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ε-almost Contact Metric Manifolds

Rozmaitości metryczne ε-prawie kontaktowe

Abstract. Let M^{4n+1} be a Riemannian manifold with a certain structure called "ε-almost contact structure". In this paper we study properties of this structure and the existence of a metric and a linear connection compatible with an ε-almost contact metric structure. Further we will consider structures induced on the hypersurfaces immersed in a manifold with given an ε-almost contact metric structure.

1. Introduction. The purpose of this paper is to introduce and study a new general class of almost contact metric manifolds, which will be called ε-almost contact metric manifolds.

In the section 2 we recall some definitions and properties of the ε-almost contact metric structures on a manifold. We will show the existence of the almost contact metric.

In the section 3 we discuss affine connections in an ε-almost contact metric manifold. In the section 4 we study the structure induced on certain hypersurfaces immersed in manifold with the ε-almost contact metric structure.

2. ε-almost contact metric structures.

Definition 2.1. An odd dimensional differentiable manifold M^{4n+1} will be called the manifold with an ε-almost contact structure (F,ω,η,λ), if there exist on M^{4n+1} a tensor field F of the type (1,1), a vector field η, a 1-form ω and a function λ satisfying

$$(2.1)\qquad \begin{cases} F^2 = \varepsilon(I-\omega\otimes\eta) \\ F(\eta) = -\varepsilon\lambda\eta \\ \omega(\eta) = 1-\varepsilon\lambda^2 \neq 0 \\ \omega\circ F = -\varepsilon\lambda\omega \end{cases}$$

where I is the identity mapping on TM^{4n+1} and $\varepsilon=\pm1$.

Definition 2.2. Let (F,ω,η,λ) be an ε-almost contact structure on M^{4n+1} and $\widetilde{g}$ be a Riemannian metric on M^{4n+1} such that

$$\text{(2.2)} \qquad \begin{cases} \widetilde{g}(FX,FY)=\widetilde{g}(X,Y)-\omega(X)\omega(Y) \\ \widetilde{g}(X,\eta)=\omega(X) \end{cases}$$

for any vector fields X, Y on M^{4n+1}; then M^{4n+1} is called an ε-almost contact metric manifold an it will be denoted by $M^{4n+1}(F,\omega,\eta,\lambda,\widetilde{g})$ (see for exp. [1]).

Theorem 2.1. *For any ε-almost contact structure (F,ω,η,λ) on a manifold M^{4n+1} there exists a metric $\widetilde{g}$ satisfying the conditions* (2.2).

Proof. Let g be an arbitrary Riemannian metric on M^{4n+1} and put

$$\begin{aligned}\bar{g}(X,Y)&=g(X,Y)+g(FX,FY)+A[\omega(X)g(Y,\eta)+\omega(Y)g(X,\eta)]\\&+B[\omega(X)g(FY,\eta)+\omega(Y)g(FX,\eta)]+C\omega(X)\omega(Y)\,,\end{aligned}$$

where A, B, C are arbitrary constants.

Then we have

$$\begin{aligned}\bar{g}(FX,FY)&=g(FX,FY)+g(X,Y)-\omega(X)g(Y,\eta)-\omega(Y)g(X,\eta)\\&+\omega(X)\omega(Y)g(\eta,\eta)+A[-\varepsilon\lambda\omega(X)g(FY,\eta)-\varepsilon\lambda\omega(Y)g(FX,\eta)]\\&+B[-\lambda\omega(X)g(Y,\eta)-\varepsilon\omega(Y)g(X,\eta)+2\lambda\omega(X)\omega(Y)g(\eta,\eta)]\\&+C\lambda^2\omega(X)\omega(Y)\,.\end{aligned}$$

The condition

$$\bar{g}(FX,FY)=\bar{g}(X,Y)-\omega(X)\omega(Y)$$

implies the relations

$$\begin{cases} A+\lambda B=-1 \\ \varepsilon\lambda A+B=0 \\ g(\eta,\eta)(2\lambda B+1)+(\lambda^2-1)C=-1\,. \end{cases}$$

Hence we get

$$A=\frac{-1}{1-\varepsilon\lambda^2} \qquad B=\frac{\varepsilon\lambda}{1-\varepsilon\lambda^2}$$

(in view of the condition (2.1) we have $1-\varepsilon\lambda^2=\omega(\eta)\neq 0$) and

$$C=\frac{g(\eta,\eta)(1+\varepsilon\lambda^2)+(1-\varepsilon\lambda^2)}{(1-\varepsilon\lambda^2)(1-\lambda^2)}\,.$$

Thus we have

$$\begin{aligned}\bar{g}(X,Y)&=g(X,Y)+g(FX,FY)-\frac{1}{1-\varepsilon\lambda^2}[\omega(X)g(Y,\eta)+\omega(Y)g(X,\eta)]\\&+\frac{\varepsilon\lambda}{1-\varepsilon\lambda^2}[\omega(X)g(FY,\eta)+\omega(Y)g(FX,\eta)]\\&+\frac{g(\eta,\eta)(1+\varepsilon\lambda^2)+(1-\varepsilon\lambda^2)}{(1-\varepsilon\lambda^2)(1-\lambda^2)}\omega(X)\omega(Y)\end{aligned}$$

and

$$\bar{g}(X,\eta) = \omega(X)\frac{1-\varepsilon\lambda^2}{1-\lambda^2} = \omega(X)$$

(if $\varepsilon = -1$, then $\lambda = 0$, [2]).

We see that the metric $\bar{g}$ defined by (2.3) satisfies the conditions (2.2) and we can take

$$\widetilde{g}(X,Y) = \bar{g}(X,Y)\,.$$

It completes the proof.

3. Existence of a linear connection compatible with ε–almost contact metric structure.

Definition 3.1. Let $\overset{*}{\nabla}$ denote the covariant differentiation with respect to linear connection $\overset{*}{\Gamma}$ on manifold M^{4n+1}. We say that connection $\overset{*}{\Gamma}$ is compatible with ε-almost contact metric structure $(F,\omega,\eta,\lambda,\widetilde{g})$ on M^{4n+1} if

$$\overset{*}{\nabla} F = 0\,, \quad \overset{*}{\nabla}\eta = 0 \quad \text{and} \quad \omega(\overset{*}{\nabla}_X Y) = 0$$

for any vector fields X, Y on M^{4n+1}.

Theorem 3.1. *Let ε–almost contact metric manifold M^{4n+1} $(F,\omega,\eta,\lambda,\widetilde{g})$ be given. There exists on M^{4n+1} a linear connection $\overset{*}{\Gamma}$ compatible with ε-almost contact metric structure $(F,\omega,\eta,\lambda,\widetilde{g})$.*

Proof. Let ∇ denote the covariant differentiation with respect to Riemannian connection defined by the metric $\widetilde{g}$ satisfying the conditions (2.2). We define a connection $\overset{*}{\Gamma}$ as follows

$$\begin{aligned} \overset{*}{\nabla}_X Y =& a[\nabla_X Y + \varepsilon F(\nabla_X FY)] + b[\nabla_X FY + F(\nabla_X Y)] \\ &+ \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\omega(\nabla_X Y)\eta + \omega(Y)\nabla_X\eta] \\ &+ \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\omega(\nabla_X FY)\eta + \omega(Y)F(\nabla_X\eta)]\,, \end{aligned} \tag{3.1}$$

where $a, b \in R$. Then for any vector fields X, Y on M^{4n+1} we have

$$\begin{aligned} (\overset{*}{\nabla}_X F)(Y) = \overset{*}{\nabla}_X FY - F(\overset{*}{\nabla}_X Y) = a[\, \nabla_X FY + \varepsilon F(\nabla_X F^2 Y)\,] \\ + b[\, \nabla_X F^2 Y + F(\nabla_X FY)] + \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\, \omega(\nabla_X FY)\eta + \omega(FY)\nabla_X\eta\,] \\ + \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\, \omega(\nabla_X F^2 Y)\eta + \omega(FY)F(\nabla_X\eta)\,] \\ - a[\, F(\nabla_X Y) + \varepsilon F^2(\nabla_X FY)\,] - b[\, F(\nabla_X FY) + F^2(\nabla_X Y)\,] \\ - \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\, \omega(\nabla_X Y)F\eta + \omega(Y)F(\nabla_X\eta)\,] \\ - \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\, \omega(\nabla_X FY)F\eta + \omega(Y)F^2(\nabla_X\eta)\,] = \end{aligned}$$

$$
\begin{aligned}
&= a\,[\,\nabla_X FY + F(\nabla_X Y) + \varepsilon\lambda(\nabla_X\omega(Y))\eta - \omega(Y)F(\nabla_X\eta)\,] \\
&+ b\,[\,\varepsilon\nabla_X Y - \varepsilon(\nabla_X\omega(Y))\eta - \varepsilon\omega(Y)\nabla_X\eta + F(\nabla_X FY)\,] \\
&+ \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\,\omega(\nabla_X FY)\eta - \varepsilon\lambda\omega(Y)\nabla_X\eta\,] \\
&+ \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\,\varepsilon\omega(\nabla_X Y)\eta - \varepsilon(1-\varepsilon\lambda^2)(\nabla_X\omega(Y))\eta \\
&- \varepsilon\omega(Y)\omega(\nabla_X\eta)\eta - \varepsilon\lambda\omega(Y)F(\nabla_X\eta)\,] \\
&- a\,[\,F(\nabla_X Y) + \nabla_X FY - \omega(\nabla_X FY)\eta\,] \\
&- b\,[\,F(\nabla_X FY) + \varepsilon\nabla_X Y - \varepsilon\omega(\nabla_X Y)\eta\,] \\
&- \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\,-\varepsilon\lambda\omega(\nabla_X Y)\eta + \omega(Y)F(\nabla_X\eta)\,] \\
&- \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\,-\varepsilon\lambda\omega(\nabla_X FY)\eta + \varepsilon\omega(Y)\nabla_X\eta \\
&- \varepsilon\omega(Y)\omega(\nabla_X\eta)\eta\,] = 0
\end{aligned}
$$

and hence

$$\overset{\bullet}{\nabla} F = 0\,.$$

Moreover, for any vector fields X, Y on M^{4n+1} we get

$$
\begin{aligned}
\overset{\bullet}{\nabla}_X\eta &= a\,[\,\nabla_X\eta + \varepsilon F(\nabla_X F\eta)\,] + b\,[\,\nabla_X F\eta + F(\nabla_X\eta)\,] \\
&+ \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\,\omega(\nabla_X\eta)\eta + \omega(\eta)\nabla_X\eta\,] \\
&+ \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\,\omega(\nabla_X F\eta)\eta + \omega(\eta)F(\nabla_X\eta)\,] \\
&= a\,[\,\nabla_X\eta + \varepsilon\lambda(\partial_X\lambda)\eta - \lambda F(\nabla_X\eta)\,] \\
&+ b\,[\,-\varepsilon(\partial_X\lambda)\eta - \varepsilon\lambda\nabla_X\eta + F(\nabla_X\eta)\,] \\
&+ \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\,\omega(\nabla_X\eta)\eta + (1-\varepsilon\lambda^2)\nabla_X\eta\,] \\
&+ \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\,-\varepsilon(\partial_X\lambda)(1-\varepsilon\lambda^2)\eta - \varepsilon\lambda\omega(\nabla_X\eta)\eta \\
&+ (1-\varepsilon\lambda^2)F(\nabla_X\eta)\,] = \frac{2\varepsilon\lambda b - a(1+\varepsilon\lambda^2)}{1-\varepsilon\lambda^2}\,\omega(\nabla_X\eta)\eta
\end{aligned}
$$

and

$$
\begin{aligned}
(\overset{\bullet}{\nabla}_X\omega)(Y) &= \partial_X\omega(Y) - \omega(\overset{\bullet}{\nabla}_X Y) = \partial_X\omega(Y) \\
&- a\,[\,\omega(\nabla_X Y) - \lambda\omega(\nabla_X FY)\,] - b\,[\,\omega(\nabla_X FY) - \varepsilon\lambda\omega(\nabla_X Y)\,] \\
&- \frac{\varepsilon\lambda b - a}{1-\varepsilon\lambda^2}\,[\,\omega(\nabla_X Y)(1-\varepsilon\lambda^2) + \omega(Y)\omega(\nabla_X\eta)\,] \\
&- \frac{\lambda a - b}{1-\varepsilon\lambda^2}\,[\,\omega(\nabla_X FY)(1-\varepsilon\lambda^2) - \varepsilon\lambda\omega(Y)\omega(\nabla_X\eta)\,] \\
&= \partial_X\omega(Y) - \frac{2\varepsilon\lambda b - a(1+\varepsilon\lambda^2)}{1-\varepsilon\lambda^2}\,\omega(Y)\omega(\nabla_X\eta)\,.
\end{aligned}
$$

If
$$a = \frac{2\varepsilon\lambda b}{1+\varepsilon\lambda^2}$$
then
$$\dot{\nabla}_X \eta = 0 \quad \text{and} \quad (\dot{\nabla}_X \omega)(Y) = \partial_X \omega(Y)$$
for any vector fields X, Y on M^{4n+1}.

The above considerations imply that
$$\omega(\dot{\nabla}_X Y) = 0$$
for any vector fields X, Y on M^{4n+1}.

Corollary . *If for any vector fields X, Y on M^{4n+1} and $a, b \in \mathbf{R}$*

$$\begin{aligned}\tilde{\nabla}_X Y = {} & \frac{2\varepsilon\lambda b}{1+\varepsilon\lambda^2}\left[\nabla_X Y + \varepsilon F(\nabla_X FY)\right] + b[\nabla_X FY + F(\nabla_X Y)] \\ & - \frac{\varepsilon\lambda b}{1+\varepsilon\lambda^2}\left[\omega(\nabla_X Y)\eta + \omega(Y)\nabla_X \eta\right] \\ & - \frac{b}{1+\varepsilon\lambda^2}\left[\omega(\nabla_X FY)\eta + \omega(Y)F(\nabla_X \eta)\right],\end{aligned} \tag{3.2}$$

then

$$\tilde{\nabla} F = 0\,, \quad \tilde{\nabla}_X \eta = 0\,, \quad \omega(\tilde{\nabla}_X Y) = 0\,. \tag{3.3}$$

Let W denote the distribution orthogonal to vector field η with respect to $\tilde{g}$. Then for arbitrary vector field $X \in W$ we have
$$\tilde{g}(X, \eta) = \omega(X) = 0$$
(see (2.2)).

Theorem 3.2. *The distribution W is integrable.*

Proof. Let $X, Y \in W$. Then $\omega(X) = \omega(Y) = 0$. From (3.3) we get
$$\omega([X, Y]) = \omega(\tilde{\nabla}_X Y) - \omega(\tilde{\nabla}_Y X) = 0\,,$$
what implies that $[X, Y] \in W$.

4. The structures induced on hypersurfaces. Let us assume that M^{4n+1} is a Riemannian manifold with a metric g satisfying the conditions (2.2) and ε-almost contact metric structure $(F, \omega, \eta, \lambda, \tilde{g})$ satisfying (2.1). Moreover, let M^{4n} be smooth, oriented hypersurface immersed in M^{4n+1}. By N we denote the local vector field such that N is orthogonal to TM^{4n} and $\tilde{g}(N, N) = 1$. Then for each vector field $X \in TM^{4n+1}$ we have the following decomposition

$$FX = \varphi X + \omega(X)N \tag{4.1}$$

where φ is a tensor field of the type (1,1), $\varphi X \in TM^{4n}$, Ω is a tensor field of the type (0,1) (see, [2]). Then

$$(\omega \circ F)(X) = (\omega \circ \varphi)(X) + \varepsilon\Omega(X)\omega(N) \tag{4.2}$$

We introduce the notations

$$\begin{aligned} \varphi(N) &= \xi \in TM^{4n} \\ \Lambda &= \Omega(N) \end{aligned} \tag{4.3}$$

The conditions (2.1) imply

$$-\varepsilon\lambda\omega(X) = (\omega \circ \varphi)X + \varepsilon\Omega(X)\omega(N) . \tag{4.4}$$

For any vector field X we have

$$F^2(X) = F(\varphi X + \varepsilon\Omega(X)N) = \varphi^2 X + \varepsilon\Omega(X)\xi + \varepsilon(\Omega \circ \varphi)(X)N + \Lambda\Omega(X)N$$

The above relation and (1.1) lead us to the following condition

$$\varepsilon X - \varepsilon\omega(X)\eta = \varphi^2 X + \varepsilon\Omega(X)\xi + \varepsilon(\Omega \circ \varphi)(X)N + \Lambda\Omega(X)N \tag{4.5}$$

I. If $N = \eta$, then $\tilde{g}(\varphi X, \eta) = (\omega \circ \varphi)(X) = 0$. Substituting $X = \eta$ to the condition (4.4) and using (1.1) we obtain

$$-\varepsilon\lambda(1 - \varepsilon\lambda^2) = (\omega \circ \varphi)(\eta) + \varepsilon\Omega(\eta)(1 - \varepsilon\lambda^2).$$

Hence we get

$$\Omega(\eta) = \Lambda = -\lambda \tag{4.6}$$

and

$$\Omega(X) = -\frac{\lambda}{1 - \varepsilon\lambda^2}\,\omega(X) = 0 \quad \text{for} \quad X \in TM^{4n}$$

Making use of (4.6) we can write

$$\varphi^2 = \varepsilon I \quad \text{on} \quad M^{4n} \tag{4.7}$$

The above implies that if M^{4n} is an integral manifold of the distribution W (Theorem 3.2), then we have the following:

Theorem 4.1. *The ε-almost contact metric structure $(F, \omega, \eta, \lambda, \tilde{g})$ given on the $(4n+1)$-dimensional Riemannian manifold M^{4n+1} induces the structure φ on a hypersurface M^{4n} which satisfies the condition* (4.7).

II. If $\tilde{g}(N, \eta) = 0$, then $\omega(N) = 0$ and $(\omega \circ \varphi)(X) = -\varepsilon\lambda\omega(X)$ for $X \in TM^{4n}$, $\omega(\xi) = -\varepsilon\lambda\omega(N) = 0$. Moreover, from (4.5) we get on M^{4n}:

$$\begin{cases} \varphi^2 = \varepsilon(I - \omega \otimes \eta - \Omega \otimes \xi) \\ \Omega \circ \varphi = -\varepsilon\Lambda\Omega \\ \varphi\xi = -\varepsilon\Lambda\xi \\ \Omega(\xi) = 1 - \varepsilon\Lambda^2 \end{cases} \tag{4.8}$$

The above implies that if the manifold M^{4n} is orthogonal to the field N and $\widetilde{g}(N,\eta) = 0$, then we have

Theorem 4.2. *The ε-almost contact metric structure $(F,\omega,\eta,\lambda,\widetilde{g})$ on the $(4n+1)$-dimensional Riemannian manifold M^{4n+1} induces the structure $(\varphi,\omega,\Omega,\xi,\eta,\Lambda)$ on a hypersurface M^{4n} which satisfies the conditions* (4.8).

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STRESZCZENIE

Niech M^{4n+1} będzie rozmaitością Riemannowską z pewną strukturą zwaną "strukturą ε-prawie kontaktową". W pracy badamy własności tej struktury oraz istnienie metryki i koneksji liniowej zgodnych ze strukturą metryczną ε-prawie kontaktową. Ponadto rozważamy struktury indukowane na hiperpowierzchniach zanurzonych w rozmaitości z zadaną ε-prawie kontaktową strukturą metryczną.