{M+1}\Big)^n - \frac{1}{2^n}}$$
$$=: \rho(M) = \begin{cases} O(\sqrt{M-1}) & as\ M \to 1^+ \\ O(M) & as\ M \to \infty\,. \end{cases}$$

**Proof.** If (3.1) holds then

$$\frac{1}{\pi}\int_0^{2\pi} [\,\sigma(x+h) - \sigma(x-h)\,]^2\, dx = 4\sum_{n=1}^{\infty} \rho_n^2 \sin^2 nh$$

cf. [4 ; p.241].
Hence for any $k \in \mathbf{N}$ and $h = \dfrac{\pi}{2^{n+1}}$

$$\int_0^{2\pi}\Big[\sigma\Big(x+\frac{k\pi}{2^n}\Big) - \sigma\Big(x+\frac{(k-1)\pi}{2^n}\Big)\Big]^2 dx = 4\pi \sum_{k=1}^{\infty} \rho_k^2 \sin^2 \frac{\pi k}{2^{n+1}}$$

Moreover, in view of (2.16) we have

$$\sum_{k=1}^{2^{n+1}}\left[\sigma\left(x+\frac{k\pi}{2^n}\right)-\sigma\left(x+\frac{(k-1)\pi}{2^n}\right)\right]^2$$

$$\le 2\pi\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]\sum_{k=1}^{2^{n+1}}\left|\sigma\left(x+\frac{k\pi}{2^n}\right)-\sigma\left(x+\frac{(k-1)\pi}{2^n}\right)\right|$$

$$2\pi\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]V[\sigma]\,,$$

where $V[\sigma]$ stands for the total variation of $\sigma$ over $[0;2\pi]$.

After integrating both sides of the above inequality over $[0;2\pi]$ we obtain

$$2^{n+1}4\pi\sum_{k=1}^{\infty}\rho_k^2\sin^2\frac{\pi k}{2^{n+1}}\le 4\pi^2\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]\cdot V[\sigma]\,.$$

Hence, following [4 ; p.242] we obtain

$$\sum_{k=2^{n-1}+1}^{2^n}\rho_k^2\sin^2\frac{\pi k}{2^{n+1}}\le\pi\cdot 2^{-n-1}\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]\cdot V[\sigma]\,.$$

and hence

$$\sum_{k=2^{n-1}+1}^{2^n}\rho_k^2\le 2\pi\cdot 2^{-n-1}\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]\cdot V[\sigma]$$

Next, in view of the obvious inequality $V[\sigma]\le 4\pi$,

$$\sum_{k=2^{n-1}+1}^{2^n}\rho_k\le\Big(\sum_{k=2^{n-1}+1}^{2^n}\rho_k^2\Big)^{1/2}\Big(\sum_{k=2^{n-1}+1}^{2^n}1\Big)^{1/2}$$

$$\le 2\pi\cdot 2^{-n/2}\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]^{1/2}\cdot 2^{(n-1)/2}$$

$$=\pi\sqrt{2}\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right]^{1/2}.$$

Hence

$$\sum_{n=2}^{\infty}\rho_n=\sum_{n=1}^{\infty}\sum_{k=2^{n-1}+1}^{2^n}\rho_k\le\pi\sqrt{2}\sum_{n=2}^{\infty}\left[\left(\frac{M}{M+1}\right)^n-\frac{1}{2^n}\right]^{1/2}$$

which proves (3.2), in view of the inequality $\rho_1\le\pi\dfrac{M-1}{M+1}$, cf. [3 ; p.235].

We now derive asymptotic behaviour of the bound $\rho(M)$ as $M\to 1^+$ and $M\to\infty$.

For $0<b<a$ we have $a^n-b^n\le n(a-b)a^{n-1}$ and hence

$$\sum_{n=2}^{\infty}\sqrt{\left(\frac{M}{M+1}\right)^n-\frac{1}{2^n}}\le\sqrt{\frac{M}{M+1}-\frac{1}{2}}\sum_{n=1}^{\infty}\sqrt{n+1}\left(\frac{M}{M+1}\right)^{n/2}$$

$$\le\sqrt{\frac{M-1}{2(M+1)}}\left(\sqrt{\frac{M}{M+1}}+(2M+1)\sqrt{M(M+1)}+M(M+1)\right),$$

i.e.

$$\rho(M) = O(\sqrt{M-1}) \quad \text{as } M \to 1^+ .$$

We have

$$\sum_{n=2}^{\infty} \sqrt{\left(\frac{M}{M+1}\right)^n - \frac{1}{2^n}} < \sqrt{\frac{M}{M+1}} \sum_{n=1}^{\infty} \left(\frac{M}{M+1}\right)^{n/2}$$

$$= \sqrt{\frac{M}{M+1}}(M + \sqrt{M(M+1)}) = O(M)$$

and this implies $\rho(M) = O(M)$ as $M \to \infty$.

**Corollary 3.2.** *If $\sigma \in E(M)$ then its Fourier conjugate $\widetilde{\sigma}$ has the bound*

(3.3) $$\sup\{|\widetilde{\sigma}(x)| : x \in \mathbf{R}\} \le \rho(M) = O(\sqrt{M-1})$$

*as $M \to 1^+$.*

The method applied above and due to Zygmund [4] enables us to prove the convergence of the series $\sum \rho_n^\beta$ for $2 > \beta > 2/(2+\alpha)$ where

(3.4) $$\alpha = \log_2\left(1 + \frac{1}{M}\right)$$

and estimate its sum.

We have the following generalization of Theorem 2.1.

**Theorem 3.3.** *If $\sigma \in E(M)$ and (3.1) holds then for $2 > \beta > 2/(2+\alpha)$, where $\alpha$ is defined by* (3.4), *we have*

(3.5) $$\sum_{n=1}^{\infty} \rho_n^\beta \le \frac{1}{2}(2\sqrt{2}\pi)^\beta \sum_{n=1}^{\infty} 2^{n(1-\beta)} \left[\left(\frac{M}{M+1}\right)^{n+1} - \frac{1}{2^{n+1}}\right]^{\beta/2} =: \rho_\beta(M) .$$

*Moreover, $\rho_\beta(M) = O((M-1)^{\beta/2})$ as $M \to 1^+$ and $\rho_\beta(M) = O(1)$ as $M \to \infty$ for fixed $\beta > 1$.*

We omit the proof since it is quite analogous to the proof of Theorem 3.1. We use the following estimate :

$$\sum_{k=2^{n-1}+1}^{2^n} \rho_k^\beta \le \left(\sum_{k=2^{n-1}+1}^{2^n} \rho_k^2\right)^{\beta/2} \left(\sum_{k=2^{n-1}+1}^{2^n} 1\right)^{1-\beta/2}$$

which is a special case of Hölder's inequality with $p = 2/\beta$ , $q = 2/(2-\beta)$.

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## STRESZCZENIE

Jak wykazano w pracach [2], [3] quasikonforemne odwzorowania obszarów jordanowskich w siebie mogą być reprezentowane przez $M$-quasisymetryczne funkcje postaci $x \to x + \sigma(x)$, $x \in \mathbf{R}$, gdzie $\sigma$ jest funkcją okresową, o okresie $2\pi$. W pracy poprawiono niektóre, przedstawione w [3], oszacowania zależne od $M$ dla pewnych norm i współczynników Fouriera funkcji $\sigma$.