ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLIII, 8 SECTIO A 1989

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On the First Order Natural Operators Transforming 1-forms on Manifold to the Tangent Bundle

O operatorach naturalnych pierwszego rzędu transformujących 1-formy na rozmaitości do wiązki stycznej

Abstract. In this paper all first order natural operators transforming 1-forms on a manifold to tangent bundle, are determined. Fundamental operators of this type are a complete lift and a vertical lift of a 1-form. All first order natural operators form a 3-parameter family with coefficients being smooth functions of one variable.

The aim of this paper is to determine all first order natural operators transforming 1-forms on a manifold to the tangent bundle.

We deduce that the fundamental operator here are a complete lift and a vertical lift of a 1-form.

In the paper we use an invariant function theorem developed by I. Kolář, [2].

The author is grateful to Professor I.Kolář for suggesting the problem, valuable remarks and usrful discussions.

1. Let M be a smooth n-dimensional manifold. We denote by $p_M : TM \to M$ a tangent bundle and by $q_M : T^*M \to M$ a cotangent bundle.

A classical field of 1-forms ω on the manifold M can be interpreted as a linear map $\omega : TM \to R$ with respect to the vector bundle structure $p_M : TM \to M$. If a 1-form ω has in a local chart (U, x^i) on M the local expression $\omega = b_i(x)dx^i$, then the linear map $\omega : TM \to R$ in a local induced chart $(p_M^{-1}(U), x^i, X^i)$ on TM is of the form $\omega = b_i(x)X^i$.

Consider the tangent map $T\omega : TTM \to TR = R \times R$. The second component of the tangent map $\Omega = pr_2 \circ T\omega$, where $pr_2 : R \times R \to R$ is a projection on the second factor, defines a linear map $\Omega = pr_2 \circ T\omega : TTM \to R$ with respect to the vector bundle structure $p_{TM} : TTM \to TM$.

Definition 1. A field of 1-forms Ω on TM defined by the second component of the tangent map $T\omega$, i.e.

$$\Omega = pr_2 \circ T\omega : TTM \to R \tag{1.1}$$

is called a complete lift of a field of 1-forms ω on M.

Definition 2. A field of 1-forms Ω on TM defined as the image of a field of 1- forms ω on M under a dual map p_M^* of the projection $p_M : TM \to M$, i.e.

$$\Omega = p_M^* \omega = \omega \circ Tp_M \ : \ TTM \to R \tag{1.2}$$

is called a vertical lift of a field of 1-forms ω on M.

If a field of 1-forms ω on M has in a local chart (U, x^i) a local expression $\omega = b_i(x)\ dx^i$, then the complete lift $\Omega = \omega^c$ and the vertical lift $\Omega = \omega^v$ in the local induced chart $p_M^{-1}(U), x^i, X^i)$ on TM are of the form

$$\omega^c = b_{ij} X^i dx^j + b_i dX^i \tag{1.3}$$

$$\omega^v = b_i dx^i \ . \tag{1.4}$$

If $\tau_M : TTM \to TTM$ is a canonical involution, then we have a field of 1-forms $\omega^c \circ \tau_M$ on TM

$$\omega^c \circ \tau_M = b_{ji} X^i dx^j + b_i dX^i \tag{1.5}$$

We need the following invariant function theorem developed by I. Kolář, [2].

Theorem 1. *Let* $f : \underbrace{R^n \times \ldots \times R^n}_{k\text{-times}} \times \underbrace{R^{n\bullet} \times \ldots \times R^{n\bullet}}_{l\text{-times}} \to R$ *be a smooth and* $Gl(n, R)$ *invariant map. Then there exists a smooth function* $\varphi : R^{k \cdot l} \to R$ *such that*

$$f(x_i, y^p)_{\substack{i = 1, \ldots, l \\ p = 1, \ldots, k}} = \varphi(\langle x_i, y^p \rangle)_{\substack{i = 1, \ldots, l \\ p = 1, \ldots, k}} \ . \tag{1.6}$$

We will use the following

Lemma 2. *Every* $Gl(n, R)$ *invariant smooth map* $G : R^n \times R^n \times R^{n\bullet} \times \otimes^2 R^{n\bullet} \to R$ *is of the form*

$$G(y^i, X^i, b_i, b_{ij}) = \\ = \psi(y^i b_i, X^i b_i, y^i y^j b_{ij}, y^i X^j b_{ij}, X^i y^j b_{ij}, X^i X^j b_{ij}) \ , \tag{1.7}$$

where $\psi : R^6 \to R$ *is a smooth function.*

Proof. Consider any $Gl(n, R)$ invariant smooth map $\widetilde{G} : R^n \times R^n \times R^{n\bullet} \times R^{n\bullet} \times R^{n\bullet} \to R$ such that $\widetilde{G} = G \circ \otimes$, i.e.

$$\widetilde{G}(y^i, X^i, b_i, u_i, v_i) = G(y^i, X^i, b_i, u_i, v_j) \ . \tag{1.8}$$

By the invariant function theorem there exists a smooth function $\varphi : R^6 \to R$ such that

$$\widetilde{G}(y^i, X^i, b_i, u_i, v_i) = \varphi(y^i b_i, y^i u_i, y^i v_i, X^i b_i, X^i u_i, X^i v_i) \ . \tag{1.9}$$

Taking into account the invariance with respect to $u_i \mapsto k \cdot u_i$, $v_i \mapsto \frac{1}{k} \cdot v_i$ for $k \in R \setminus \{0\}$, we obtain the relation

$$\varphi(\alpha,\beta,\gamma,\delta,\varepsilon,\omega) = \varphi(\alpha, k\cdot\beta, \tfrac{1}{k}\cdot\gamma, \delta, k\cdot\varepsilon, \tfrac{1}{k}\cdot\omega) \,. \tag{1.10}$$

By the invariant function theorem for $n=1$, there exists a smooth function $\psi : R^4 \to R$ depending on two parameters α, δ such that

$$\varphi(\alpha,\beta,\gamma,\delta,\varepsilon,\omega) = \psi(\alpha,\delta,\beta\cdot\gamma,\beta\cdot\omega,\varepsilon\cdot\gamma,\varepsilon\cdot\omega) \,. \tag{1.11}$$

Thus, we obtain

$$G(y^i, X^i, b_i, u_i \cdot v_j) = \psi(y^i b_i, X^i b_i, y^i y^j u_i v_j, y^i X^j u_i v_j, X^i y^j u_i v_j, X^i X^j u_i v_j) \,. \tag{1.12}$$

This proves the lemma, if we put $b_{ij} = u_i \cdot v_j$.

2. In this part we determine all first order natural operators transforming 1-forms on manifold M to the tangent bundle TM.

Theorem 3. *All first order natural operators $F : T^*M \to T^*TM$ set up a 3-parameter family of the form*

$$\begin{aligned} F : b_i dx^i \mapsto & a(b_k X^k)[b_i dx^i] + b(b_k X^k)[b_{ij} X^j dx^i + b_i dX^i] + \\ & + c(b_k X^k)[b_{ji} X^j dx^i + b_i dX^i] \,, \end{aligned} \tag{2.1}$$

where a, b, c are three arbitrary smooth functions of one variable.

Proof. Any map $F : T^*M \to T^*TM$ in local coordinates (x^i) on M and (x^i, X^i) on TM is of the form

$$F : b_i(x) dx^i \mapsto e_i(x^k, X^k) dx^i + g_i(x^k, X^k) dX^i \,. \tag{2.2}$$

The first order natural operators $F : T^*M \to T^*TM$ are in bijection with natural transformations $F : J^1T^*M \to T^*TM$ and L^2_n-equivariant maps of standard fibres $F : (J^1T^*R^n)_0 \to (T^*TR^n)_0$.

The group L^2_n acts on the standard fibre $S = (J^1T^*R^n)_0$ in the form

$$\begin{aligned} \overline{b}_i &= b_j \widetilde{a}^j_i \\ \overline{b}_{ij} &= b_{kl} \widetilde{a}^k_i \widetilde{a}^l_j + b_k \widetilde{a}^k_{ij} \end{aligned} \tag{2.3}$$

We denote by $(\widetilde{a}^i_j, \widetilde{a}^i_{jk})$ the coordinates of the inverse element a^{-1} of an element $a \in L^2_n$ with coordinates (a^i_j, a^i_{jk}).

The group L^2_n acts on the standard fibre $W = (T^*TR^n)_0$ by formula

$$\begin{aligned} \overline{X}^i &= a^i_j X^j \\ \overline{e}_i &= e_j \widetilde{a}^j_i + g_k \widetilde{a}^k_{li} a^l_j X^j \\ \overline{g}_i &= g_j \widetilde{a}^j_i \,. \end{aligned} \tag{2.4}$$

Any map $F : (J^1T^*R^n)_0 \to (T^*TR^n)_0$ in coordinates (b_i, b_{ij}) and (X^i, e_i, q_i) is of the form

$$e_i = e_i(X^i, b_i, b_{ij}) \qquad (2.5)$$
$$g_i = g_i(X^i, b_i, b_{ij})$$

Our aim is to find a general form of an L^2_n–equivariant smooth maps $e_i : R^n \times R^{n\bullet} \times \otimes^2 R^{n\bullet} \to R^{n\bullet}$ and $g_i : R^n \times R^{n\bullet} \times \otimes^2 R^{n\bullet} \to R^{n\bullet}$.

We define an L^1_n–invariant smooth map $G : R^n \times R^n \times R^{n\bullet} \times \otimes^2 R^{n\bullet} \to R$ by formula

$$G(y^i, X^i, b_i, b_{ij}) = g_i(X^i, b_i, b_{ij}) \cdot y^i \ . \qquad (2.6)$$

Considering equivalence with respect to homotheties $y^i \mapsto ky^i$ of L^1_n–invariant map $G = g_i y^i$ of the form (1.7), we get

$$\begin{aligned}\psi(ky^ib_i, X^ib_i, k^2y^iy^jb_{ij}, ky^iX^jb_{ij}, kX^iy^jb_{ij}, X^iX^jb_{ij}) = \\ = k\cdot\psi(y^ib_i, X^ib_i, y^iy^jb_{ij}, y^iX^jb_{ij}, X^iy^jb_{ij}, X^iX^jb_{ij})\ .\end{aligned} \qquad (2.7)$$

From this, the map ψ is linear in y^ib_i , $y^iX^jb_{ij}$, $X^iy^jb_{ij}$ and is independent of $y^iy^jb_{ij}$, where coefficients are three arbitrary smooth functions p, q, r of two variables depending on b_iX^i and $b_{ij}X^iX^j$.

Thus, every L^1_n–invariant map $g_i : R^n \times R^{n\bullet} \times \otimes^2 R^{n\bullet} \to R^{n\bullet}$ is of the form

$$\begin{aligned}g_i(X^k, b_k, b_{kl}) = p(b_kX^k, b_{kl}X^kX^l)b_i + q(b_kX^k, b_{kl}X^kX^l)b_{ij}X^j + \\ + r(b_kX^k, b_{kl}X^kX^l)b_{ji}X^j\end{aligned} \qquad (2.8)$$

with arbitrary smooth function p, q, r of two variables. In the same way, we obtain L^1_n–invariant map $e_i : R^n \times R^{n\bullet} \times \otimes^2 R^{n\bullet} \to R^{n\bullet}$ of the form

$$\begin{aligned}e_i(X^k, b_k, b_{kl}) = a(b_kX^k, b_{kl}X^kX^l)b_i + b(b_kX^k, b_{kl}X^kX^l)b_{ij}X^j + \\ + c(b_kX^k, b_{kl}X^kX^l)b_{ij}X^j\end{aligned} \qquad (2.9)$$

with arbitrary smooth functions a, b, c of two variables.

We will consider L^2_n–equivariance of the map $F : (J^1T^*R^n)_0 \to (T^*TR^n)_0$. If the map F is L^2_n–equivariant, then for every vector $A = (A^i_j, A^i_{jk})$ of the Lie algebra l^2_n of L^2_n the corresponding fundamental vector fields A_s on $S = (J^1T^*R^n)_0$ and A_w on $W = (T^*TR^n)_0$ must be F–related. This gives the following system of partial differential equations for maps g_i and e_i with parameters A^i_j, A^i_{jk} :

$$\begin{aligned}-A^j_i g_j = {} & \frac{\partial p}{\partial u^2}(-b_kA^k_{lm}X^lX^m)b_i - pA^j_ib_j + \\ & + \frac{\partial q}{\partial u^2}(-b_kA^k_{lm}X^lX^m)b_{ij}X^j - qb_{kl}A^k_iX^l - qb_kA^k_{ij}X^j + \\ & + \frac{\partial r}{\partial u^2}(-b_kA^k_{lm}X^lX^m)b_{ij}X^j - rb_{kl}A^l_iX^k - rb_kA^k_{ji}X^j ,\end{aligned} \qquad (2.10)$$

$$\begin{aligned}-A^j_ie_j - A^k_{ji}X^jg_k = {} & \frac{\partial a}{\partial u^2}(-b_kA^k_{lm}X^lX^m) - aA^j_ib_j + \\ & + \frac{\partial b}{\partial u^2}(-b_kA^k_{lm}X^lX^m)b_{ij}X^j - bb_{kl}A^k_iX^l - bb_kA^k_{ij}X^j + \\ & + \frac{\partial c}{\partial u^2}(-b_kA^k_{lm}X^lX^m)b_{ji}X^j - cb_{kl}A^l_iX^k - cb_kA^k_{ji}X^j\ .\end{aligned} \qquad (2.11)$$

First, we consider the differential equation (2.10). Setting $A^i_j = 0$ in (2.10), we obtain

$$\frac{\partial p}{\partial u^2} = 0 \quad , \quad \frac{\partial q}{\partial u^2} = 0 \quad , \quad \frac{\partial r}{\partial u^2} = 0 \tag{2.12}$$

$$q + r = 0 \; . \tag{2.13}$$

By means of (2.12), we get that smooth functions p, q, r of two variables are independent of the second variable. Thus, the map $g_i : R^n \times R^{n*} \times \otimes^2 R^{n*} \to R^{n*}$ is of the form

$$g_i(X^k, b_k, b_{kl}) = p(b_k X^k) b_i - q(b_k X^k) b_{ij} X^j - q(b_k X^k) b_{ji} X^j \tag{2.14}$$

Now, setting $A^i_j = 0$ in (2.11) and using (2.13), we obtain

$$\frac{\partial a}{\partial u^2} = 0 \quad , \quad \frac{\partial b}{\partial u^2} = 0 \quad , \quad \frac{\partial c}{\partial u^2} = 0 \tag{2.15}$$

$$p = b + c \quad , \quad q = 0 \; . \tag{2.16}$$

By means of (2.15), we get that smooth functions a, b, c of two variables are independent of the second variable. Thus, the maps $e_i : R^n \times R^{n*} \times \otimes^2 R^{n*} \to R^{n*}$ and $g_i : R^n \times R^{n*} \times \otimes^2 R^{n*} \to R^{n*}$ are of the form

$$\begin{aligned} e_i(X^k, b_k, b_{kl}) &= a(b_k X^k) b_i + b(b_k X^k) b_{ij} X^j + c(b_k X^k) b_{ji} X^j \\ g_i(X^k, b_k, b_{kl}) &= [\, b(b_k X^k) + c(b_k X^k) \,] b_i \; . \end{aligned} \tag{2.17}$$

Finally, using (2.17) in (2.2), we obtain the 3-parameter system of natural operators of the form (2.1). This proves theorem.

The geometrical interpretation of the 3-parameter system (2.1) of first order natural operators $F : T^*M \to T^*TM$ is

$$F : \omega \mapsto a(b_i X^i) \cdot \omega^v + b(b_i X^i) \cdot \omega^c + c(b_i X^i) \cdot \omega^c \circ \tau_M$$

where ω^v and ω^c are the vertical lift and the complete lift of ω.

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STRESZCZENIE

W pracy wyznacza się wszystkie operatory naturalne pierwszego rzędu transformujące 1-formy na rozmaitości do wiązki stycznej. Podstawowymi operatorami tego typu są podniesienie zupełne i podniesienia wertykalne 1-formy. Wszystkie operatory naturalne pierwszego rzędu stanowią 3-parametrową rodzinę ze współczynnikami będącymi funkcjami gładkimi jednej zmiennej.